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Approximation of Partial Sums of i.i.d.r.v.s when the Summands Have Only Two Moments

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Given a sequence of partial sums $S_1, S_2, \ldots$ of i.i.d.r.v.s; we approximate them by the partial sums of independent normal variables. We show that our construction is optimal if nothing more than the existence of the first two moments of the summands is assumed. We generalize the construction to the case when the time parameter set is multi-dimensional.

1. Introduction

An essential part in the proof of Strassen's law of iterated logarithm is the following

Theorem 1. *Given a d.f.* $F(x)$, $\int x\,dF(x)=0$, $\int x^2\,dF(x)=1$ *one can construct two infinite sequences of i.i.d.r.v.s* $X_1, X_2, \ldots$ *and* $Y_1, Y_2, \ldots$ *with distribution* $F(x)$ *resp.* $\Phi(x)$ *in such a way that the partial sums* $S_n=\sum_{i=1}^{n} X_i$, $T_n=Y_i$, $n=1,2\ldots$ *satisfy*

$$|S_n-T_n|=o(\sqrt{n\log\log n}) \quad \textit{with} \text{ pr. } 1.$$

($\Phi(x)$ denotes the standard normal distribution function).

Strassen [5] proved this result by applying the so-called Skorohod embedding. In the present paper we give a direct proof to Theorem 1. Then we show that this result cannot be improved. More precisely the following statement holds.

Theorem 2. *Let* $f(n)$ *be any positive function tending to infinity. Then there exists a distribution function* $F(x)$, $\int x\,dF(x)=0$, $\int x^2\,dF(x)=1$ *with the following property: for any pair of sequences of i.i.d.r.v.s* $X_1, X_2, \ldots$ *and* $Y_1, Y_2, \ldots$ *with d.f.* $F(x)$ *resp.* $\Phi(x)$ *one has*

$$P\left(\limsup f(n)\frac{|S_n-T_n|}{\sqrt{n\log\log n}}=\infty\right)=1$$

where $S_n=\sum_{i=1}^{n} X_i$, $T_n=\sum_{i=1}^{n} Y_i$.

Theorem 1 can be slightly generalized. Let the time-parameter set T^d consist of all the lattice points with positive coordinates in the d-dimensional Euclidian space R^d. We say that $m \leqq n$, $m=(m_1, \dots m_d)$, $n=(n_1, \dots n_d)$ if $m_i \leqq n_i$ for $i=1, \dots d$. We define $|n| = \prod_{i=1}^{d} n_i$.

We state the following

Theorem 1'. *Given a distribution function* $F(x)$, $\int x\,dF(x)=0$, $\int x^2\,dF(x)=1$ *and a monotone sequence* $n_1 < n_2 < \cdots$, $n_i \in T^d$. *One can then construct two sets of i.i.d.r.v.s* X_n, Y_n, $n \in T^d$ *with distribution* $F(x)$ *and* $\Phi(x)$ *in such a way that the variables* $S_n = \sum_{i \leqq n} X_i$, $T_n = \sum_{i \leqq n} Y_i$ *satisfy*

$$\sup_{i \leqq n_k} |S_i - T_i| = o\left(\sqrt{|n_k| \log\log |n_k|}\right) \quad \textit{with} \ \text{pr. } 1.$$

Theorem 1' implies that the set of variables S_n, $n \in T^d$ satisfies similar laws of iterated logarithm as the Wiener process with d-dimensional parameter set. Thus using Theorem 1' we can reduce some proofs of Wichura [7] to the investigation of the Wiener process with d-dimensional parameter set.

Finally we make some remarks about the approximation of sums of i.i.d.r.v.s, comparing the cases when the summands have only two moments, and when they have more.

Denote the set of distributions $F(x)$: $\int x\,dF(x)=0$, $\int x^2\,dF(x)=1$ and

$$\int |x|^r\,dF(x) < \infty \quad \text{by } K_r, \quad r \geqq 2.$$

If $F(x) \in K_2$, then two sequences $X_1, X_2, \dots X_n$ and $Y_1, Y_2, \dots, Y_n$ of i.i.d.r.v.s can be constructed so that

$$P\left(\frac{1}{\sqrt{n}} \sup_{p \leqq n} |S_k - T_k| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty \text{ for every } \varepsilon > 0.$$

(This statement is equivalent to the functional central limit theorem for sums of i.i.d.r.v.s.)

