RENORMALIZATION OF DYSON'S VECTOR-VALUED HIERARCHICAL MODEL AT LOW TEMPERATURES

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In this paper we discuss Dyson's vector-valued model and explain how its so-called large-scale limit can be calculated. Formally, this is a limit theorem for the distribution of sums of random variables. Hence one would expect that the classical methods of probability theory, worked out for the investigation of such problems, can be applied in this case too. However, this model has a much more complex behaviour than the analogous models in classical probability, and the usual methods of probability theory are not appropriate for its investigation. The aim of the present paper is to explain the main motivations (statistical physical and probabilistic) for studying Dyson's model and the method of of investigation. Here we restrict ourselves to vector-valued models. A detailed discussion of scalar-valued models is given in [2]. On the other hand, vector-valued models have a peculiar behaviour which cannot be guessed from the behaviour of scalar models. The behaviour of these models strongly depend on a physical parameter, on the temperature T. In scalar-valued case there is a special, so-called critical value of the parameter, where the large-scale limit exists with a different scaling. This means a non-classical limit theorem with an unusual normalization.

In vector-valued case a similar phenomenon appears not only at the critical, but at all low temperatures. On low temperatures there is a phase transition, hence first we have to construct the random field, called pure state in the literature, which we want to renormalize. This field has a spontaneous magnetization, and a different normalization is needed in the direction of the spontaneous magnetization and in the direction orthogonal to it. In the direction orthogonal to the spontanous magnetization one has to normalize similarly to the normalization at the critical temperature. In the direction of the magnetization the situation is even more complex. Here the right normalization and the large-scale limit depends on another parameter of the model which measures the strength of the interaction. Let us emphasize again that the above statements hold not only for a singular critical value of the parameter T, but for all sufficiently low temperatures T. A detailed description of Dyson's vector-valued hierarchical model together with a complete proof is contained in the works [1], [4] and [5]. Since these papers are burdened with many technical details we have found it useful to discuss separately the most important analytical problems one has to study during these investigations.

The peculiar behaviour of vector-valued models is related to their invariance with respect to rotations. This is a symmetry with respect to a continuous group. Actually, our main motivation for studying this problem is to understand the role of continuous symmetries.

The most important part of the problem we are dealing with can be translated to purely analytical questions. One has to investigate the action of large powers of a certain integral operator to a starting function. This operator is very similar to the convolution operator (i.e. to the convolution of a function with itself), which is a well-known object in classical probability. There is however one essential difference between these operators. The operator appearing in our problem has, unlike the convolution operator, the following instability property: When a large power of this operator is applied to a starting function then the result strongly depends on the starting function. This instability property is the main reason for the peculiar behaviour of models in statistical physics. (See paper [3] for a heuristic explanation of this question.) Actually this instability is behind the different behaviour of scalar and vector-valued models too.

Let us discuss our problem in some more detail. We are interested in the behaviour of equilibrium states. Equilibrium states are appropriately defined probability measures on the space of all configurations $\sigma = \{\sigma(j),\ j \in \mathbf{Z}\}$. The set $\mathbf{Z}$ is chosen generally in translation invariant models as the integer lattice of the d-dimensional Euclidean space R^d with some $d \geq 1$. In Dyson's model we choose $\mathbf{Z} = \{1, 2, \dots\}$. The variables $\sigma(j)$, $j \in \mathbf{Z}$, which are called in the literature spin variables, take values in the Euclidean space R^p with some $p \geq 1$. We speak of scalar-valued models if $p = 1$ and vector-valued models if $p \geq 2$. The probability measure that we call equilibrium space is a Gibbs measure which is a well-known object in statistical physics. It depends on the following three quantities: A Hamiltonian function $\mathcal{H}(\sigma)$ defined on the configurations σ, a probability measure ν, called the free measure, on R^p, a physical parameter T, called the temperature. Formally it is defined as the probability measure μ given by the formula

$$\mu(d\sigma) = const. \exp\left\{-\frac{1}{T}\mathcal{H}(\sigma)\right\} \prod_{j\in\mathbf{Z}} \nu(d\sigma(j))\,. \tag{1}$$

Formula (1) would require a more detailed explanation. We omit it in this discussion, because it is needed only to translate the problems we are interested in into analytical questions. A precise definition of equilibrium states can be found also in Appendix D of [4].

