LIMIT THEOREMS IN STATISTICAL PHYSICS; ON DYSON'S HIERARCHICAL MODEL

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Problems about the limit distribution of appropriately normalized partial sums of strongly dependent random variables appear in statistical physics in a natural way. Formally the problem is very similar to that appearing in classical probability theory, but there are very essential differences. In classical probability theory one generally knows the asymptotic behaviour of the correlation function at the beginning. In statistical physics we are looking for the limit distribution of partial sums of Gibbs distributed random variables. This Gibbs distribution depends on a physical parameter, the temperature T. This dependence is very intricate, and our main goal is to understand how the behaviour of the partial sums depends on the paramter. In interesting cases there is a special value of the parameter, the so-called critical parameter T_{cr}, where the partial sums satisfy a limit theorem with a different norming than at any other parameter. This particular role of the critical parameter is also reflected in the behaviour of the correlation function by its slow decrease in infinity. But we cannot describe the behaviour of the correlation function at the beginning. What we are able to do is first to solve the limit problem and then to describe the asymptotic behaviour of the correlation function with its help. The study of these questions is not carried out completely in the case of general models because of some very deep mathematical difficulties. Hence in this report we restrict ourselves to a special case, the so-called Dyson's hierarchical model, where we can give a fairly complete picture about the behaviour of the model at the critical parameter. This also helps us to understand what is happening in the general situation.

In order to define Dyson's hierarchical model first we have to introduce its Hamiltonian function H and a free measure . Let $\sigma(j)$, $j\in \mathbb{N}$, $\mathbb{N}=\{1,2,\dots\}$ $\sigma(j)\in R$ be an arbitrary sequence. For a fixed number n we define the Hamiltonian function of the subsequence $\bar\sigma(n)=\{\sigma(1),\dots,\sigma(2^n)\}$ in the following way:

$$H(\bar\sigma(n)) = -\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} d(i,j)^{-a}\sigma(i)\sigma(j),$$

where a, $1<a<2$, is an important parameter of the model, $d(i,j)$, the so-called hierarchical distance, is a modified version of the usual distance $\bar d(i,j)=|i-j|$, and it is defined by the formulas $d(i,j)=2^{n(i,j)}$,

$n(i,j)=\{\min k, \exists\, \ell$ such that $\ell 2^k \leq i,j\ (\ell+1)2^k\}$.

We also define the so-called free measure ν on the real line given by the formula

$$\frac{d\nu(x)}{dx} = p(x)=p(x,u)= \exp(-\frac{x^2}{2} - \frac{u}{4}x^4),$$

where $u>0$ is sufficiently small. What is important for us in this definition is that $p(x)$ is close to a Gaussian density, and it tends to zero at infinity faster than any Gaussian density. With the help of the function H the measure ν and a parameter T, $T>0$, we introduce the following probability measures $\mu_n(T)$, $n=0,1,2,\dots$ on R^{2^n}:

$$\mu_{n,T}(d\bar\sigma(n)) = \frac{1}{Z_n(T)}\{\exp -\frac{1}{T}H(\bar\sigma(n)\}\prod_{i=1}^{2^n}\mu(d\sigma(i)),$$

where $Z_n(T)$ is an appropriate norming constant. The measure $\mu_{n,T}$ is called the Gibbs distribution with Hamiltonian H at temperature T in the volume $\{1,2,\dots,2^n\}$.

Let $\sigma(1),\dots,\sigma(2^n)$ be a $\mu_{n,T}$ distributed random vector, and let $p_n(x,T)$ denote the density function of the average $\frac{1}{2^n}\sum_{i=1}^{2^n}\sigma(i)$. One of the main problem in the study of Dyson's hierarchical model is the description of the asymptotic behaviour of $p_n(x,T)$ for large n. In particular, with what kind of rescaling (in the variable x) has it a limit as $n\to\infty$? How does the answer depend on the parameter T? In the present report we restrict ourselves to this problem.

