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RENORMALIZING THE VOTER MODEL. SPACE AND SPACE-TIME RENORMALIZATION

by

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Summary

In a recent paper Bramson and Griffeath have determined the large scale limit of the equilibrium state of voter models. First we give another proof for this result. We show that in spite of the strong correlation between distant sites it can be deduced from the central limit theorem for sums of independent random variables, if a duality equation is combined with an appropriate conditioning. Our method works also in the case of space-time renormalization. We get some results closely related to those of Holley and Stroock.

1. Introduction

In a recent paper [1] BRAMSON and GRIFFEATH have investigated the large scale limit of the equilibrium state of 3-dimensional voter models. In this paper we give another proof for this result, and investigate also space-time renormalization. First we formulate the main result of [1] in detail.

Let $\mathbf{Z}_d$ denote the integer lattice of the d-dimensional space R_d, and let $\Xi=\{0, 1\}^{\mathbf{Z}_d}$. By a d-dimensional voter model we mean a Ξ valued Markov process $(\xi_n)=(\xi_n(i))$, $n=0, 1, 2, \dots, i\in\mathbf{Z}_d$, $\xi_n(i)=0$ or 1, with transition probabilities defined by a distribution p on a finite subset of $\mathbf{Z}_d$ in the following way:

$$\mathsf{P}(\xi_n(j)=1|\xi_{n-1})=\Sigma\,\mathsf{p}(i)\xi_{n-1}(i+j),$$

$$\mathsf{P}(\xi_n(j_1)=1, \dots, \xi_n(j_m)=1|\xi_{n-1})=\prod_{l=1}^{m}\mathsf{P}(\xi_n(j_l)=1|\xi_{n-1}).$$

Let us consider the case when the initial distribution of ξ_0 is given by the formula

$$\mathsf{P}(\xi_0(j_1)=1, \dots, \xi_0(j_m)=1)=\lambda^m, \quad 0\leqq\lambda\leqq 1 \tag{1.1}$$

and the distribution p satisfies the following conditions:

(i) the group generated by $\{i, \mathsf{p}(i)>0\}$ is d-dimensional. Let $Z=(Z_1, \dots, Z_d)$ denote a random variable with distribution p. Then

(ii) $\mathsf{E}Z_k=0, \quad k=1, 2, \dots, d$

(iii) $\mathsf{E}Z_kZ_l=0, \ \mathsf{E}Z_k^2=\mathsf{E}Z_l^2=m, \quad 1\leqq k<l\leqq d.$

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In paper [5] it is proven that the distributions of the configurations ξ_n tend weakly to a measure $\mu=\mu_\lambda$ as $n\to\infty$. The measure μ is translation invariant, and in case $d\geqq 3$ it is ergodic. (This was proven for continuous models, but the proof applies for discrete models as well.)

Let $\xi=(\xi(i))$, $i\in\mathbf{Z}_d$, have the distribution $\mu=\mu_\lambda$. For all $r>0$ and $\varphi\in\mathcal{S}$ ($\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions) we define the random variables

(1.2) $$F_\lambda(\varphi)=\sum_{i\in\mathbf{Z}_d}[\xi(i)-\mathsf{E}\xi(i)]\varphi(i)$$

and

$$F_{\lambda,r}(\varphi)=F(\varphi_r),$$

where $\varphi_r(x)=r^{-\frac{\alpha d}{2}}\varphi\left(\frac{x}{r}\right)$, and $\alpha>0$ will appropriately be chosen. Set $g(j)=\sum_i p(i)p(i+j)$. Let $\mathcal{G}$ denote the group generated by the set $\{j: q(j)>0\}$, and let (Y_n) be a random walk starting from the origin and having transition probabilities $q(j)$. Bramson and Griffeath have proved the following

THEOREM 1. *Let (ξ_n) be a d-dimensional voter model, $d\geqq 3$, satisfying* (i), (ii), (iii) *and* (1.1). *Then for all $\varphi\in\mathcal{S}$ the random variables $F_{\lambda,r}(\varphi)$, with the choice $\alpha=\frac{d+2}{2}$ tend in distribution to a normal random variable with expectation zero and covariance $C_\gamma B(\varphi,\varphi)$ as $r\to\infty$ where*

$$B(\varphi,\psi)=\int_{R_d}\int_{R_d}\frac{\varphi(x)\psi(y)}{|x-y|^{d-2}}\,dx\,dy,$$

and $C_\gamma>0$ is an appropriate constant. If $\mathcal{G}=\mathbf{Z}_d$ then

$$C_\lambda=\frac{\lambda(1-\lambda)\gamma}{2m}(2\pi)^{-d/2}B(d)$$

with

$$\gamma=\mathsf{P}(Y_n\neq 0 \text{ for all } n\geqq 1)$$

and

$$B(d)=\begin{cases}\sqrt{2\pi}(1\cdot 3\cdot 5\ldots(d-3)) & \text{if } d \text{ is odd}\\ 2(2\cdot 4\ldots(d-4)) & \text{if } d \text{ is even.}\end{cases}$$

(This result was proved in [1] in the case $d=3$, but the proof works without any difficulty for $d\geqq 3$.) Using the terminology of [2] this result can be interpreted in the following way. In formula (1.2) a generalized field was defined. Theorem 1 says that its large scale limit is a Gaussian self-similar field with covariance function $C_\lambda B(\varphi,\varphi)$. It follows from the general theory that the large-scale limit of a generalized field, if this limit exists at all, must be self-similar. But it may be also non-Gaussian. Our original aim was to find an intuitive explanation why the limit is Gaussian in the present case. In [1] the convergence to a normal law was proved by the method of semi-invariants. We will show that combining a duality equation with an appropriate conditioning this result can be deduced from the central limit theorem for sums of independent random variables. Our method works also in the case of space-time renormalization.

This paper consists of three sections. In Section 2 a new proof is given for Theorem 1, in Section 3 space-time renormalization is considered.

2. Proof of Theorem 1

We need a duality equation which slightly differs from that of paper [1]. Let a system of coalescing random walks $X(j)=(X_n(j)), j\in \mathbf{Z}_d$, $n=0, 1, 2, \ldots$, be defined on $\mathbf{Z}_d$ in the following way: $X_0(j)=j$, $\mathsf{P}(X_{n+1}(j)-X_n(j)=i)=\mathsf{p}(i)$, and let the random walks $X_n(j)$ evolve independently for different j except for the following collision rule. Whenever two or more walks attempt to occupy the same site at the same time they merge into one. Let $\alpha(j), j\in\mathbf{Z}_d$, be i.i.d. random variables with distribution $\mathsf{P}(\alpha(j)=1)=1-\mathsf{P}(\alpha(j)=0)=\lambda$, and let moreover the α be independent also of the random walks $X(j)$. Set $\hat{\xi}_n(j)=\alpha(X_n(j))$. We claim that the following duality equation holds true:

$$\mathsf{P}(\xi_n(j_1)=1, \ldots, \xi_n(j_k)=1)=\mathsf{P}(\hat{\xi}_n(j_1)=1, \ldots, \hat{\xi}_n(j_k)=1) \tag{2.1}$$
$$\text{for all} \quad n=0, 1, 2, \ldots, k=1, 2, \ldots \quad \text{and} \quad j_1, \ldots, j_k\in\mathbf{Z}_d;$$

i.e. the joint distribution of the random variables $\xi_n(j)$ and that of the random variables $\hat{\xi}_n(j)$ agree for a fixed n. The following argument shows the validity of (2.1). A voter model satisfying (1.1) can be interpreted as follows: At time zero a site j has the opinion 1 with probability λ and the opinion 0 with probability $1-\lambda$. The opinions of different sites at time zero are independent. At time m the site j takes the opinion of the site $j+i$ at time $m-1$ with probability $\mathsf{p}(i)$. These changes of opinion take place at different sites independently of each other. Let us reverse the time, i.e. let us look at the process $\xi_n, \xi_{n-1}, \ldots, \xi_0$. By observing where the opinions $\xi_n(j)$ stem from we can see that there exists a system of coalescing random walks $X_m(j)$, $m=1, 2, \ldots, n$ such that $\xi_n(j)=\xi_0(X_n(j))$. This relation implies (2.1).