This means that by approximating the partial sums in K_2 a better rate can be achieved if we only want stochastic convergence instead of convergence with probability 1.

In K_r, $r > 2$ the situation is somewhat different. Here an approximation with the property

$$\lim n^{-\frac{1}{r}} (S_n - T_n) = 0 \quad \text{with pr. } 1$$

can be reached, see [2] and [3]. On the other hand for any $f(n) \to \infty$ a distribution $F(x) \in K_r$ and a sequence n_k, $k=1, 2, \dots n_k \to \infty$ can be found so that

$$P\left(f(n_k) n_k^{-\frac{1}{r}} \sup_{j \leqq n_k} |S_j - T_j| > 1\right) \to 1.$$

In fact, choosing a distribution $F(x) \in K_r$ and a sequence n_k in such a way that

$$1 - F(3 n_k^{\frac{1}{r}} / f(n_k)) > f(n_k)/n_k$$

the relation

$$P(\sup_{j\leqq n_k} f(n_k)\, n_k^{-\frac{1}{r}}\, |S_j-T_j|>1)\leqq P(\sup_{j\leqq n} |X_j-Y_j|>n_k^{\frac{1}{r}}/2f(n_k))\to 1$$

holds.

2. Proofs

Our proofs are based on the following theorem of Heyde (see [1]).

Theorem A (Heyde). *Consider a sequence $X_1, X_2, \dots$ of i.i.d.r.v.s and its partial sums $S_n=\sum_{i=1}^{n} X_i$. Let $EX_1=0$, $EX_1^2=1$, $F(x)=P(X_1<x)$ and $F_n(x)=P(S_n<x\sigma_n\sqrt{n})$ where*

$$\sigma_n^2=\int_{|x|<\sqrt{n}} x^2\,dF(x)-\Big[\int_{|x|<\sqrt{n}} x\,dF(x)\Big]^2.$$

Then if $K>0$, $C>1$ and n_k, $K\geqq 1$ is a sequence of integers with $n_k\sim KC^{2k}$ as $k\to\infty$, we have

$$\sum \sup_x |F_{n_k}(x)-\Phi(x)|<\infty.$$

Proof of Theorem 1. Let us first remark that it is enough to prove the following somewhat weaker statement:

For any $\varepsilon>0$ there is a construction satisfying the relation:

$$P\left(\limsup \frac{|S_n-T_n|}{\sqrt{n\log\log n}}\leqq\varepsilon\right)=1. \tag{2.1}$$

To prove this remark we make the following construction. Let the sequences $S_n^{(k)}$, $T_n^{(k)}$ $n=1,2,\dots$ satisfy (2.1) with $\varepsilon=\frac{1}{k}$. We may assume that the pairs $S_n^{(k)}$, $T_n^{(k)}$ for different k-s are independent. Let us now consider a sequence of integers $n_1<n_2<\cdots$ with the following properties:

$n_k\sim 2^{2^k}$, $\quad n_k$ is of the form 2^{m_k} where m_k is a positive integer and

$$P\left(\sup_{n\geqq n_k} \frac{|S_n^{(k)}-T_n^{(k)}|}{\sqrt{n\log\log n}}>\frac{2}{k}\right)<\frac{1}{k^2}$$

Let the sequence S_n and T_n be such that $S_n-S_{n_k}=S_n^{(k)}-S_{n_k}^{(k)}$, $T_n-T_{n_k}=T_n^{(k)}-T_{n_k}^{(k)}$ if $n_k<n\leqq n_{k+1}$. These relations define the S_n's and T_n's and we claim that these S_n's and T_n's satisfy Theorem 1.

First we state that

$$|S_{n_k}-T_{n_k}|=o(\sqrt{n_k\log\log n_k}).$$

Since the sequence n_k is very rare, we even have

$$S_{n_k}=o(\sqrt{n_k\log\log n_k}) \quad\text{and}\quad T_{n_k}=o(\sqrt{n_k\log\log n_k}). \tag{2.2}$$

In fact, applying Theorem A, we have for any $\varepsilon>0$

$$P(|S_{n_k}|>\varepsilon\sqrt{n_k\log\log n_k})$$
$$\leqq \sum_{k=1}^{\infty}\sup_x |F_{n_k}(x)-\Phi(x)|+2\sum_{k=1}^{\infty}(1-\Phi(\varepsilon\sqrt{\log\log n_k})<\infty$$

and the Borel-Cantelli lemma proves (2.2) for the sequence S_{n_k}. The proof for T_{n_k} is similar.