It is relatively simple to show that the relation

$$H((\sigma(n+1))=H(\bar{\sigma}(n))+H(\bar{\bar{\sigma}}(n))-2^{-(n+1)a}\sum_{i=1}^{2^n}\sigma(i)\sum_{j=2^n+1}^{2^{n+1}}\sigma(j)$$

holds with $\bar{\bar{\sigma}}(n)=\{\sigma(2^n+1),\dots,\sigma(2^{n+1})\}$, and it implies the formula

$$p_{n+1}(x,T)=C_n(T)\int\exp\left(\frac{c^n}{T}(x^2-v^2)\right)p_n(x-v,T)p_n(x+v,T)dv \quad (1)$$

with $c=2^{2-a}$ and an appropriate norming constant $C_n(T)$. On the other hand

$$p_o(x,T)=C(u)\exp\left(-\frac{x^2}{2}-\frac{u}{4}x^4\right). \quad (2)$$

Thus the problem we are interested in is equivalent to the description of the asymptotic behaviour of the sequence $p_n(x,T)$ defined by the recursive formulas (1) and (2). In order to simplify the latter problem we introduce the functions

$$q_n(x)=q_n(x,T)=B_n\exp\left(\frac{a_o}{2a_1}x^2\right)p_n\left(c^{-\frac{n}{2}}\sqrt{\frac{T}{a_1}}\,x,T\right),\quad a_o=\frac{2}{2-c},$$

$a_1=a_o+1$.

with some appropriately chosen constant B_n. A simple calculation, based on formulas (1) and (2) shows that

$$q_{n+1}(x) = Sq_n(x) \tag{3}$$

with

$$Sq(x) = \frac{1}{\sqrt{\pi}} \int e^{-v^2} q\left(\frac{x}{\sqrt{c}} + v\right) q\left(\frac{x}{\sqrt{c}} - v\right) dv \tag{3'}$$

$$q_o(x) = q_o(x,T) = C(u)\exp\left(\frac{a_o - T}{2a_1} x^2 - \frac{u}{4}\frac{T^2}{a_1^2} x^4\right) \tag{4}$$

and

$$p_n(x,T) = \bar{C}_n \exp\left(-\frac{a_o}{2T} c^n x^2\right) q_n\left(c^{n/2}\sqrt{\frac{a_1}{T}}\, x\right). \tag{5}$$

Therefore it is enough to study the asymptotic behaviour of $S^n q_o(x)$ as $n \to \infty$. Let us observe that the operator S is very similar to the convolution operator T,

$$Tq(x) = \int q\left(\frac{x}{\sqrt{c}} + v\right) q\left(\frac{x}{\sqrt{c}} - v\right) dv,$$

which plays a very important role in classical probability theory. (We have applied the same scaling in the operators T and S.) Moreover both S and T transform a Gaussian density function into a Gaussian density function again. To understand the behaviour of the operator S it is worth while to compare the action of its powers for a starting function $q_o(x)$ with that of the powers of T. We shall consider the following two special cases:

Case a) $q_o(x) = \text{const}\cdot\exp(-Ax^2)$.

Case b) $q_o(x) = \frac{1}{2}(\delta(x-M) + \delta(x+M))$, where $\delta(x) \geq 0, \int \delta(x)dx = 1$, $\delta(x)$ has a compact support, and M is sufficiently large.

In case a) $S^n q_o(x)$ and $T^n q_o(x)$ behave alike. Both are Gaussian densities with variance $\frac{1}{A}\left(\frac{c}{2}\right)^n$. (We are not interested in the multiplying factor before the exponent, because at the end we normalize in such a way that we get a density function.) Moreover if $q_o(x)$ is close to a Gaussian density then both

$S^n q_o((\frac{c}{2})^{n/2}x)$ and $T^n q_o((\frac{c}{2})^{n/2}x)$ tend to a Gaussian density function as $n \to \infty$.