Let $\sigma = \{\sigma(j),\ j \in \mathbf{Z}\}$ be a μ distributed random field. We are interested in its large-scale limit. Let us explain the meaning of this problem. Since we restrict ourselves to Dyson's model we shall assume that $\mathbf{Z} = \{1, 2, \dots\}$. For all $n = 1, 2, \dots$ define the fields

$$Y_n(j) = \frac{1}{A_n} \sum_{k=(j-1)2^n+1}^{j2^n} \sigma(k) - E\sigma(k)\,, \quad j \in \mathbf{Z}\,. \tag{2}$$

Let us choose the norming constant A_n in formula (2) in such a way that the finite dimensional distributions of the fields $Y_n(j)$ have a nontrivial limit as $n \to \infty$. We are interested in the right choice of the norming factor A_n and the distribution of the limit field which is called the large-scale limit of the original field $\sigma(j)$ in the literature. In

case of vector-valued models, i.e. when $\sigma(j) = (\sigma^{(1)}(j), \dots, \sigma^{(p)}(j))$ with some $p \geq 2$ we normalize in each coordinate independently. This means that we define

$$Y_n^{(l)}(j) = \frac{1}{A_n^{(l)}} \sum_{k=(j-1)2^n+1}^{j2^n} \sigma^{(l)}(k) - E\sigma^{(l)}(k)\,, \quad j \in \mathbf{Z},\ l = 1, \dots, p\,. \tag{2'}$$

In formulas (2) and (2′) we have formulated a problem about the limit distribution of partial sums of random variables, and this is a very natural problem of classical probability theory. If the variables $\sigma(j)$ are independent or very weakly dependent then $A_n = 2^{n/2}$ is the right choice, and the large-scale limit consists of independent Gaussian random variables. But for certain choice of the Hamiltonian function $\mathcal{H}(\sigma)$ and the temperature T we have to apply a different normalization, and also the large-scale limit has a different structure. Our main goal is to obtain a possibly complete picture of different possibilities.

It is quite natural that the large-scale limit strongly depends on the Hamiltonian function of the equilibrium state. The surprising fact, which requires an explanation, is the dependence of the large-scale limit and of the norming constant on the temperature T. In interesting models the following picture holds true: There is a so-called critical temperature T_{cr}, where the model has a peculiar behaviour. For all $T \neq T_{cr}$ the large-scale limit is a field of independent Gaussian variables, and the norming constant is the classical one, i.e. one has to divide by the square-root of the number of summands in a bloc. The only exceptional case is when $T = T_{cr}$. In this case one has to divide by a different power, and the large-scale limit may be non-Gaussian. Moreover, the behaviour of translation invariant models at the critical temperature strongly depends on the dimension of the lattice where they are defined.

It is an outstanding problem of the statistical physics to find a satisfactory explanation for the above mentioned phenomena. Our paper [2] was devoted mainly to this problem. We have also understood that in vector-valued models some even more complex phenomena appear. Such models can show some phenomena similar to the critical behaviour for *all* low temperatures. The discussion of this question is the main topic of the present paper. Let us remark that since here a model living on the integer lattice is investigated, i.e. a model whose dimension is always one, it is not possible to investigate the dependence of the behaviour of different models on their dimension directly with the help of this model. Nevertheless, this model can give some useful information also about this problem.