The Borel-Cantelli lemma also implies that

$$\lim_k \sup_{n_k\leqq n\leqq n_{k+1}} \frac{|(S_n-S_{n_k})-(T_n-T_{n_k})|}{\sqrt{n\log\log n}}=0 \quad \text{with pr. 1.}$$

This formula together with (2.2) proves our remark.

Let us now turn to the proof of (2.1).

Set $C=1+\frac{\varepsilon}{8}$, $n_k=[C^k]$ and $r_k=\sum_{j=1}^{k} n_j$. Let $\alpha_1, \alpha_2, \dots$ be i.i.d.r.v.s uniformly distributed in [0, 1].

Define

$$\frac{S_{r_k}-S_{r_{k-1}}}{\sigma_{n_k}\sqrt{n_k}}=F_{n_k}^{-1}(\alpha_k)$$

and

$$\frac{T_{r_k}-T_{r_{k-1}}}{\sqrt{n_k}}=\Phi^{-1}(\alpha_k).$$

(Here $F^{-1}(t)$ is defined as $F^{-1}(t)=\sup(x\colon F(x)\leqq t)$, $F_{n_k}(x)$ and σ_{n_k} as in Theorem A.)

The sequences S_{r_k}, T_{r_k}, $k=1,2\dots$ have the required distribution. We can complete them into sequences $S_1, S_2, \dots; T_1, T_2, \dots$ so that $S_1, S_2, \dots$ be i.i.d.r.v.s with distribution $F(x)$, $T_1, T_2, \dots$ i.i.d.r.v.s with distribution $\Phi(x)$. We claim that these sequences satisfy (2.1).

First we show that

$$S_{r_k}-T_{r_k}=o(\sqrt{r_k\log\log r_k}). \tag{2.3}$$

According to the Borel-Cantelli lemma it is enough to prove that

$$\sum P(|(S_{r_k}-S_{r_{k-1}})-(T_{r_k}-T_{r_{k-1}})|>\delta\sqrt{n_k\log\log n_k})<\infty$$

for any $\delta>0$, since then

$$S_{r_k}-T_{r_k}<K(\omega)+\sum_{j=1}^{k}\delta\sqrt{n_j\log\log n_j}$$
$$<K(\omega)+\frac{2\delta}{\sqrt{C}-1}\sqrt{r_k\log\log r_k} \quad \text{with pr. 1.}$$

But the convergence of the above sum is a direct consequence of the following estimation

$$P\left(|(S_{r_k}-S_{r_{k-1}})-(T_{r_k}-T_{r_{k-1}})|>\delta\sqrt{n_k\log\log n_k}\right)$$
$$\leqq\frac{1}{\log^2 n_k}+8\sup_x|F_{n_k}(x)-\Phi(x)|. \tag{2.4}$$

Now we need to prove only (2.4).

$$P\left(|(S_{r_k}-S_{r_{k-1}})-(T_{r_k}-T_{r_{k-1}})|>\delta\sqrt{n_k\log\log n_k}\right)$$
$$\leqq P\left(\sigma_{n_k}|F_{n_k}^{-1}(\alpha_k)-\Phi^{-1}(\alpha_k)|>\frac{\delta}{2}\sqrt{\log\log n_k}\right)$$
$$+P\left(|\Phi^{-1}(\alpha_k)|>\frac{\delta\sqrt{\log\log n_k}}{2(1-\sigma_{n_k})}\right).$$

The second term is obviously less than $\frac{1}{\log^2 n_k}$ if k is sufficiently large. Now we estimate the first term. Denoting $\Phi^{-1}(\alpha_k)$ by y, a simple calculation shows that

$$P\left(\sigma_{n_k}|F_{n_k}^{-1}(\alpha_k)-\Phi^{-1}(\alpha_k)|>\frac{\delta}{2}\sqrt{\log\log n_k}\right)$$
$$\leqq P(|F_{n_k}^{-1}(\Phi(y))-y|>1)$$
$$\leqq P\left(|\Phi(y+1)-\Phi(y)|\leqq\sup_x|F_{n_k}(x)-\Phi(x)|\right)$$
$$+P\left(|\Phi(y)-\Phi(y-1)|\leqq\sup_x|F_{n_k}(x)-\Phi(x)|\right).$$