Let us first restrict ourselves to the case $\varphi\in\mathcal{D}$, where $\mathcal{D}$ denotes the class of infinite many times differentiable functions on R_d with compact support. Put

$$Q(r)=\{j\,|\,j\in\mathbf{Z}_d,\ j=(j_1, \ldots, j_d),\ |j_i|<r,\ i=1, 2, \ldots, d\}.$$

For all $\varphi\in\mathcal{D}$ there exists an $A>0$ such that $\varphi_r(j)=0$ for all $r>1$ and $j\in\mathbf{Z}_d-Q(Ar)$. For every positive integer n the coalescing random walks $X(j)$, $j\in Q(Ar)$ induce a random partition $B_1(n), \ldots, B_{k(n)}(n)$ of $Q(Ar)$ (the number $k(n)$ is also random) defined in the following way: $j\in Q(Ar)$ and $j'\in Q(Ar)$ belong to the same element of the partition if and only if $X_n(j)=X_n(j')$.

Let $\mathcal{P}_N$ denote the distribution of these partitions. Fix a partition $B_1(n), \ldots, B_{k(n)}(n)$, and observe the following fact: Under the condition that this partition took place the random variables $\hat{\xi}_n(j)$ with argument j in the same element of the partition agree, and with arguments in different elements of the partition are conditionally independent. Introduce the notation

$$\varphi(B, r)=\sum_{j\in B}\varphi_r(j), \quad B\subset\mathbf{Z}_d.$$

The duality equation (2.1) implies that

$$\Sigma[\xi_n(j)-\mathsf{E}\xi_n(j)]\varphi_r(j) \stackrel{\triangle}{=} \Sigma[\hat{\xi}_n(j)-\mathsf{E}\hat{\xi}_n(j)]\varphi_r(j) \stackrel{\triangle}{=} \sum_{l=1}^{k(n)}(\alpha_l-\lambda)\varphi(B_l(n),r) \tag{2.2}$$

where the α are i.i.d. random variables, independent also of the partition $B_1(n), \dots, B_{k(n)}(n)$, $\mathsf{P}(\alpha_l=1)=1-\mathsf{P}(\alpha_l=0)=\lambda$, the partitions $B_1(n), \dots, B_{k(n)}(n)$ are $\mathscr{P}_n$ distributed, and $\stackrel{\triangle}{=}$ denotes equality in distribution. The validity of (2.2) can be seen by observing that the right-hand side and the middle expression in (2.2) have the same conditional distribution with respect to a partition $B_1(n), \dots, B_{k(n)}(n)$.

Let us also consider the (random) partitions $B_1, \dots, B_k$ of $Q(Ar)$ defined by the following rule: $j\in Q(Ar)$ and $j'\in Q(Ar)$ belong to the same element of the partition if and only if $X_n(j)=X_n(j')$ for $n>n(j,j')$. Let $\mathscr{P}_\infty$ denote the distribution of these partitions. It is not difficult to see that $\mathscr{P}_n$ tends to $\mathscr{P}_\infty$ as $n\to\infty$. Indeed, a simple monotonicity argument shows that for every partition $\mathscr{B}=\{B_1, \dots, B_k\}$ of $Q(Ar)$ the $\mathscr{P}_n$ probability of the event that some partition rougher than $\mathscr{B}$ appears; i.e. the $\mathscr{P}_n$ probability of the event that the points of B_1 belong to the same element of the partition, and so do the points of $B_2, B_3, \dots$ and B_k; tends to the $\mathscr{P}_\infty$ probability of the same event. But this relation is equivalent to $\mathscr{P}_n\to\mathscr{P}_\infty$. Exploiting the relation $\mathscr{P}_n\to\mathscr{P}_\infty$ we get, letting n tend to infinity in (2.2), that

$$F_{\lambda,r}(\varphi) \stackrel{\triangle}{=} \sum_{l=1}^{k}(\alpha_l-\lambda)\varphi(B_l,r), \tag{2.3}$$

where the random variables α_l are the same as in (2.2), they are independent of the partitions, and the partitions $B_1, \dots, B_k$ are $\mathscr{P}_\infty$ distributed.

Let us fix a partition of $Q(Ar)$, and consider the conditional distribution of the right-hand side of (2.3) with respect to the condition that this partition took place. We are going to show that for typical partitions this conditional distribution is asymptotically normal with expectation zero and with a variance which is almost the same for different partitions.

Let $A(j_1, \dots, j_k)$ denote the event that the random walks $X(j_1), \dots, X(j_k)$ coalesce. Let $R(j_1, \dots, j_k)=\mathsf{P}(A(j_1, \dots, j_k))$. (In these definitions $j_l=j_{l'}$ for $l\neq l'$ is allowed.) It is easy to see that

$$\Sigma\varphi(B_l,r)^2 = \sum_{\substack{j_1\in Q(Ar)\\ j_2\in Q(Ar)}} I(A(j_1,j_2))\varphi_r(j_1)\varphi_r(j_2),$$

where $I(A)$ denotes the indicator function of the set A. Hence

$$\mathsf{E}[\Sigma\varphi(B_l,r)^2] = \sum_{j_1,j_2} R(j_1,j_2)\varphi_r(j_1)\varphi_r(j_2), \tag{2.4}$$

and

$$\mathsf{D}[\Sigma\varphi(B_l,r)^2] = \tag{2.5}$$

$$= \Sigma\big[\mathsf{P}(A(j_1,j_2)\cap A(j_3,j_4))-R(j_1,j_2)R(j_3,j_4)\big]\varphi_r(j_1)\varphi_r(j_2)\varphi_r(j_3)\varphi_r(j_4).$$

Similarly

$$\mathsf{E}[\Sigma\varphi(B_l, r)^4] = \Sigma R(j_1, j_2, j_3, j_4)\varphi_r(j_1)\varphi_r(j_2)\varphi_r(j_3)\varphi_r(j_4) \leqq \leqq Kr^{-2(d+2)} \sum_{\substack{j_i\in Q(Ar)\\ i=1,2,3,4}} R(j_1, j_2, j_3, j_4). \tag{2.6}$$

Let $\tilde{A}(j_1, j_2, j_3, j_4)$ denote the event that the random walks $X(j_1)$ and $X(j_2)$ coalesce, and so do the random walks $X(j_3)$ and $X(j_4)$, but the random walks $X(j_1)$ and $X(j_2)$ do not meet the random walks $X(j_3)$ and $X(j_4)$. We claim that

$$\mathsf{P}(A(j_1, j_2)\cap A(j_3, j_4)) \leqq \mathsf{P}(\tilde{A}(j_1, j_2, j_3, j_4)) + R(j_1, j_2, j_3, j_4) \tag{2.7}$$

and

$$\mathsf{P}(\tilde{A}(j_1, j_2, j_3, j_4)) \leqq R(j_1, j_2)R(j_3, j_4). \tag{2.8}$$

Relation (2.7) is trivial, and it is enough to check (2.8) in the case when j_1, j_2, j_3, j_4 are different. Let $\tilde{X}(j_2)$ and $\tilde{X}(j_4)$ denote the random walks $X(j_2)$ and $X(j_4)$ which vanish after hitting $X(j_1)$ resp. $X(j_3)$. If the random walks $X(j_i)$ move independently also after hitting each other then the right-hand side of (2.8) equals the probability of the event that $X(j_1)$ hits $X(j_2)$ and $X(j_3)$ hits $X(j_4)$. The left-hand side of (2.8) equals the probability of the event that the same collisions take place, but the random walks $X(j_1)$ and $\tilde{X}(j_2)$ do not meet the random walks $X(j_3)$ and $\tilde{X}(j_4)$. These relations imply (2.8).