Our model has the Hamiltonian function

$$\mathcal{H}(\sigma) = -\sum_{i \in \mathbf{Z}} \sum_{\substack{j \in \mathbf{Z} \\ j > i}} d(i,j)^{-a} \sigma(i)\sigma(j)\,, \tag{3}$$

where $1 < a < 2$, $d(i,j) = 2^{n(i,j)-1}$, and

$$n(i,j) = \min n,\ \exists \text{ some } k \text{ such that } (k-1)2^n < i, j \leq k2^n\,.$$

(In vector-valued models $\sigma(i)\sigma(j)$ denotes scalar product in formula (3).) For the sake of convenience we work with the number $c = 2^{2-a}$ instead of the parameter a. Here

the function $d(i,j)$, called hierarchical distance, is a modification of the usual distance $|i-j|$. It behaves very similarly, and we have introduced it, because models defined with the help of such a distance are simpler to handle because of their symmetry. Let us observe that in formula (3) there is an interaction $d^{-a}(i,j)$ between all pairs (i,j). This interaction is power-like decreasing with respect to the hierarchical distance, and the exponent a of this power is a most important parameter of the model. It plays a role similar to the dimension in translation invariant models.

The description of the large-scale limit of an equilibrium state is a very natural probabilistic problem, but it needs some justification from the point of view of statistical physics. It seems very unlikely that some physical effect can be thought out which depends on what kind of limit distribution appears in the large-scale limit of an equilibrium state. In statistical physics one would be satisfied with the answer to such at the first sight much simpler questions like the order of decrease of the correlation function $E\sigma(i)\sigma(j)-E\sigma(i)E\sigma(j)$ as $|i-j|\to\infty$. But, disregarding some special solvable models, there seems to be no way to determine the decrease of the covariance function without solving the more complex problem about the large-scale limit. This is a very important peculiarity of the problem we are dealing with, and it also indicates the special character of the probabilistic problem one has to solve when investigating the large-scale limit of an equilibrium state. In formula (2) the norming constant A_n has to be chosen in such a way that the variance of $Y_n(j)$ be separated both from zero and infinity. If one knows the right choice of A_n one can also determine the decrease of the correlation function. But we do not know the behaviour of the correlation function at the start. Moreover, the correlation function behaves essentially differently for $T=T_{cr}$ and $T\neq T_{cr}$, and no method is known to give an analytic formula for the critical temperature. Probably, it is principally impossible to give such an explicit formula. What we are able to do is to carry out a procedure which enables us to approximate the critical temperature at each step better and better. We can get the information we are interested in with the help of such a procedure. This fact also indicates an essential difference between the determination of the large-scale limit of equilibrium states and the usual limit theorems in classical probability. In problems of probability theory one generally knows at the start whether one has to deal with strongly or weakly dependent random variables. Here we learn it only after solving the limit problem.

To give a more detailed discussion we have to translate our problem to purely analytical questions. To do it first we have to define our model completely. We consider a model with the Hamiltonian function $\mathcal{H}(\sigma)$ defined in (3) and the free measure ν defined by the formula

$$p_0(x)=\frac{d\nu}{dx}(x)=C(t)\exp\left\{-\frac{x^2}{2}-\frac{t}{4}|x|^4\right\}, \tag{5}$$

where $t>0$ is another parameter of the model, and $C(t)$ is an appropriate norming constant. We shall assume that $t>0$ is sufficiently small. What is important for us is that $p_0(x)$ is a small perturbation of the normal density function, and it tends to zero very fast.

Our first problem is to construct an equilibrium state with the above defined Hamiltonian function and free measure at some temperature T and then to describe its large-scale limit. Both problems can be solved with the help of some limiting procedure if one solves first the following two problems:

Problem 1. Put $V_n = \{1, 2, \dots, 2^n\}$, and

$$\mathcal{H}_{V_n}(x_1, \dots, x_{2^n}) = -\sum_{i \in V_n} \sum_{\substack{j \in V_n \\ j > i}} d(i,j)^{-a} x_i x_j.$$