In case b) the behaviour of $S^n q_o$ and $T^n q_o$ are essentially different. (See figure 1.)

$$Tq(x) = \int q(\tfrac{x}{\sqrt{c}}+v)\cdot q(\tfrac{x}{\sqrt{c}}-v)\,dv \qquad Sq(x) = \tfrac{1}{\sqrt{\pi}}\int e^{-v^2}\cdot q(\tfrac{x}{\sqrt{c}}+v)\cdot q(\tfrac{x}{\sqrt{c}}-v)\,dv$$

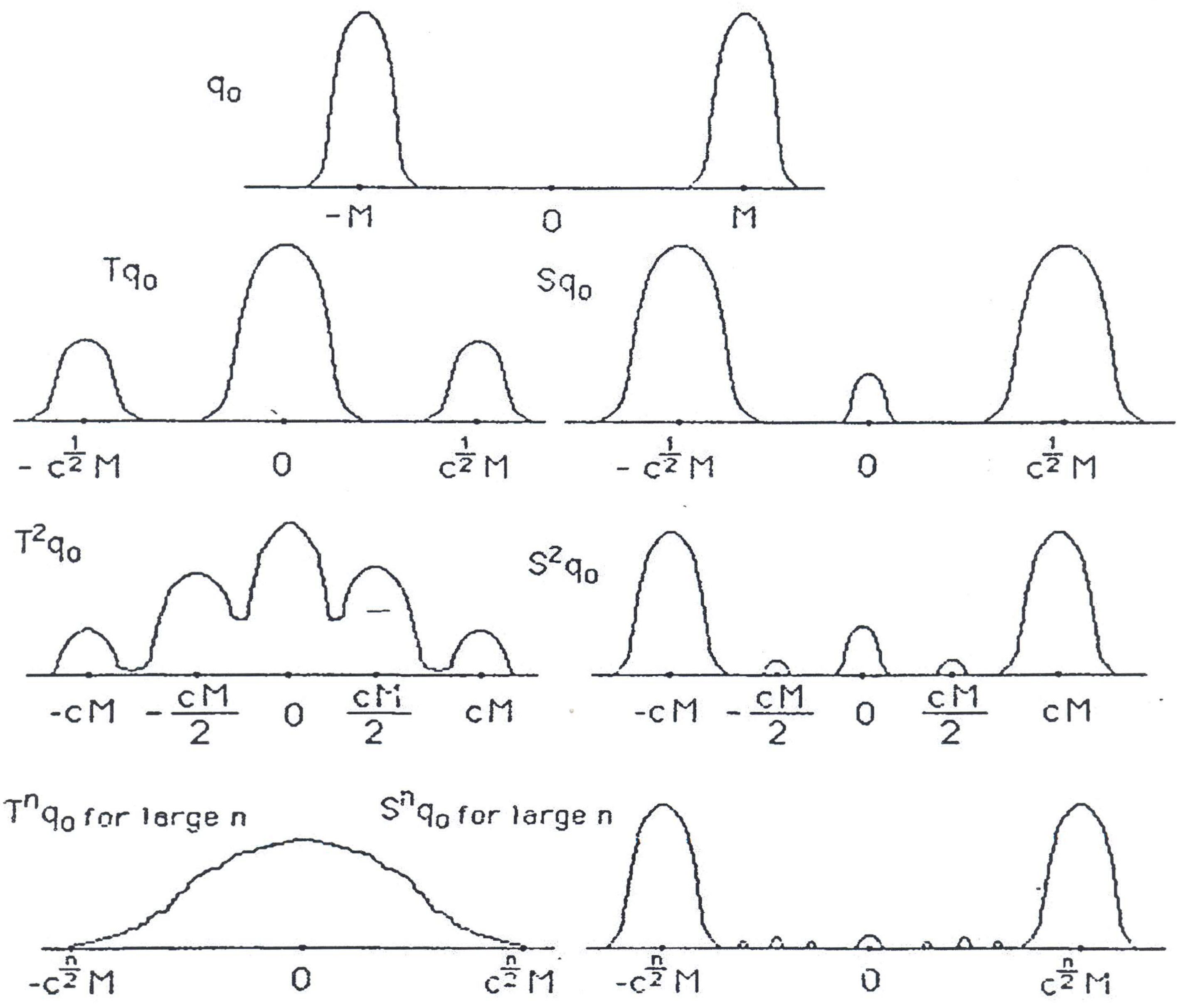

figure 1

The function $Tq_o(x)$ has three large peaks in $\pm\sqrt{c}$ M and zero. $Sq_o(x)$ has two large peaks in the points $\pm\sqrt{c}M$ and a small one in zero. The last peak is very small because of the effect of the multiplying term $\exp(-v^2)$ in the operator S. In the functions $T^n q_o(x)$ new high peaks appear after each step, for large n they merge together, and by the central limit theorem they draw out a Gaussian density. The function $S^n q_o(x)$ has two high peaks in $\pm c^{n/2}M$, and there are many small peaks in between, which are all negligible because of the effect of the multiplying term $\exp(-v^2)$ in S. The peaks in $c^{n/2}M$ and $-c^{n/2}M$ evolve independently, and for large n $S^n q((\frac{c}{2})^{n/2}x \pm c^{n/2}M)$ is asymptotically normal.

The above examples show the essential difference between the probabilistic problems appearing in statistical physics and classical probability theory. In statistical physics we have to deal with operators which are very sensitive for the starting functions.