Formulae (2.5), (2.7) and (2.8) imply that

$$\mathsf{D}(\Sigma\varphi(B_l, r)^2) = Kr^{-2(d+2)} \sum_{\substack{j_i\in Q(Ar)\\ i=1,2,3,4}} R(j_1, j_2, j_3, j_4). \tag{2.9}$$

$R(j_1, j_2)$ equals the probability of the event that a random walk starting from the origin and having transition probabilities $\mathsf{q}(i)$ hits the point j_1-j_2.

Some modification in the proofs in Sections 2, 6 and 7 in [7] shows that there exists a constant $C>0$ such that

$$R(j_1, j_2) \sim \begin{cases} \dfrac{C}{|j_1-j_2|^{d-2}} & \text{as } |j_1-j_2| \to \infty,\ j_1-j_2\in\mathscr{G} \\ 0 & \text{if } j_1-j_2\in \mathbf{Z}_d-\mathscr{G} \end{cases}$$

with $C=\frac{\gamma}{2m}(2\pi)^{-d/2}B(d)$ if $\mathscr{G}=\mathbf{Z}_d$. It can be proved with the help of this relation that

$$\lambda(1-\lambda)\sum_{j_1,j_2} R(j_1, j_2)\varphi_r(j_1)\varphi_r(j_2) \sim C_\lambda B(\varphi, \varphi) \tag{2.10}$$

as $r\to\infty$. We omit the details of the proof, since it can be found in Proposition 1 of [1] in the case $d=3$, and the proof in the case $d>3$ can be done in the same way. We claim that

$$\sum_{\substack{j_i\in Q(Ar)\\ i=1,2,3,4}} R(j_1, j_2, j_3, j_4) = o(r^{2(d+2)}). \tag{2.11}$$

First we prove Theorem 1 with the help of (2.11). Because of (2.9), (2.11) and Chebyshev's inequality

(2.12) $$\Sigma\varphi(B_l, r)^2 - \mathsf{E}\,\Sigma\varphi(B_l, r)^2 \Rightarrow 0 \quad \text{as} \quad r \to \infty$$

where $\Rightarrow$ denotes convergence in probability.

Relations (2.6) and (2.11) imply that

(2.13) $$\Sigma\varphi(B_l, r)^4 \Rightarrow 0 \quad \text{as} \quad r \to \infty.$$

The conditional distribution of the right-hand side of (2.3) with respect to a partition $B_1, \ldots; B_k$ has the variance $\lambda(1-\lambda)\Sigma\varphi(B_1, r)^2$, which tends, because of (2.12), (2.4) and (2.10) to $C_\lambda B(\varphi, \varphi)$ in probability. This conditional distribution can be represented as the distribution of the sum of the independent random variables $(\alpha_1-\lambda)\varphi(B_1, r)$. Hence (2.13) shows that the central limit theorem applies for it. More precisely, the proof of the central limit theorem for sums of independent random variables shows that

$$\mathsf{E}\big[\exp\big[it\,\Sigma(\alpha_l-\lambda)\varphi(B_l, r)\big]\big|B_1, \ldots, B_k\big] \Rightarrow \exp\left(-\frac{C_\lambda(\varphi, \varphi)}{2}t^2\right)$$

for all t. Hence Theorem 1 holds for all $\varphi\in\mathscr{D}$.

The duality equation (2.1) also implies that

$$\operatorname{Cov}(\xi(j_1), \xi(j_2)) = \lim_{n\to\infty} \operatorname{Cov}(\xi_n(j_1), \xi_n(j_2)) = \lambda(1-\lambda)R(j_1, j_2).$$

Hence

(2.14) $$\mathsf{E}[F_{\lambda,r}(\varphi)^2] = \sum_{j_1, j_2} R(j_1, j_2)\varphi_r(j_1)\varphi_r(j_2) \quad \text{for all} \quad \varphi\in\mathscr{S}.$$

On the other hand there exists a $K>0$ such that

(2.15) $$R(j_1, j_2) \leqq K(|j_1-j_2|+1)^{-d+2} \quad \text{for all} \quad j_1, j_2\in\mathbf{Z}_d.$$

Relations (2.14) and (2.15) imply that for all $\varphi\in\mathscr{S}$

$$\limsup_r \mathsf{E}[F_{\lambda,r}(\varphi)^2] \leqq \limsup_r \sum_{j_1, j_2} R(j_1, j_2)|\varphi_r(j_1)||\varphi_r(j_2)| \leqq$$

$$\leqq K\limsup_r \sum_{j_1, j_2} r^{-2d}\left(\frac{|j_1-j_2|+1}{r}\right)^{-d+2}\left|\varphi\left(\frac{j_1}{r}\right)\right|\left|\varphi\left(\frac{j_2}{r}\right)\right| \leqq K'B(|\varphi|, |\varphi|).$$

Given a $\varphi\in\mathscr{S}$ and $\varepsilon>0$ there exists a $\varphi_\varepsilon\in\mathscr{D}$ such that $B(|\varphi-\varphi_\varepsilon|, |\varphi-\varphi_\varepsilon|)<\varepsilon$. The last inequality implies that

(2.16) $$\limsup_r \mathsf{E}[F_{\lambda,r}(\varphi-\varphi_\varepsilon)^2] \leqq K'\varepsilon$$

and

(2.17) $$\limsup_r \{\mathsf{E}[F_{\lambda,r}(\varphi)^2] - \mathsf{E}[F_{\lambda,r}(\varphi_\varepsilon)^2]\} \leqq K'\varepsilon + 2[K'\varepsilon B(\varphi, \varphi)]^{1/2}.$$

Relations (2.16) and (2.17) reduce Theorem 1 to the already proved special case $\varphi\in\mathscr{D}$.

Relation (2.11) in the case $d=3$ has been proved in Proposition 3 of [1]. We give another proof which we think to be simpler. Let us introduce the notation

$$S_k(r) = \sum_{\substack{j_i \in Q(r) \\ i=2,3,\dots,k}} R(0, j_2, \dots, j_k), \quad k = 2, 3, \dots.$$

Obviously, $R(j_1, \dots, j_k) = R(0, j_2 - j_1, \dots, j_k - j_1)$, hence

$$\sum_{\substack{j_i \in Q(Ar) \\ i=1,2,\dots,k}} R(j_1, \dots, j_k) \leqq A^d r^d S_k(2Ar),$$

and relation (2.11) follows from the following

LEMMA 1. *If* $d \geqq 3$ *then*

$$S_k(r) \leqq A(k) r^{2k-2}$$

for all $r > 1$, $k \geqq 2$ *with an appropriate* $A(k)$.

PROOF of Lemma 1. Let $\bar{R}(j_1, \dots, j_k)$ denote the probability of the event that the random walks $X(j_1), \dots, X(j_k)$ coalesce, and none of the random walks $X(j_1), \dots, X(j_k)$ hit each other before $X(j_1)$ and $X(j_2)$ meet. Fix some points $j_1, \dots, j_k \in \mathbf{Z}_d$, and define the random walks

$$U_k = (U_k(n)), \quad \bar{U}_k = (\bar{U}_k(n)), \quad n = 0, 1, 2, \dots,$$

on the lattice $\mathbf{Z}_{kd-d}$ in the following way:

$$U_k(n) = (X_n(j_2) - X_n(j_1), \dots, X_n(j_k) - X_n(j_1)),$$

and

$$\bar{U}_k(n) = U_k(n) - U_k(0).$$

Given an $x \in \mathbf{Z}_{kd-d}$ let $A_k(x)$ denote the event that $\bar{U}_k(n) = x$ for some n, and none of the random walks $X_m(j_1), \dots, X_m(j_k)$ meet for $m < n$. We claim that