Define the probability measure $\mu_n = \mu_{n,T}$ on R^{V_n} (on $(R^p)^{V_n}$ if we have a model with p-dimensional spins) with the density function $p_n(x_1, \dots, x_{2^n})$ by the following formula:

$$p_n(x_1, \dots, x_{2^n}) = \frac{d\mu_n(x_1, \dots, x_{2^n})}{dx_1 \dots dx_{2^n}} = C_n \exp\left\{-\frac{1}{T}\mathcal{H}_{V_n}(x_1, \dots, x_{2^n})\right\}.$$

Let $(\sigma(1), \sigma(2), \dots, \sigma(2^n))$ be a μ distributed random vector, and let $p_n(x)$ denote the density function of the average $2^{-n}\sum_{i=1}^{2^n}\sigma(i)$. Give a good asymptotic formula for $p_n(x)$.

Problem 2. Let $N \geq n$, and $h > 0$. Define the probability measure μ_N^h on R^{V_N} (on $(R^p)^{V_N}$ in p-dimensional models) with density function $p_N^h(x_1, \dots, x_{2^N})$ by the formula

$$p_N^h(x_1, \dots, x_{2^N}) = C_N^h \exp\left\{-\frac{1}{T}\mathcal{H}_{V_N}^h(x_1, \dots, x_{2^N})\right\},$$

where

$$\mathcal{H}_{V_N}^h(x_1, \dots, x_{2^N}) = \mathcal{H}_{V_N}(x_1, \dots, x_{2^N}) - h\sum_{i=1}^{2^N} x_i\,,$$

and C_N^h is an appropriate norming constant. (In the vector-valued case we consider h as the vector $(h, 0, \dots, 0)$ with some $h > 0$, and product means scalar product in the last formula.) Let $\mu_{n,N}^h$ denote the restriction of the above defined measure μ_N^h to the volume V_n, and let us consider the Radon–Nikodym derivative

$$f_{n,N}^h(x_1, \dots, x_{2^n}) = \frac{d\mu_{n,N}^h}{d\mu_n}(x_1, \dots, x_{2^n})\,.$$

Give a good asymptotic formula on the function $f_{n,N}^h(x_1, \dots, x_{2^n})$.

Both problems can be translated to purely analytical questions. It can be seen e.g. with the help of Appendix A of [4] that Problem 1 is equivalent to the following

Problem 1′. Define the sequence of density functions $p_n(x) = p_n(x, T)$ by the recursive formula

$$p_{n+1}(x) = C_n(T)\int \exp\left\{\frac{c^n}{T}(x^2 - u^2)\right\} p_n(x-u)p_n(x+u)\,du\,,$$

and let $p_0(x)$ be defined by formula (5). Give a good asymptotic formula on $p_n(x)$.

Problem 2 can be translated with the help of the result in Appendix C of [4] to the following

Problem 2′. Define the functions $f^h_{n,N}(x)$, $N \ge n$, by the relations

$$f^h_{N,N}(x) = K(N,h)\exp\left(\frac{2^N h x^{(1)}}{T}\right) \tag{6}$$

$$f^h_{n,N}(x) = K(n,N,h)\mathbf{S}_n f^h_{n+1,N}(x) \tag{6'}$$

with

$$\mathbf{S}_n f(x) = \int \exp\left(\frac{c^n}{T}xy\right) f\left(\frac{x+y}{2}\right) p_n(y)\,dy\,, \tag{6''}$$

where $p_n(x)$ is the same as in Problems 1 and 1′, and $K(n,N,h)$ is an appropriate normig constant. Find a good asymptotic formula for the above defined functions $f^h_{n,N}(x)$.

It is proved in Appendix C of [4] that

$$\frac{d\mu^h_{n,N}}{d\mu_n}(x_1,\dots,x_{2^n}) = f^h_{n,N}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right), \quad n \le N\,,$$

hence Problems 2 and 2′ are equivalent. The main part of our investigation consists of solving Problems 1′ and 2′. In this paper the vector-valued case (i.e. the case when $p \ge 2$) is considered.