In our problem the starting function q_o is defined in formula (4). It is of the form $\exp(A(T)x^2+B(T)x^4)$, where $B(T)<0$ for all T, $A(T)\leq 0$ for large T, and $A(T)>0$ for small T. Hence $q_o(x)$ behaves similarly to the example in case a) for large T and to the example in case b) for small T. These examples also suggest the limit behaviour of $q_n(x)$. A detailed analysis of $S^n q$ and formula (5) yield that

Theorem 1

For large T here is some $A(T)>0$ such that

$$p_n(2^{-\frac{n}{2}}x,T) \to \frac{1}{\sqrt{2\pi}\,A(T)} \exp\left(-\frac{x^2}{2A(T)}\right)$$

Theorem 2

For small T there is some sequence $M_n(T)\to M(T)>0$ and $A(T)>0$ such that

$$p_n(2^{-\frac{n}{2}}x,T) \sim \frac{1}{2\sqrt{2\pi}A(T)}\left[\exp\left(-\frac{(x-2^{n/2}M_n(T))^2}{2A(T)}\right)+\right.$$

$$\left.\exp\left(-\frac{(x+2^{n/2}M_n(T))^2}{2A(T)}\right)\right].$$

For $T \sim a_o, q_o(x) \sim 1$. In order to describe the limit behaviour of $p_n(x,T)$ at the critical parameter T_{cr} and in its small neighbourhood one has to study $S^n q(x)$ with a starting function close to 1. Observe that the function $\bar{q}(x) = 1$ is a fixed point of the operator S. We are interested in the stability of this fixed point with respect to S. Hence we take a small perturbation of the fixed point $1+\varepsilon g(x)$, and consider

$$S(1+\varepsilon g(x)) = 1+\varepsilon \frac{2}{\sqrt{\pi}}\int e^{-v^2} g\left(\frac{x}{\sqrt{c}}+v\right)dv +$$

$$+\frac{\varepsilon^2}{\sqrt{\pi}}\int e^{-v^2} g\left(\frac{x}{\sqrt{c}}+v\right) g\left(\frac{x}{\sqrt{c}}-v\right)dv = 1+\varepsilon\, D_1Sg + O(\varepsilon^2)$$

with

$$D_1Sg(x) = \frac{2}{\sqrt{\pi}}\int e^{-v^2} g\left(\frac{x}{\sqrt{c}}+v\right)dv.$$