$$\mathsf{P}_k(x) = \mathsf{P}(A_k(x)) \leqq C(k)(|x|+1)^{-kd+d+2}. \tag{2.18}$$

Indeed, if the random walks $X(j_i)$, $i = 1, 2, \dots, k$ move independently also after hitting each other then $(X_n(j_2) - X_n(j_1) - (j_2 - j_1), \dots, X_n(j_k) - X_n(j_1) - (j_k - j_1))$ is a non-degenerate isotropic random walk starting from the origin in the $kd-d$ dimensional lattice. Hence it hits the point x with a probability less than $C(k)(|x|+1)^{-kd+d+2}$. This probability clearly majorizes $\mathsf{P}(A_k(x))$, hence (2.18) holds true. The quantity $R(j_1, \dots, j_k)$ equals the probability of the following event: There exist some

$$x = (j_1 - j_2, u_3 + j_1 - j_3, \dots, u_k + j_1 - j_k), \quad u_i \in \mathbf{Z}_d, \quad i = 2, 3, \dots, k,$$

such that at first the event $A_k(x)$ takes place, and then the random walks $X_n(j_2) - X_n(j_1), \dots, X_n(j_k) - X_n(j_1)$ coalesce. The Markov property of the random walk $(X(j_1), \dots, X(j_k))$ implies that

$$\bar{R}(j_1, \dots, j_k) = \sum_{\substack{u_i \in \mathbf{Z}_d \\ i=3,\dots,k}} P_k((j_1 - j_2, u_3 + j_1 - j_3, \dots, u_k + j_1 - j_k)) R(0, u_3, \dots, u_k). \tag{2.19}$$

(The function R on the right-hand side has $k-1$ arguments.)

Now we want to make a good recursion formula for $S_k(r)$ with the help of (2.19). Obviously,

$$(k-2)!\,R(j_1, \dots, j_k) \leqq \sum_{\pi\in\Pi_k} \bar{R}(j_{\pi(1)}, \dots, j_{\pi(k)}) =$$

$$= \sum_{\pi\in\Pi_k} \bar{R}(0, j_{\pi(2)}-j_{\pi(1)}, \dots, j_{\pi(k)}-j_{\pi(1)}),$$

where Π_k denotes the set of all permutations of the numbers $1, 2, \dots, k$. Let us sum up the last inequality for all $(0, j_2, \dots, j_k)$, $j_i\in Q(r)$, $i=2, 3, \dots, k$. The left-hand side equals $(k-2)!S_k(r)$ and the right-hand side can be estimated from above by $k! \sum_{\substack{j_i\in Q(2r)\\ i=2,3,\dots,k}} \bar{R}(0, j_2, \dots, j_k)$. To see the last relation one has to observe that $j_{\pi(i)}-j_{\pi(1)}\in Q(2r)$ if $j_i\in Q(r)$, $i=2, 3, \dots, k$, and for all $\pi\in\Pi_k$ and $j_i\in Q(r)$, $i=2, 3, \dots, k$, the equation

$$(0, j_2, \dots, j_k) = (x_1, x_{\pi(2)}-x_{\pi(1)}, \dots, x_{\pi(k)}-x_{\pi(1)})$$

has only one solution. Thus we obtain that

$$S_k(r) \leqq k(k-1) \sum_{\substack{j_i\in Q(2r)\\ i=2,3,\dots,k}} \bar{R}(0, j_2, \dots, j_k). \tag{2.20}$$

Summing up the identity (2.19) for all $(0, j_2, \dots, j_k)$, $j_i\in Q(2r)$, $i=2, 3, \dots, k$, and exploiting (2.20) we get that

$$\begin{aligned} S_k(r) \leqq k(k-1) &\sum_{\substack{u_i\in \mathbf{Z}_d\\ i=2,3,\dots,k}} R(0, u_3, \dots, u_k)\times \\ &\times \sum_{\substack{j_l\in Q(2r)\\ l=2,\dots,k}} \mathsf{P}_k((-j_2, u_3-j_3, \dots, u_k-j_k)). \end{aligned} \tag{2.21}$$

Define the sets

$$D(m) = D(m, r, k) =$$

$$= \{(x_3, \dots, x_k)\in \mathbf{Z}_{(k-2)d},\ x_l\in \mathbf{Z}_d,\ l = 3, \dots, k,\ 2^m r \leqq \max_{3\leqq l\leqq k} |x_l| < 2^{m+1}r\}$$

for $m=1, 2, \dots,$ and

$$D(0) = D(0, r, k) = \{(x_3, \dots, x_k),\ x_l\in \mathbf{Z}_d,\ l = 3, \dots, k,\ \max_{3\leqq l\leqq k} |x_l| < 2r\}.$$

Let $j_l\in Q(2r)$, $l=2, 3, \dots, k$, $u_p\in \mathbf{Z}_d$, $p=3, \dots, k$, and $(u_3, \dots, u_k)\in D(m)$ for some $m\geqq 0$. If $m\geqq 2$ then

$$\mathsf{P}_k((-j_2, u_3-j_3, \dots, u_k-j_k)) \leqq C(k)(2^{m-1}r)^{-dk+d+2}$$

because of (2.18) and the relation $|u_l-j_l|\geqq 2^{m-1}r$ for some $3\leqq l\leqq k$. Hence

$$\sum_{\substack{j_l\in Q(2r)\\ l=2,3,\dots,k}} \mathsf{P}_k((-j_2, u_3-j_3, \dots, u_k-j_k)) \leqq C'(k)2^{(-kd+d+2)m}r^2 \tag{2.22}$$

if $(u_3, \ldots, u_k) \in D(m)$ with $m \geqq 2$. On the other hand if $(u_3, \ldots, u_k) \in D(m)$ with $m=0$ or $m=1$ then

$$(2.23) \qquad \sum_{\substack{j_l \in Q(2r) \\ l=2,3,\ldots,k}} \mathsf{P}_k((-j_2, u_3-j_3, \ldots, u_k-j_k)) \leqq \sum_{\substack{u \in Z_{kd-d} \\ |u| \leqq 6kr}} (1+|u|)^{-kd+d+2} \leqq C''(k)r^2.$$

Splitting up the outer sum in (2.21) in the form

$$\sum_{\substack{u_i \in Z_d \\ i=3,\ldots,k}} = \sum_{m=0}^{\infty} \sum_{(u_3,\ldots,u_k) \in D(m)}$$

and exploiting (2.22) and (2.23) we obtain that

$$(2.24) \qquad S_k(r) \leqq \bar{C}(k) \sum_{m=0}^{\infty} 2^{(-kd+d+2)m} r^2 \sum_{(u_3,\ldots,u_k) \in D(m)} R((0, u_3, \ldots, u_k)),$$

or

$$S_k(r) \leqq \bar{C}(k) r^2 \sum_{m=0}^{\infty} 2^{(-kd+d+2)m} S_{k-1}(2^{m+1}r).$$

$S_2(r) \leqq A(2)r^2$ because of (2.15). Substituting this inequality into (2.24) with $k=3$ we obtain that $S_3(r) \leqq A(3)r^4$. A simple induction by k in (2.24) shows that $S_k(r) \leqq A(k)r^{2k-2}$. Lemma 1 is proven.

At a first sight Proposition 3 of [1] seems to be slightly more general than Lemma 1 of this paper. Nevertheless it easily follows from the present Lemma because of the fast decrease of the functions $\varphi \in \mathcal{S}$ at infinity. Relation (2.19) makes possible also to estimate the individual terms R in $S_k(r)$, but a direct estimation of $S_k(r)$ turned out to be simpler.