Formally the problems change very little when scalar-valued models are replaced with vector-valued ones. Thus in Problem 1′ the only change is that $|x|$ means the absolute value of a vector, and xy denotes scalar product. Nevertheless, and this is the most striking feature of the problem we are investigating, these seemingly unessential modifications radically change the behaviour of the model. Thus the functions $p_n(x)$ defined in Problem 1′ have the following behaviour for small T in the scalar-valued case: Since $p_n(x) = p_n(-x)$, it is enough to consider $p_n(x)$ for $x \ge 0$. There is some sequence $M_n = M_n(T)$, $M_n > 0$, $M_n \to M$ with some $M = M(T) > 0$ such that $2^{-n/2}p(2^{-n/2}x + M_n)$ tends to a normal density function with expectation zero and some positive variance. (The number M_n is called the spontaneous magnetization in the literature.) This means a central limit theorem with the usual normalization.

The behaviour of the model in the vector-valued case is more complex. It is not difficult to see that $p_n(x)$ depends on x only through its absolute value $|x|$, i.e. there is a function $P_n(z) = P_n(z,T)$, $z \in R^1$ and $z \ge 0$ such that $p_n(x) = P_n(|x|)$ for all $x \in R^p$. Hence, it is natural to investigate the function $P_n(z)$ instead of $p_n(x)$. For small T the behaviour of the function $P_n(z,T)$ is essentially different for $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$. For $\sqrt{2} < c < 2$ there is a sequence M_n such that $2^{-n/2}P_n(2^{-n/2}z + M_n)$ tends to a Gaussian density function with expectation zero and positive variance, i.e. the situation is similar to the scalar-valued case. On the other hand, for $1 < c < \sqrt{2}$ we have to normalize otherwise. In this case $c^{-n}P(c^{-n}z + M)$ has a non-trivial limit with some $M > 0$ which can be described as the solution of an integral equation. This means a non-central limit theorem with an unusual normalization. (In scalar-valued cases there is only at $T = T_{cr}$ such a big difference between the cases $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$.)

The main goal of our investigations is to give an explanation for the above discussed phenomena and some related questions. For this aim let us rewrite Problem 1′ in a form more appropriate for us. Let us introduce the functions

$$\bar{q}_n(x) = \bar{q}_n(x,T) = B_n \exp\left\{ \frac{a_0}{2a_1} c^n x^2 \right\} p_n\left(\sqrt{\frac{T}{a_1}} x \right)$$

and

$$q_n(x) = c^{-n/2} \bar{q}_n\left(c^{-n/2} x \right)$$

with $a_0 = \frac{2}{2-c}$ and $a_1 = a_0 + 1$, where the function $p_n(x)$ is the same as in Problems 1 and 1′. Simple calculation shows that

$$\bar{q}_{n+1}(x) = \int \exp\{-c^n u^2\} \bar{q}_n(x-u) \bar{q}_n(x+u)\, du \tag{7}$$

and

$$q_{n+1}(x) = \mathbf{R} q_n(x)$$

with

$$\mathbf{R}q(x) = \int \exp\{-u^2\} q\left(\frac{x}{\sqrt{c}} - u \right) q\left(\frac{x}{\sqrt{c}} + u \right) du \tag{7′}$$

with some B_n. (We are not really interested in the value of B_n, because it influences only the norming factor when $p_n(x)$ is expressed through $q_n(x)$. On the other hand, the norming constant in $p_n(x)$ is determined by the fact that $p_n(x)$ is a density function.) We also have

$$\bar{q}_0(x) = q_0(x) = B_0 \exp\left\{ \frac{a_0 - T}{2a_1} x^2 - \frac{tT^2}{4a_1^2} |x|^4 \right\}.$$