The operator D_1S is well-known in classical analysis. Its eigen-functions are the Hermite polynomials with weight function $\exp(-\frac{c-1}{c}x^2)$, and the corresponding eigenvalues are $\frac{2}{c}, \frac{2}{c^2}, \ldots$. (In our investigation of $S^n q_o(x)$ we can restrict ourselves to the space of even functions, hence the Hermite polynomials of odd order are not interesting for us.) For $c > \sqrt{2}$ the operator D_1S has exactly one unstable eigen function h_2 with the eigenvalue $\frac{2}{c} > 1$, and the subspace orthogonal to h_2 is a D_1S invariant subspace, where D_1S is a contraction. Since $S(g(x)+\bar{q}(x)) - S\bar{q}(x)$ is very close to $D_1Sg(x)$ for small functions $g(x)$, $(\bar{q}(x)=1)$, this suggests the following picture: (See figure 2)

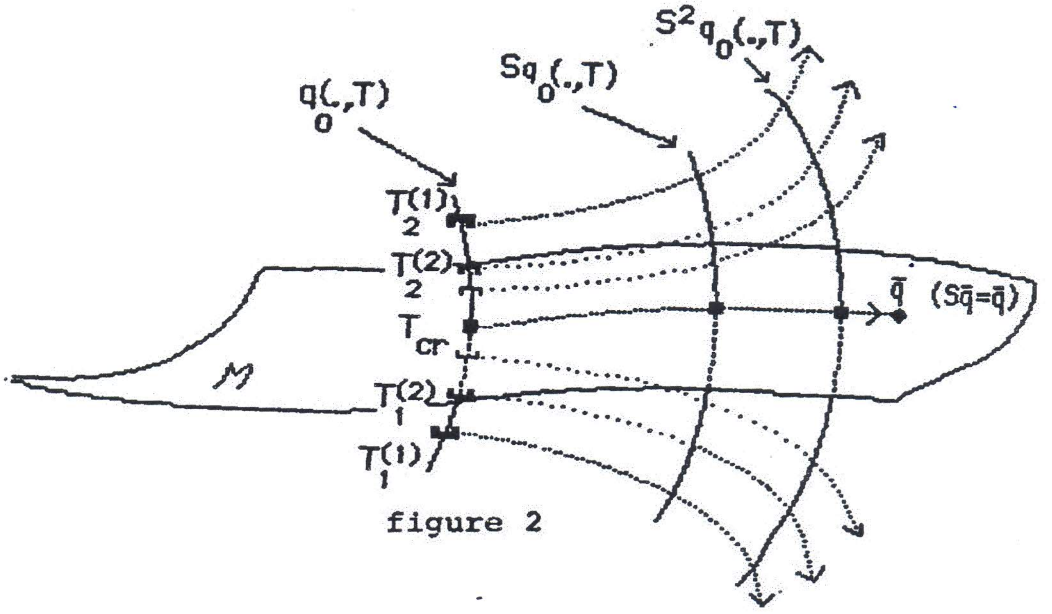

figure 2

In a small neighbourhood of q(x) there is an S invariant mainfold M of codimension 1, where S is a contraction. For $q\in M$ $S^n q(x)\to q(x)$ exponentially fast, and for $q\notin M$ the scalar product $(S^n q-q,h_2)$ tends exponentially fast either to plus or to minus infinity. The one-dimensional curve $q_o(x,T)$ intersects the manifold M in a point T_{cr}, and for this value T_{cr} $S^n q_o(x,T_{cr})\to q(x)$. We are looking for this value T_{cr}. We can exploit that for $T>T_{cr}$ $(S^n q_o(x,T)-q(x),h_2)\to\infty$, and for $T<T_{cr}$ $(S^n q_o(x,T)-q(x),h_2)\to-\infty$. These relations enable us to find a sequence of intervals $[T_1^{(n)},T_2^{(n)}]$ with exponentially fast decreasing length such that they are embedded in each other, i.e. $T_1^{(1)}<T_1^{(2)}<T_1^{(3)}\ldots$ and $T_1^{(2)}>T_2^{(2)}>T_2^{(3)}\ldots$, and they all contain the point T_{cr} we are looking for. The critical parameter can be found as the intersection of these intervals.