In paper [2] DOBRUSHIN described the covariance function of a Gaussian self-similar field in a different form, he gave a so-called spectral representation. We finish Section 2 by rewriting $B(\varphi, \psi)$ in this form. More generally, we prove the identity

$$(2.25) \qquad \int_{R_d} \int_{R_d} \varphi(x)\psi(y)|x-y|^{-\varkappa}\, dx\, dy = C(\varkappa, d) \int_{R_d} \tilde{\varphi}(t)\bar{\tilde{\psi}}(t)|t|^{\varkappa-d}\, dt$$

for all d-dimensional $\varphi, \psi \in \mathcal{S}$ and $\varkappa > 0$, where

$$C(\varkappa, d) = \frac{2^{d-\varkappa}\pi^{d/2}\Gamma\left(\frac{d-\varkappa}{2}\right)}{\Gamma\left(\frac{\varkappa}{2}\right)},$$

and $\tilde{}$ denotes Fourier transform. In our case $\varkappa = d-2$, and the expressions in formula (2.25) are the covariance function of the so-called d-dimensional 0-mass free field.

To prove (2.25) we remark that the Fourier transform of $|x|^{-\varkappa}$, considering it as a generalized function equals $C(\varkappa, d)|t|^{d-\varkappa}$ (see e.g. [4]). Let $*$ denote convolution, and let $\psi_-(x)=\psi(-x)$. Then

$$\iint \varphi(x)\psi(y)|x-y|^{-\varkappa}\,dx\,dy = \int |t|^{-\varkappa} \int \varphi(x)\psi(x-t)\,dx\,dt =$$
$$= \int |t|^{-\varkappa}\varphi * \psi_-(t)\,dt = C(\varkappa, d)\int |t|^{\varkappa-d}\varphi \tilde{*} \psi_-(t)\,dt =$$
$$= C(\varkappa, d)\int |t|^{\varkappa-d}\tilde{\varphi}(t)\bar{\tilde{\psi}}(t)\,dt$$

as we claimed.

3. On space-time renormalization

Given a positive integer N, a number $r>0$ and a function $\varphi\in\mathcal{S}$ we define the random variables

$$F_{\lambda,N}(\varphi) = \sum_{i\in \mathbf{Z}_d} [\xi_N(i)-\mathsf{E}\xi_N(i)]\varphi(i)$$

and

$$F_{\lambda,N,r}(\varphi)=F_{\lambda,N}(\varphi_{N,r}),$$

where $\varphi_{N,n}(x)=h(N,r)\varphi\left(\frac{x}{r}\right)$, and the norming factor $h(N,r)$ is appropriately chosen. We are interested in the limiting behaviour of $F_{\lambda,N,r}(\varphi)$ when both N and r tend to infinity. The most interesting case is when $N\sim\alpha r^2$ with some $\alpha>0$. We investigate this case, then we briefly describe the situation when either $Nr^{-2}\to 0$ or $Nr^{-2}\to\infty$ without working out the details. We also determine the limit distribution of the random vector $F_{\lambda,[t_1N],r}(\varphi^{(1)}), \dots, F_{\lambda,[t_kN],r}(\varphi^{(k)})$, when $0<t_1<\dots<t_k$, $\varphi^{(1)}, \dots, \varphi^{(k)}\in\mathcal{S}$ are arbitrary, $N=r^2$, $[\ ]$ denotes integer part, and $N\to\infty$. We determine this limiting distribution also in the case when ξ_0 has an equilibrium distribution μ_λ.

Finally we show that the limit distributions we have obtained in the multi-dimensional case determine, after an appropriate rescaling, a so-called generalized Ornstein-Uhlenbeck process. Our first result is the following

THEOREM 2. *Let $N\sim\alpha r^2$, $\alpha>0$, $r\to\infty$, and $h(N,r)=r^{-\frac{d+2}{2}}$. Then $F_{\lambda,N,r}(\varphi)$ tends in distribution to a normal random variable with expectation zero and covariance $\bar{C}_\lambda B_\alpha(\varphi,\varphi)$, with some $\bar{C}_\lambda>0$, where*

$$B_\alpha(\varphi,\psi) = \int_{R_d}\int_{R_d} \varphi(x)\psi(y)G_\alpha(|x-y|)\,dx\,dy$$

with

$$G_\alpha(t) = \int_0^{2m\alpha} \exp\left(-\frac{t^2}{2u}\right)u^{-d/2}\,du.$$

If $\mathcal{G}=\mathbf{Z}_d$ then

$$\bar{C}_\lambda = \frac{\gamma\lambda(1-\lambda)}{2m}(2\pi)^{-d/2}.$$

Let $A_N(j_1,\dots,j_k)$ denote the event that the random walks $X(j_1),\dots,X(j_k)$ meet up to time N. Let $R_N(j_1,\dots,j_k)=\mathsf{P}(A_N(j_1,\dots,j_k))$. Obviously,

$$R_N(j_1,\dots,j_k)\leqq R(j_1,\dots,j_k). \tag{3.1}$$

The proof of Theorem 2 goes essentially on the same line as that of Theorem 1. The main difference is that now we have to estimate $R_N(j_1, j_2)$ instead of $R(j_1, j_2)$. Obviously,

$$R_N(j_1, j_2) = \mathsf{P}(Y_n = j_1 - j_2 \text{ for some } 0 \leqq n \leqq N).$$

We need the following

LEMMA 2. *For all* $\varepsilon > 0$, $x \in \mathbf{Z}_d$, $|x| > \varepsilon \sqrt{N}$

$$\pi_N(x) = \mathsf{P}(Y_n = x \text{ for some } 0 \leqq n \leqq N) =$$

$$= \begin{cases} C \int_0^{2Nm} \exp\left(-\frac{|x|^2}{2u}\right) u^{-d/2}\, du + o(N^{-d-2/2}) & \text{if } x \in \mathcal{G} \\ 0 & \text{if } x \notin \mathcal{G}, \end{cases}$$

$C = \frac{\bar{C}_\lambda}{\lambda(1-\lambda)}$, *where* $\bar{C}_\lambda$ *is the same constant as in Theorem* 2, *and* $o(\cdot)$ *is uniform in* x *and* N.

The proof is a natural modification of that of P1 in Chapter 26 of [7].

PROOF of Lemma 2. Let $P(n, x) = P(Y_n = x)$, $G_M(x) = \sum_{n=0}^{M} \mathsf{P}(n, x)$, $G(0) = G_\infty(0) = \sum_{n=0}^{\infty} P(n, 0)$. Given a $\delta > 0$ a $k = k(\delta)$ can be chosen in such a way that $G_k(0) > G(0) - \delta$. First we show that

$$\frac{G_N(x)}{G(0)} \leqq \pi_N(x) \leqq \frac{G_{N+k}(x)}{G(0)-\delta}. \tag{3.2}$$

Let $\mathsf{p}_j(x) = \mathsf{P}(Y_j = x,\ Y_l \neq x \text{ for } l < j)$. Then

$$P_n(x) = \sum_{j=0}^{n} \mathsf{p}_j(x) P(n-j, 0),$$

and

$$G_{N+k}(x) = \sum_{n=0}^{N+k} \sum_{j=0}^{n} \mathsf{p}_j(x) P(n-j, 0) = \sum_{j=0}^{N+k} \mathsf{p}_j(x) \sum_{l=0}^{N+k-j} P(l, 0) \geqq$$

$$\geqq \sum_{j=0}^{N} \mathsf{p}_j(x) \sum_{l=0}^{k} P(l, 0) \geqq \pi_N(x)[G(0) - \delta],$$

what is the right-hand side of (3.2).