Since the function $p_n(x)$ can be simply expressed by $q_n(x)$ or $\bar{q}_n(x)$, their investigations are equivalent problems. It is more convenient to work with the function $q_n(x)$ or $\bar{q}_n(x)$ than directly with $p_n(x)$. Let us emphasize that the formulas expressing $q_{n+1}(x)$ and $\bar{q}_{n+1}(x)$ through $q_n(x)$ and $\bar{q}_n(x)$ do not depend on the parameter T. Moreover, in the case of the function $q_n(x)$ it depends neither on n. (This was our main reason for introducing the function $q_n(x)$.) All dependence on T is contained in the starting function $q_0(x)$. But this dependence is very essential, because the operator $\mathbf{R}$, unlike its classical probability counterpart the convolution operator, "remembers" of the starting function. The limit of $\mathbf{R}^n q_0(x)$, as $n \to \infty$, is a small perturbation of the starting function. In our paper [3] we discussed this property and pointed out that this is the final cause of phase transitions and critical phenomena in statistical physics. Here we show that it has other far-reaching consequences. It also implies that vector-valued models with continuous symmetries have some properties which have no scalar-valued counterpart. Let us explain this in more detail.

The function $\bar{q}_n(x)$ depends on x only through its absolute value $|x|$, i.e. it remains invariant if we rotate x. This property, called $\mathbf{O}(p)$ symmetry in the literature, is a continuous symmetry, and it has far-reaching consequences. For the sake of simpler notations let us assume that $p = 2$. Because of this property it is natural to introduce the function $Q_n(x)$ defined by the relation $Q_n(x) = \bar{q}_n((x,0))$ and to work with this

function instead of the original function $\bar{q}_n(x)$. Let us rewrite formula (7) for the functions $Q_n(x)$. We get that

$$Q_{n+1}(x) = C_n \int e^{-c^n(u^2+v^2)} Q_n\left(\sqrt{(x+u)^2+v^2}\right) Q_n\left(\sqrt{(x-u)^2+v^2}\right) du\, dv \quad (8)$$

The argument of the function Q_n in formula (8) is rather complicated. Hence it is natural to substitute it by a simpler expression and to control the error caused by this substitution. This can be done in both cases $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$, but the right substitutions are different in the two cases. This is the reason of their different behaviour. We give a short informal explanation for this.

In formula (8) we have to integrate in the whole space R^2, but because of the kernel $\exp\{-c^n(u^2+v^2)\}$ in the integral an essential contribution for the integral is supplied only if u and v are small. (Actually this is the cause why the operator $\mathbf{R}$ "remembers" to the starting function.) The function $Q_n(x)$ has a maximum $M = M_n$ which depends very weakly on n, hence at this heuristical level we can simply disregard this dependence. Actually it is enough to give a good asymptotic formula for $Q_n(x)$ only in a small neighbourhood of M. When $x \sim M$ and u and v are small it is natural to try to make one of the following approximations in the argument of Q_n in formula (8):

$$\sqrt{(x \pm u)^2+v^2} \sim x \pm u + \frac{v^2}{2(x \pm u)} \sim x \pm u + \frac{v^2}{2M}\,,$$

or

$$\sqrt{(x \pm u)^2+v^2} \sim x \pm u\,.$$

The first approximation would suggest the formula

$$\begin{aligned} &Q_{n+1}(x) \\ &\quad = C_n \int \exp\left\{-c^n(u^2+v^2)\right\} Q_n\left(x+u+\frac{v^2}{2M}\right) Q_n\left(x-u+\frac{v^2}{2M}\right) du\, dv + \varepsilon_n^1(x) \\ &\quad = \int \exp\left\{-c^n u^2\right\} Q_n\left(x+u+\frac{v^2}{2M}\right) Q_n\left(x-u+\frac{v^2}{2M}\right) du\, dv + \varepsilon_n(x)\,, \end{aligned} \quad (9)$$

and the second one

$$\begin{aligned} Q_{n+1}(x) &= C_n \int \exp\left\{-c^n(u^2+v^2)\right\} Q_n(x+u) Q_n(x-u)\, du\, dv + \bar{\varepsilon}_n^1(x) \\ &= \int Q_n(x+u) Q_n(x-u)\, du + \bar{\varepsilon}_n(x)\,. \end{aligned} \quad (9')$$

(If the approximation in the first row is allowed in the latter formulas then the approximation in the second row is also permitted.) Whether the first approximation (9) has to be applied or the second rougher approximation (9′) is also permitted that depends on whether the error terms $\varepsilon_n(x)$ and $\bar{\varepsilon}_n(x)$ are negligible or not. A detailed analysis

shows that in the case $1 < c < \sqrt{2}$ the first approximation (9) is the right choice and in the case $\sqrt{2} < c < 2$ the second one (9′).