On the basis of the above argument the following Theorem 3 can be proved:

Theorem 3

For $c>\sqrt{2}$ there is some $T_{cr}=a_o+0(u)$ such that $S^n q_o(x,T_{cr})\to 1$. Hence formula (5) implies that

$$p_n(c^{-n/2}x,\ T_{cr})\to\sqrt{\frac{a_o}{2\pi T_{cr}}}\exp\left(\frac{-a_o}{2\,T_{cr}}x^2\right).$$ The function

$p_n(x,T)$ satisfies Theorem 1 for $T>T_{cr}$ and Theorem 2 for $T<T_{cr}$.

Theorem 3 implies that $\frac{c^{n/2}}{2^n}\sum_{j=0}^{2^n}\sigma(j)$ is asymptotically normal with a finite variance for $T=T_{cr}$, $2^{-\frac{n}{2}}\cdot\sum_{j=1}^{2^n}\sigma(j)$ is asymptotically normal with finite variance for $T>T_{cr}$, and its density is asymptotically the mixture of two Gaussian density functions with finite variance and expectations $\pm\sqrt{2^n}\,M(T)$ for $T<T_{cr}$.

For $c<\sqrt{2}$ the operator D_1S has two unstable eigen-function $h_2(x)$ and $h_4(x)$ with eigenvalues $\frac{2}{c}$ and $\frac{2}{c^2}$ respectively. Hence the fixed point of S $\bar{q}(x)=1$ is not stable enough, and Theorem 3 does not hold in this case. For $c=\sqrt{2}-\varepsilon$, $\varepsilon>0$ is small, one can find with the help of bifurcation theory another fixed point $q^*(x)=1+c\varepsilon(1-x^2)+0(\varepsilon^2)$ of S such that the linearization of the operator S in the point $q^*(x)$ has only one unstable eigenfunction. In this case the following Theorem 4 holds:

Theorem 4

There is some $\varepsilon_o>0$ such that for $c=\sqrt{2}-\varepsilon$, $0<\varepsilon<\varepsilon_o$ the fixed point equation $Sq=q$ has a solution of the form $q^*(x)=1+C\varepsilon(1-x^2)+0(\varepsilon^2)$, $C>0$, such that the linearization of the operator S in the point q^*

$$D_{q^*}\,g(x)=\int e^{-v^2}q^*\left(\frac{x}{\sqrt{c}}+v\right)g\left(\frac{x}{\sqrt{c}}-v\right)dv$$

has exactly one eigenvalue larger than 1. There is a critical parameter T_{cr} such that $S^n q_o(x,T_{cr})\to q^*(x)$ and hence $p_n(c^{-n/2}x,T_{cr})\to \text{const.}\exp\left(\frac{a_o}{2T_{cr}}x^2\right)q^*\left(\sqrt{\frac{a_1}{T_{cr}}}\,x\right)$.

With some extrawork it can be proved that for

$T=T_{cr}$ $\frac{2^{2n}}{c^n} E(\sum_{j=1}^{2^n} \sigma(j))^2 \sim$ const. and $E\sigma(i)\sigma(j) \sim$ $\sim \text{const}\, d(i,j)^{-\alpha}$

with $\alpha=\log_2 c$, for $T>T_{cr}$. $E\sigma(i)\sigma(j) \sim \text{const}\ d(i,j)^{-a}$

and for $T>T_{cr}$, $E\sigma(i)\sigma(j)-M_n^2 \sim \text{const}\ d(i,j)^{-a}$, where M_n is defined in Theorem 2.

Generally one is interested in translation invariant models. In this case only partial results are available. They are the natural analogues of the results proved for Dyson's model, and the method of investigation is also strongly motivated by this model.

In this report we only could give a short introduction to this subject. A more detailed discussion together with a list of literature will appear in our paper [1].

Reference

[1] Bleher P.M., Major P. (1987) Critical phenomena and universal exponents in statistical physics. On Dyson's hierarchical model, Annals of Probability 2.

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