On the other hand $G_N(x)$ is the expected value of the number $n, n \leqq N$ such that $Y_n = x$. Because of the Markov property the conditional expectation of this number under the condition that the first hitting of the point x took place at time l is less than or equal to $G(0)$, the expected number of returns to x. Hence

$$G_N(x) \leqq \sum_{l=0}^{N} \mathsf{p}_l(x) G(0).$$

It remains to prove that

$$(3.3)\quad G_N(x) = C\gamma^{-1}\int_0^{2Nm} \exp\left(-\frac{|x|^2}{2u}\right) u^{-d/2}\,du + o(N^{-(d-2)/2}) \quad \text{if} \quad x\in\mathcal{G},\ |x| > \varepsilon\sqrt{N}.$$

If Y_n is a strongly aperiodic random walk over G (cf. [7] D1 on p. 42 for the definition of strongly aperiodic random walk) then the local central limit theorem holds for $\mathsf{P}(Y_n=x)$; $n\leqq N$, $|x|>\varepsilon N^{1/2}$; with an error term $o(N^{-d/2})$. Hence

$$G_N(x) \sim 2mC\gamma^{-1}\sum_{n=1}^{N}(2mn)^{-d/2}\exp\left(-\frac{|x|^2}{4mn}\right) =$$

$$= C\gamma^{-1}\sum_{n=1}^{N}|x|^{-d+2}\exp\left(-\frac{1}{2\Delta n}\right)(\Delta n)^{-d/2}\Delta \sim$$

$$\sim C\gamma^{-1}|x|^{-d+2}\int_0^{2Nm|x|^{-2}}\exp\left(-\frac{1}{2u}\right)u^{-d/2}\,du = C\gamma^{-1}\int_0^{2Nm}\exp\left(-\frac{|x|^2}{2u}\right)u^{-d/2}\,du,$$

where $\Delta=2m|x|^{-2}$.

If Y_n is not strongly aperiodic then the local central limit theorem may not hold true. But a modification of the argument in [7] can reduce the general case to the strongly aperiodic one. We introduce the strongly aperiodic random walks $Y^\alpha=(Y_n^\alpha)$, $0<\alpha<1$, defined by the formulae

$$Y_0^\alpha = 0,\quad \mathsf{P}(Y_{n+1}^\alpha = y\,|\,Y_n^\alpha = x) = \alpha q(y-x)+(1-\alpha)\delta(x, y).$$

Let $G_N^\alpha(x)=\sum_{n=0}^{\infty}\mathsf{P}(Y_n^\alpha=x)$. We claim that

$$(3.4)\qquad \alpha G_N^\alpha(x) \leqq G_N(x) \leqq \frac{N}{N-1}G_{\bar\alpha N}^\alpha(x),$$

with $\bar\alpha=(1+(1-\alpha)^{1/2})/\alpha$. Letting α go to 1 in (3.4) we get that (3.3) holds in the general case. To prove (3.4) let us consider a sequence $\eta_1=\eta_1(\alpha)$, $\eta_2=\eta_2(\alpha)$, ... of i.i.d. random variables with distribution $\mathsf{P}(\eta_1=k)=\alpha(1-\alpha)^{k-1}$, $k=1, 2, \ldots$, which are independent also of the random walk Y_n. Set $S(n)=\sum_{j=1}^{n}\eta_j$, $n=1, 2, \ldots$. $S(0)=0$ and $Y_n^\alpha=y_j$ if $S(j-1)\leqq n<S(j)$. Then the random walk Y_n^α has the prescribed distribution, hence

$$G_N(x) = \sum_{n=0}^{N}\mathsf{E}(\delta(x, Y_n)),\quad G_N^\alpha(x) = \sum_{n=0}^{N}\mathsf{E}(\delta(x, Y_n^\alpha)).$$

As $S(N)\geqq N$,

$$G_N^\alpha(x) = \mathsf{E}\sum_{n=0}^{S(N)}\mathsf{E}(\delta(x, Y_n^\alpha)) = \mathsf{E}\eta_1\sum_{n=0}^{N}\mathsf{E}(\delta(x, Y_n)) = \alpha^{-1}G_N(x),$$

what is the left-hand side of (3.4).

Define the set $A=\{\eta_1+\ldots+\eta_N<N\bar{\alpha}\}$. As $\mathsf{E}\eta_1=\alpha^{-1}$; $\mathsf{D}\eta_1=\frac{1-\alpha}{\alpha^2}$, Chebyshev's inequality yields that

$$1-\mathsf{P}(A)\leqq\mathsf{P}\left(\left|\sum_{j=1}^{N}(\eta_j-\mathsf{E}\eta_j)\right|>N\frac{(1-\alpha)^{1/2}}{\alpha}\right)\leqq N^{-1}. \tag{3.5}$$

Since the event A is independent of the random walk Y_n,

$$\int_A\sum_{n=0}^{N}\delta(x,Y_n)\,d\mathsf{P}=\mathsf{P}(A)G_N(x). \tag{3.6}$$

On the other hand

$$\begin{aligned}\int_A\sum_{n=0}^{N}\delta(x,Y_n)\,d\mathsf{P}&\leqq\int_A\sum_{n=0}^{N}\eta_n\delta(x,Y_n)\,d\mathsf{P}\leqq\\&\leqq\int_A\sum_{n=0}^{N\bar{\alpha}}\delta(x,Y_n^{\alpha})\,d\mathsf{P}\leqq G^{\alpha}_{\bar{\alpha}N}(x).\end{aligned} \tag{3.7}$$

Relations (3.5), (3.6) and (3.7) imply the right-hand side of (3.4). Lemma 2 is proven.

Now some slight change in the proof of Theorem 1 yields the proof of Theorem 2. First we show that

$$\lim_{r\to\infty}\mathsf{E}[F_{\lambda,N,r}(\varphi)^2]=B_\alpha(\varphi,\varphi),\quad\varphi\in S. \tag{3.8}$$

We can write, similarly to (2.14),

$$\begin{aligned}\mathsf{E}[F_{\lambda,N,r}(\varphi)^2]&=\lambda(1-\lambda)\sum_{j_1,j}R_N(j_1,j_2)\varphi_{N,r}(j_1)\varphi_{N,r}(j_2)=\\&=\lambda(1-\lambda)\Big[\sum_{|j_1-j_2|<\varepsilon r}+\sum_{|j_1-j_2|\geqq\varepsilon r}\Big].\end{aligned} \tag{3.9}$$

The first sum on the right-hand side of (3.9) is $O(\varepsilon^2)$ for $r>r(\varepsilon)$ because of (3.1) and (2.15), and the second sum is $B_\alpha(\varphi,\varphi)+O(\varepsilon^2)$ because of Lemma 2. Letting ε go to zero in (3.9) we obtain (3.8).

In the proof of Theorem 2 we apply formula (2.2). Observe that because of relation (3.1) Lemma 1 remains valid if we substitute $R(0,j_2,\ldots,j_k)$ with $R_N(0,j_2,\ldots,j_k)$ in the definition of $S_k(r)$. We get the proof of Theorem 2 simply by writing R_N instead of R, A_N instead of A everywhere in the proof of Theorem 1.

If $Nr^{-2}\to\infty$, and $r\to\infty$ then $F_{\lambda,N,r}$ tends, with the choice of $h(N,r)=r^{-(d+2)/2}$, to a normal random variable with expectation zero and variance $C_\lambda B(\varphi,\varphi)$. This can be seen by exploiting the relation $R_N(j_1,j_2)\sim R(j_1,j_2)$ if $|j_1-j_2|=o(N^{1/2})$. If $Nr^2\to0$, $N\to\infty$ then $F_{\lambda,N,r}(\varphi)$ tends, with the choice of $h(N,r)=N^{-1/2}r^{-d/2}$, to a Gaussian random variable with expectation zero and variance $C'_\lambda\int\varphi^2(x)dx$ with an appropriate $C'_\lambda>0$, i.e. the generalized fields $F_{\lambda,N,r}$ tend to the white noise field as $r\to\infty$. The reason for this result is that for $|j_1-j_2|\gg N^{1/2}$ the function $R_N(j_1,j_2)$ is very small; hence the interaction between such pairs (j_1,j_2) is negligible. More explicitely the following inequality holds true:

$$R_N(j_1,j_2)\leqq C_1\exp\left(-C_2\frac{|j_1-j_2|^2}{N}\right) \tag{3.10}$$

with some appropriate $C_1>0$ and $C_2>0$, hence

$$\mathsf{E}[F_{\lambda,N,r}(\varphi)^2] = \lambda(1-\lambda)\,\Sigma R_N(j_1,j_2)\varphi_{N,r}(j_1)\varphi_{N,r}(j_2) \sim C'_\lambda \int \varphi^2(x)\,dx.$$