Now $1-\Phi(y)>2(1-\Phi(y+1))$ if y is large enough, so we have

$$P\left(\sigma_{n_k}|F_{n_k}^{-1}(\alpha_k)-\Phi^{-1}(\alpha_k)|>\frac{\delta}{2}\sqrt{\log\log n_k}\right)$$
$$\leqq 4P\left(\Phi(y)\leqq\sup_x|F_{n_k}(x)-\Phi(x)|\right)$$
$$+4P\left(1-\Phi(y)\leqq\sup_x|F_{n_k}(x)-\Phi(x)|\right)$$
$$\leqq 8\sup_x|F_{n_k}(x)-\Phi(x)|.$$

Thus (2.4) holds true.

Finally we show that

$$P\left(\limsup\frac{\sup_{r_k\leqq r\leqq r_{k+1}}|S_n-S_{r_k}|}{\sqrt{r_k\log\log r_k}}\leqq\frac{\varepsilon}{2}\right)=1 \tag{2.5}$$

and a similar estimation holds for the T's. These relations together with (2.3) imply (2.1). Applying a well-known estimation about the partial sums of independent summends see e.g. [4] p. 248. We get

$$P(\sup_{r_k \leqq n < r_{k+1}} |S_n - S_{r_k}| > 3\sqrt{n_k \log\log n_k})$$
$$\leqq 2\sum P(|S_{r_{k+1}} - S_{r_k}| > 2\sqrt{n_k \log\log n_k})$$
$$\leqq 2\sum (1-\Phi(2\sqrt{\log\log n_k})) + 2\sum \sup_x |F_{n_k}(x) - \Phi(x)| < \infty$$

which implies (2.5).

To prove Theorem 1′ we need a Theorem which states that the variable $\max_{k \leqq n} S_k$ is not essentially larger than S_n.

Theorem B (Wichura [6]). *Let $n \in T^d$ and let $(X_m)\, m \leqq n$ be a d-dimensional array of independent random variables with* 0 *mean and finite variances. Put $S_m = \sum_{k \leqq m} X_k$ and set $M_n = \max_{m \leqq n} |S_m|$. Then*

$$P(M_n \geqq 2^q a) \leqq \left[1 - \left(\frac{\sigma}{a}\right)^2\right]^{-q} P(|S_n| \geqq a)$$

if $a^2 \geqq \sigma^2 = ES_n^2$.

Proof of Theorem 1′. As in the proof of Theorem 1, the proof can be reduced to the construction of two sets of random variables S_n, T_n, $n \in T^d$ satisfying the relation

$$P\left(\limsup \frac{\sup_{i \leqq n_k} |S_i - T_i|}{\sqrt{|n_k| \log\log |n_k|}} > \varepsilon\right) = 0.$$

Fix a number $C > 1$, and consider the set H of points of the form

$$(|C^{m_1}|, |C^{m_2}|, \dots |C^{m_d}|)$$

where the numbers m_i range over all positive integers. For any n_k consider its smallest upper bound in H. We get a new monotone sequence. Let us consider every point of the sequence with multiplicity 1. Let us embed this sequence into a new sequence r_k $k = 1, 2, \dots$ in such a way that the subsequent members of the new sequence differ only in one coordinate, and the exponent of C in this coordinate grows with one. It is sufficient to make such a construction that

$$P\left(\limsup_k \frac{\sup_{i \leqq r_k} |S_i - T_i|}{\sqrt{r_k \log\log r_k}} > C(\varepsilon)\right) = 0 \tag{2.6}$$

where $C(\varepsilon) \to 0$ as $C \to 1$.

We may assume that every coordinate of r_k tends to infinity. Otherwise the dimension of T^d in Theorem 1′ can be decreased.