For $1 < c < \sqrt{2}$ let us introduce the function $\bar{Q}_n(x) = c^{-3/2n} Q_n(c^{-n}x + M)$. Formula (9) can be rewritten as

$$\bar{Q}_{n+1}(x) = \mathbf{T}\bar{Q}_n(x) + \bar{\varepsilon}_n(x) \tag{10}$$

with

$$\mathbf{T}f(x) = \mathbf{T}_c f(x) = \int \exp(-v^2) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) du\, dv\,.$$

In this case $\bar{Q}_n(x)$ tends to the unique solution of the fixed point equation $Q = \mathbf{T}Q$, and this fact supplies the answer to Problem 1′. To prove it we have to show that the solution $Q(x)$ of the fixed point equation is sufficiently stable for our purposes, i.e. the relation $\lim_{n\to\infty} \mathbf{T}^n\big(Q(x) + \varepsilon(x)\big) = Q(x)$ holds for a small perturbation of the fixed point $Q(x)$, and this convergence is sufficiently fast. This stability property holds only in the case $1 < c < \sqrt{2}$, and this is the reason why the solution of Problem 1′ is different in the cases $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$. To give a complete proof of the convergence $\bar{Q}_n(x) \to Q(x)$ we have to overcome several technical difficulties. The most important one among them is to show that the error term $\bar{\varepsilon}_n(x)$ is really negligibly small. This question is discussed in the Preface and in the first three Sections of Part I of paper [4] in more detail.

In the case $\sqrt{2} < c < 2$ it is more natural to work with the function $\tilde{Q}_n(x) = 2^{-n/2} Q_n(2^{-n/2}x + M)$. Relation (9′) implies that

$$\tilde{Q}_{n+1}(x) = \mathbf{U}\tilde{Q}_n(x) + \tilde{\varepsilon}'_n(x) \tag{10′}$$

with

$$\mathbf{U}f(x) = \int f\left(\frac{x}{\sqrt{2}} - u\right) f\left(\frac{x}{\sqrt{2}} + u\right) du\,.$$

The fixed point of the operator $\mathbf{U}$, i.e. of the convolution operator, are normal density functions with expectation zero. In the case $\sqrt{2} < c < 2$ the error term $\bar{\varepsilon}_n(x)$ is negligibly small for $\sqrt{2} < c < 2$. This implies that in this case $\sqrt{2} < c < 2$ $\tilde{Q}_n(x)$ tends to a normal density function, and this yields the solution of Problem 1′ in this case too.

We discuss the investigation of Problem 2 more briefly. We have introduced the dependence of the measure μ_N^h on h in order to investigate the influence of an external magnetic field to the model. We need such an approach in the investigation of the model at low temperatures. At low temperatures there is a phase transition, i.e. several different equilibrium states exist with the same Hamiltonian function, free measure and temperature. Hence first we must clarify which equilibrium state we are working with. We select out a pure state, i.e. an equilibrium state which cannot be written as the mixture of different equilibrium states. A natural way to construct pure states, and we choose this approach, is to introduce an external field h_N in the volume V_N and take the limit of the measures $\mu_N^{h_N}$ in V_N as $N \to \infty$ and $h_N \to 0$. To carry out this program we have to solve Problem 2. The pure state we construct in this way

has a spontaneous magnetization in the direction $e^{(1)} = (1, 0)$, i.e. $E\sigma^{(1)}(j) = M > 0$ and $E\sigma^{(2)}(j) = 0$ for a random field $\sigma = \{\sigma(j), j \in \mathbf{Z}\}$ with the distribution of the pure state we have constructed. In the large-scale limit defined by formula $(2')$ we have to normalize differently in the direction of the magnetization and in the direction orthogonal to it. Our main interest in this work is the description of the large-scale limit in both directions.