Our conditioning argument shows that the limit is normal if

$$\sum_{\substack{j_l\in Q(r)\\ l=1,2,3,4}} R_N(j_1,j_2,j_3,j_4) = o(N^2 r^{2d}).$$

This inequality can be deduced from Lemma 1, formula (3.1) and the inequality

$$(3.10') \qquad R_N(j_1,j_2,j_3,j_4) \le C_1 \exp\left[-\frac{C_2}{N}\max_{l_1,l_2=1,2,3,4}|j_{l_1}-j_{l_2}|^2\right].$$

Inequalities (3.10) and (3.10′) follow from standard large deviation results. If N is fixed, and $r\to\infty$ then a similar argument shows that $F_{\lambda,N,r}(\varphi)$ tends, with the choice $h(N,r)=r^{-1/2}$, to a normal random variable with variance $\lambda(1-\lambda)C(N)\int\varphi^2(x)dx$ and expectation zero, where $C(N)$ is appropriately chosen. Now we turn to the investigation of the limit distribution of the vector valued random variables $(F_{\lambda,N_1,r}(\varphi^{(1)}),\dots,F_{\lambda,N_k,r}(\varphi^{(k)}))$, where $\varphi^{(1)},\dots,\varphi^{(k)}\in\mathcal{S}$, and $0<N_1<\dots<N_k$. We need a multi-dimensional version of the identities (2.1) and (2.2).

Given some integers $0\le N_1<N_2<\dots<N_k$ we define a system of coalescing random walks $X(j,l)=(X_n(j,l)),\ j\in\mathbf{Z}_d,\ l=1,2,\dots,k$ in the following way: $X_{N_k-N_l}(j,l)=j$, and $X_n(j,l)$ is not defined for $n<N_k-N_l$; i.e. the random walk $X(j,l)$ starts at time N_k-N_l from point j; $\mathsf{P}(X_{n+1}(j,l)-X_n(j,l)=i)=\mathsf{p}(i)$, and let the random walks $X(j,l)$ evolve independently for different (j,l) except for the following collision rule. Whenever two or more walks attempt to occupy the same site at the same time they merge into one. Let $\alpha(j),\ j\in\mathbf{Z}_d$, be i.i.d. random variables with distribution $\mathsf{P}(\alpha(j)=1)=1-\mathsf{P}(\alpha(j)=0)=\lambda$, which are independent also of the random walks $X(j,l)$. Put $\hat\xi_{N_l}(j)=\alpha(X_{N_k}(j,l))$. The same argument which proved (2.1) shows that for any integer p and pairs $(n_1,j_1),\dots,(n_p,j_p)$, where $j_l\in\mathbf{Z}_d$ and n_l is one of the numbers $N_1,\dots,N_k,\ l=1,2,\dots,p$,

$$(3.11) \qquad \mathsf{P}(\hat\xi_{n_1}(j_1)=1,\dots,\hat\xi_{n_p}(j_p)=1) = \mathsf{P}(\xi_{n_1}(j_1)=1,\dots,\xi_{n_p}(j_p)=1).$$

Let us first restrict ourselves to the case when $\varphi^{(1)}\in\mathcal{D},\dots,\varphi^{(k)}\in\mathcal{D}$. Then there exists an $A>0$ such that all of the functions $\varphi^{(1)}\left(\frac{j}{r}\right),\dots,\varphi^{(k)}\left(\frac{j}{r}\right)$ are zero for $j\in\mathbf{Z}_d-Q(Ar)$. Let $Q_1(Ar),\dots,Q_k(Ar)$ be k replicas of the set $Q(Ar)$, and put $\bar Q=\bigcup_{l=1}^{k}Q_l(Ar)$. We define the following random partition of $\bar Q$: $j\in Q_l(Ar)$ and $j'\in Q_{l'}(Ar)$ belong to the same element of the partition if and only if $X_{N_k}(j,l)=X_{N_k}(j',l'),\ j,j'\in Q(Ar);\ l,l'=1,2,\dots,k$. Let $\mathcal{P}(N_1,\dots,N_k)$ denote the distribution of these partition of $\bar Q$. Let us fix some $\varphi^{(1)}\in\mathcal{D},\dots,\varphi^{(k)}\in\mathcal{D}$, some real numbers $d_1,\dots,d_k$ and a partition of $\bar Q$. If B is an element of this partition then we define

$$B_l = B\cap Q_l(Ar), \quad l=1,2,\dots,k$$

and

$$\Phi(B) = \Phi(\varphi^{(1)}, \ldots, \varphi^{(k)}, d_1, \ldots, d_k, r, B) = \sum_{l=1}^{k} d_l \sum_{j \in B(l)} \varphi^{(l)}\left(\frac{j}{r}\right).$$

Let $\alpha_1, \alpha_2, \ldots$ be i.i.d. random variables, $\mathrm{P}(\alpha_1=1)=1-\mathrm{P}(\alpha_1=0)=\lambda$. With the help of (3.11) it can be proved, similarly to (2.2), that

$$\sum_{l=1}^{k} d_l \sum_{j \in \mathbf{Z}_d} \varphi^{(l)}\left(\frac{j}{r}\right)[\xi_{N_l}(j)-\mathrm{E}\xi_{N_l}(j)] \triangleq \sum_i \Phi(B_i)(\alpha_i-\lambda) \tag{3.12}$$

if the partitions $\{B_i\}$ of $\bar{Q}$ are $\mathscr{P}(N_1, \ldots, N_k)$ distributed, and they are independent of the sequence $\alpha_1, \alpha_2, \ldots$.

Let us choose some $0<t_1<\ldots<t_k$, and define $N_l=[t_l N]$, $l=1, 2, \ldots, k$. Put

$$R_{N,l,\bar{l}}(j_1, j_2) = \mathrm{P}\big(X_{N_k}(j_1, l) = X_{N_k}(j_2, \bar{l})\big), \quad j_1, j_2 \in \mathbf{Z}_d \quad l, \bar{l} = 1, 2, \ldots, k.$$

We are going to show that for all $\varepsilon>0$.

$$R_{N,l,\bar{l}}(j_1, j_2) = \tag{3.13}$$
$$= \begin{cases} C \displaystyle\int_{(N_{\bar{l}}-N_l)m}^{(N_{\bar{l}}+N_l)m} \exp\left(-\frac{|j_1-j_2|^2}{2u}\right) u^{-d/2}\,du + o(N^{-(d-2)/2}) & \text{for } j_1-j_2 \in \mathscr{G} \\ 0 \quad \text{for } j_1-j_2 \notin \mathscr{G} \end{cases}$$

if $\bar{l} \geqq l$, $|j_1-j_2| > \varepsilon N^{1/2}$, where $o(\cdot)$ is uniform in $|j_1-j_2|$ and C is the same constant as in Lemma 2. We shall also see that

$$R_{N,l,\bar{l}}(j_1, j_2) \leqq C_1 N^{-(d-2)/2} \quad \text{if} \quad l \neq \bar{l}, \text{ for all } j_1, j_2 \in \mathbf{Z}_d, \tag{3.14}$$

where the constant C_1 depends only on $t_1, t_2, \ldots, t_k$. First we prove with the help of (3.13) and (3.14) the following