The idea of the construction is the following. We want to divide T^d into a sequence $K_1, K_2, \dots$ of d-dimensional disjoint rectangles having some nice properties. By the increment of S_n and T_n on a set A, $A \in T^d$ we mean the r.v. $\sum_{i \in A} X_i$ and $\sum_{i \in A} Y_i$. First we define the variables U_i and V_i the increment of S and T on the rectangles K_i in the same way as in the proof of Theorem 1:

$$\frac{U_i}{\sigma_{m_i}\sqrt{m_i}}=F_{m_i}^{-1}(\xi_i), \qquad \frac{V_i}{\sqrt{m_i}}=\Phi^{-1}(\xi_i)$$

where m_i is the volume of K_i and $\xi_1, \xi_2, \dots$ are i.i.d.r.v.s uniformly distributed in $[0, 1]$.

Then we construct the r.v.s X_i, Y_i, $i \in T^d$ in such a way that the increment of S_i and T_i on K_i be identical with the previously defined U_i and V_i.

We want the K_i's to have the following properties:

(i) if x is a vertex of K_i, then the rectangular $A=(y\colon y \leqq x)$ is the union of some K_j's, r_n is the vertex of some K_i if $n \geqq n_0$.

(ii) $U_i - V_i = o(\sqrt{m_i \log\log m_i})$ and moreover $S_x - T_x = o(\sqrt{|x| \log\log |x|}$ if x is a vertex of some K_i's.

(iii) The vertices of the K_i's are dense enough in T^d, so that Theorem B implies (2.6).

We will construct a sequence K_i which satisfies (i)–(iii). Set $r_n=(r_1^{(n)}, \dots r_d^{(n)})$. Let the $j(n)$-th be the coordinate where r_n and r_{n+1} differ. Denote by H_n the set

$$(x\colon x=(x_1 \dots x_d),\ x_i \leqq r_i^{(n)} \text{ if } i \neq j(n),\ r_{j(n)}^{(n)} < x_{j(n)} \leqq r_{j(n)}^{(n+1)}).$$

We fix an integer $L>0$ and a number $0<\alpha<1$. First we choose a sufficiently large n_0 and we divide the rectangle $A_{n_0}=(x\colon x \leqq r_{n_0})$ into smaller ones by the hyperplanes $x_i=\left|\frac{k}{L} r_i^{(n)}\right|$ $k=1, 2, \dots, L$. These rectangles will be our first K_i's. Then we split the rectangles $H_{n_0}, H_{n_0+1}, \dots$ successively with some hyperplanes of the form $x_i=\alpha_{i,k}$, $i=1, 2 \dots d$, $i \neq j(n)$, $k=1, 2 \dots f(i, n)$, $\alpha_{i, f(i,n)}=r_i^{(n)}$. In order to satisfy (i) a hyperplane $x_i=\alpha_{i,k}$ is allowed to be one of the hyperplanes dividing H_n only if it contains a lateral face of some previously defined K. Another requirement is that $\frac{\alpha}{L} r_i^{(n)} < \alpha_{i,k} - \alpha_{i,k-1} < \frac{1}{L} r_i^{(n)}$. These requirements can be satisfied if α and L are chosen appropriately. Thus (i) is satisfied.

Theorem A remains valid if the sequence n_k need not satisfy the condition $n_k \sim KC^{2k}$, only the following weaker condition: $AC^k < n_k < BC^k$ with some $A>B>0$, $C>1$. Using the same argument as in the proof of Theorem 1, this version of Theorem A enables us to prove (ii).

Let us remark that we may choose L in such a way that $L \to \infty$ as $C \to 1$. Now given the rectangle $A_n=(x, x \leqq r_n)$, we may choose hyperplanes $x_i=\beta_{i,k}$ $i=1, 2, \dots, d$ so that $\frac{\alpha}{L} r_i^{(n)} \leqq \beta_{i,k} - \beta_{i,k-1} \leqq \frac{1}{L} r_i^{(n)}$, $\beta_{i,k} \leqq r_i^{(n)}$ and any point $x=(x_1 \dots x_d)$, $x_1=\beta_{1,j_1}, \dots, x_d=\beta_{d,j_d}$ is a vertex of some K_i, and therefore $|S_x - T_x| = o(\sqrt{|r_n| \log\log |r_n|})$. Consider a rectangle B contained in some $A_n \cap U_{i,k}$ where $U_{i,k}=(x=(x_1 \dots x_d);\ \beta_{i,k-1} < x_i \leqq \beta_{i,k_i})$. Then applying Theorem B, the above mentioned version of Theorem A and the Borel-Cantelli lemma, one obtains an increment of S and T on B which is less than $\frac{A}{L}\sqrt{|r_n| \log\log |r_n|}$ if $n > n(\omega)$ with probability 1. This estimation implies (2.6) and thus Theorem 1′ is proved.