The operator $\mathbf{S}_n$ defined in formula $(6'')$ depends on the function $p_n(x)$ appearing in Problem 1. This implies that the recursive formula expressing $f^h_{n,N}(x)$ through $f^h_{n+1,N}(x)$ has an essentially different form in the cases $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$. Nevertheless, the asymptotic behaviour of the function $f^h_{n,N}(x)$ is the same in the two cases.

More precisely, we need a good asymptotic formula for the function $f^h_{n,N}(x)$ only in a typical region, and outside this region it is enough to give some upper bound on it. Actually we are interested in the product $p_n(x)f^h_{n,N}(x)$, and not the function $f^h_{n,N}(x)$ itself. Hence the typical region, where we need a good approximaton, is a small neighbourhood of the maximum of the above mentioned product. In this domain the Radon–Nikodym derivative has the following form:

$$f^h_{n,N}(x) = C_n \exp\left\{g_n x^{(1)} + A_n x^{(2)2} + \varepsilon_n(x)\right\}, \tag{11}$$

where $\varepsilon_n(x)$ is a small error term, and the constants g_n and A_n are defined by a recursive formula. This formula holds both for $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$, and even the recursive formulas on g_n and A_n have the same structure in the two cases. This means that in Problem $2'$ there is no essential change between models with different parameters c. A heuristic explanation of this fact is contained in the last two formulas at page 466 of [2], and they are the basis for our investigations of Problem 2. When giving an upper bound outside the typical region we have to work differently if the absolute value $|x|$ is not typical and if it is typical, but the vector x is not in the typical region because of its direction. This question is discussed in Section 2 of Part II of [4] in more detail.

The solution of Problem 2 in the case $\sqrt{2} < c < 2$ is given in our paper [1]. Actually the greatest part of that work deals with this question. The solution of this problem (which also contains the investigation of the asymptotic behaviour of the sequences g_n and A_n) is the main ingredient in the description of the large-scale limit of the equilibrium state. For $\sqrt{2} < c < 2$ one has to normalize with $A^{(2)}_n = 2^n c^{-n/2}$ in the direction orthogonal to the direction of the spontaneous magnetization, and the limit is a field of dependent Gaussian random variables whose distribution we can describe explicitly. In the direction of the magnetization the classical norming $A^{(1)}_n = 2^{n/2}$ has to be applied, and the limit is a field of independent Gaussian random variables. (The bounary case $c = \sqrt{2}$ is similar to the above case $\sqrt{2} < c < 2$. (See [5]). The only difference is that in this case the normalization $A^{(1)}_n = 2^{n/2}\sqrt{n}$ has to be applied in the direction of the spontaneous magnetization.) This means that in the direction orthogonal to the spontaneous magnetization a "critical" normalization has to be applied for all low temperatures, i.e. the same normalization as at the critical temperature. This result has no equivalent in scalar-valued models.

The large-scale limit of Dyson's model in the case $1 < c < \sqrt{2}$ is similar to the case $\sqrt{2} < c < 2$ in the direction orthogonal to the spontaneous magnetization, and it is

different in the direction of the magnetization. The reason for it lies in the fact that the solution of Problem 2 is similar in the two cases, and the solution of Problem 1 is different. In the direction orthogonal to the spontaneous magnetization one has to divide again by $A_n^{(2)} = 2^n c^{-n/2}$, and the limit is a field of dependent Gaussian random variables. In the direction of the spontaneous magnetization one has to divide by $A_n^{(1)} = 2^n c^{-n}$, and the limit field is non-Gaussian. The explicit form of the large-scale limit is given in Theorem 2 of Part II in paper [4]. The proof of this result consists of a limit procedure which can be carried out if Problems 1 and 2 are already solved.

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