Proof of Theorem 2. Let us choose a monotone sequence $\bar{f}(n)$ such that $\bar{f}(n) \to \infty$ and $\bar{f}(n)/f(n) \to 0$.

There exists a distribution function $F(x)$ such that

$$\int x\,dF(x)=0, \quad \int x^2\,dF(x)=1,$$

$$\sigma_{2^n}^2 \leqq \left(1-\frac{2}{\bar{f}(2^{n+1})}\right)^2 [\log n/\log n+\log\log n]$$

where σ_n is as in Theorem A. We claim that such a distribution satisfies Theorem 2. It is enough to prove that

$$\sum_n P(S_{2^{n+1}}-S_{2^n}>x_n)<\infty,$$

$$\sum_n P(T_{2^{n+1}}-T_{2^n}>y_n)=\infty$$

where

$$x_n=\sqrt{2^{n+1}\log n}\left(1-\frac{2}{\bar{f}(2^{n+1})}\right)$$

and $y_n=\sqrt{2^{n+1}\log n}$. In fact, these inequalities and the Borel-Cantelli lemma imply that

$$(T_{2^{n+1}}-T_{2^n})-(S_{2^{n+1}}-S_{2^n})>y_n-x_n=\frac{2\sqrt{2^{n+1}\log n}}{\bar{f}(2^{n+1})} \quad \text{i.o. with pr. 1.}$$

Thus, either

$$T_{2^n}-S_{2^n}<-\frac{\sqrt{2^{n+1}\log n}}{\bar{f}(2^{n+1})},$$

or $T_{2^{n+1}}-S_{2^{n+1}}>\sqrt{2^{n+1}\log n}/\bar{f}(2^{n+1})$ which proves Theorem 2. The second sum

$$\sum P(T_{2^{n+1}}-T_{2^n}>y_n)=\sum\left(1-\Phi\left(\frac{y_n}{\sqrt{2^n}}\right)\right)$$

is equiconvergent with

$$\sum\frac{\sqrt{2^n}}{y_n}e^{-\frac{y_n^2}{2^{n+1}}}\geqq\sum\frac{1}{n\sqrt{2\log n}}=\infty.$$

By Theorem A the first sum is equiconvergent with

$$\sum\left(1-\Phi\left(\frac{x_n}{\sqrt{2^n}\,\sigma_{2^n}}\right)\right).$$

But

$$1-\Phi\left(\frac{x_n}{\sqrt{2^n}\,\sigma_{2^n}}\right)\leqq\frac{1}{\sqrt{2\pi}}\frac{\sqrt{2^n}\,\sigma_{2^n}}{x_n}\exp\left(-\frac{x_n^2}{2^{n+1}\sigma_{2^n}^2}\right)\leqq\frac{C}{n(\log n)^{\frac{3}{2}}}.$$

Therefore, the first sum is convergent.

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References

1. Heyde, C.C.: Some properties of metrics in a study on convergence to normality. Z. Wahrscheinlichkeitstheorie verw. Gebiete **11**, 181–192 (1969)
2. Komlós, J., Major, P., Tusnády, G.: An approximation of Partial Sums of Independent RV's and the Sample D.F. (II). Z. Wahrscheinlichkeitstheorie verw. Gebiete **34**, 33–58 (1976)
3. Major, P.: The approximation of partial sums of independent RV's. Z. Wahrscheinlichkeitstheorie verw. Gebiete **35**, 213–220 (1976)
4. Loève, M.: Probability Theory. Toronto-New York-London: Van Nostrand 1963
5. Strassen, V.: An invariance principle for the law of iterated logarithm. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 211–226 (1964)
6. Wichura, M.J.: Inequalities with application to weak convergence of random processes with multi-dimensional time parameters. Ann. Math. Statist. **40**, 681–687 (1969)
7. Wichura, M.J.: Some Strassen type laws of the iterated logarithm for multiparameter stochastic processes. Ann. Probab. **1**, 272–296 (1973)

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