THEOREM 3. *Let* $r^2 \sim N$, $N_l=[t_l N]$, $\varphi^{(l)} \in \mathscr{S}$, $l=1, 2, \ldots, k$, $0<t_1<\ldots<t_k$. *Then the joint distribution of* $F_{\lambda, N_l, r}(\varphi^{(l)})$, $l=1, 2, \ldots, k$, *tends, with the choice* $h(N, r) = r^{-(d+2)/2}$, *as* $N \to \infty$ *to the distribution of a vector valued normal random variable with expectation zero and covariance* $\mathrm{E}Y_l Y_{\bar{l}} = \bar{C}_\lambda B_{t_l, t_{\bar{l}}}(\varphi^{(l)}, \varphi^{(\bar{l})})$ $l, \bar{l}=1, 2, \ldots, k$, *where*

$$B_{s,t}(\varphi, \psi) = \int_{R_d} \int_{R_d} \varphi(x)\psi(y) G(t-s, t+s, |x-y|)\,dx\,dy \tag{3.15}$$

if $0 \leqq s \leqq t$, *with*

$$G(s, t, z) = \int_{ms}^{mt} \exp\left(-\frac{z^2}{2u}\right) u^{-d/2}\,du.$$

PROOF of Theorem 3. It is enough to show that

$$\sum_{l=1}^{k} \bar{d}_l F_{\lambda, N_l, r}(\varphi^{(l)}) \xrightarrow{\mathscr{D}} \sum_{l=1}^{k} \bar{d}_l Y_l$$

for arbitrary coefficients $\bar{d}_1, \dots, \bar{d}_l$, where $\xrightarrow{\mathcal{D}}$ denote convergence in distribution. First we restrict ourselves to the case $\varphi^{(l)} \in \mathcal{D}$, $l=1, 2, \dots, k$. Observe that

$$\operatorname{Cov}(\xi_{N_l}(j_1), \xi_{N_{\bar{l}}}(j_2)) = \lambda(1-\lambda) R_{N,l,\bar{l}}(j_1, j_2)$$

because of (3.11). Hence

$$\mathsf{E}\left[\sum_{l=1}^{k} \bar{d}_l F_{\lambda, N_l, r}(\varphi^{(l)})\right]^2 =$$

$$= \lambda(1-\lambda) r^{-(d+2)} \sum_{l,\bar{l}=1}^{k} \bar{d}_l \bar{d}_{\bar{l}} \sum_{j_1, j_2 \in \mathbf{Z}_d} R_{N,l,\bar{l}}(j_1, j_2) \varphi^{(l)}\left(\frac{j_1}{r}\right) \varphi^{(\bar{l})}\left(\frac{j_2}{r}\right).$$

On the other hand

$$\sum_{j_1, j_2} \lambda(1-\lambda) r^{-(d+2)} R_{N,l,\bar{l}}(j_1, j_2) \varphi^{(l)}\left(\frac{j_1}{r}\right) \varphi^{(\bar{l})}\left(\frac{j_2}{r}\right) =$$

$$= \sum_{|j_1 - j_2| > \varepsilon N^{1/2}} + \sum_{|j_1 - j_2| \leq \varepsilon N^{1/2}} \sim B_{t_l, t_{\bar{l}}}(\varphi^{(l)}, \varphi^{(\bar{l})}).$$

Indeed, the first sum in the middle term is asymptotically $B_{t_l, t_{\bar{l}}}(\varphi^{(l)}, \varphi^{(\bar{l})})$ if ε is sufficiently small because of (3.13), (the effect of the error term $o(N^{-d-2/2})$ is negligible since $\varphi^{(l)}, \varphi^{(\bar{l})} \in \mathcal{D}$) and the second sum is negligible, as it can be seen from (3.13) if $l \neq \bar{l}$ and from (2.15) and (3.1) if $l = \bar{l}$. These relations imply that

$$\lim_{N \to \infty} \mathsf{E}\left(\sum_{l=1}^{k} \bar{d}_l F_{\lambda, N_l, r}(\varphi^{(l)})\right)^2 = \sum_{l,\bar{l}=1}^{k} \bar{d}_l \bar{d}_{l'} B_{t_l, t_{\bar{l}}}(\varphi^{(l)}, \varphi^{(\bar{l})}) = \mathsf{E}(\Sigma d_l Y_l)^2.$$

Formula (3.12) enables us to prove that the limit distribution is normal. We apply the same conditioning argument as it was done in Theorems 1 and 2. Introduce the notation $d_l = d_l(r) = \bar{d}_l r^{-(d+2)/2}$, $l=1, 2, \dots, k$, and

$$\Phi_l(B) = d_l \sum_{j \in B(l)} \varphi^{(l)}\left(\frac{j}{r}\right), \quad B \subset \bar{Q}.$$

Then

$$\Phi(B) = \sum_{l=1}^{k} \Phi_l(B). \tag{3.16}$$

To carry out the conditioning arguments of Theorem 1 and 2 it is enough to show that

$$I_1 = \mathsf{E}[\Sigma \varphi(B_i)^4] \to 0$$

$$I_2 = \mathsf{E}\left[[\Sigma \varphi(B_i)^2 - \mathsf{E}\, \Sigma \varphi(B_i)^2]^2\right] \to 0 \quad \text{as} \quad r \to \infty$$

if the partitions of Q have $\mathcal{P}(N_1, \dots, N_k)$ distribution. Because of (3.16)

$$I_1 \leq k^3 \sum_{l=1}^{k} E[\Sigma \varphi_1(B_i)^4]. \tag{3.17}$$

To estimate I_2 we introduce the following notations: Let $A_N(j_1, \dots, j_p, l_1, \dots, l_p)$; $j_s \in \mathbf{Z}_d$, $l_s \in \{1, 2, \dots, k\}$, $s=1, 2, \dots, p$; denote the event that $X_{N_k}(j_1, l_1) =$

$=X_{N_k}(j_2, l_2)=\ldots=X_{N_k}(j_p, l_p)$, and let

$$R_N(j_1, \ldots, j_p, l_1, \ldots, l_p) = \mathsf{P}(A_N(j_1, \ldots, j_p, l_1, \ldots, l_p)).$$

It can be shown similarly to (2.5) that

$$I_2 = \sum_{\substack{l_i=1\\ i=1,2,3,4}}^{k} \sum_{\substack{j_{i'}\in Z_d\\ i'=1,2,3,4}} [\mathsf{P}(A_N(j_1, j_2, l_1, l_2)\cap A_N(j_3, j_4, l_3, l_4))-$$
$$-R_N(j_s, l_s;\ s=1,2,3,4)] \prod_{m=1}^{4} d_{l_m} \varphi^{(l_m)}\left(\frac{j_m}{r}\right).$$

An argument similar to the proof of the formulae (2.7) and (2.8) yields that

$$\mathsf{P}(A_N(j_1, j_2, l_1, l_2)\cap A_N(j_3, j_4, l_3, l_4))-R_N(j_s, l_s;\ s=1,2,3,4) \leqq$$
$$\leqq R_N(j_1, j_2, l_1, l_2)\, R_N(j_3, j_4, l_3, l_4).$$

Hence

$$(3.18)\qquad I_2 \leqq \Sigma R_N(j_s, l_s;\ s=1,2,3,4) \prod_{i=1}^{4} |d_i| \left|\varphi^{(l_i)}\left(\frac{j_i}{r}\right)\right| = \mathsf{E}[\Sigma|\Phi|(B_i)^4] = I_2'$$

where $|\Phi|(B)$ is defined by substituting φ_l by $|\varphi_l|$ and d_l by $|d_l|$ in the definition of $\Phi(B)$. The partitions of $Q_l(Ar)$ induced by the $\mathscr{P}(N_1, \ldots, N_k)$ distributed partitions of $\bar{Q}$ have $\mathscr{P}(N_l)$ distribution. Hence, by expressing $\mathsf{E}\Sigma\Phi_l(B_i)^4$ by means of $R_{N_l}(j_1, j_2, j_3, j_4)$ we get that $I_1\to 0$, $I_2\to 0$ because of the relations (3.17), (3.18), (3.1) and Lemma 1. The reduction of the case $\varphi^{(l)}\in\mathscr{S}$ to the case $\varphi^{(l)}\in\mathscr{D}$ can be done similarly to Theorem 1.

Now we turn to the proof of relations (3.13) and (3.14). Consider a random walk Y_n which starts from the origin and which has transition probabilities $\mathsf{p}(i)$ for $0\leqq n<N_{\bar{l}}-N_l$, and $\mathsf{q}(i)$ for $n\geqq N_{\bar{l}}-N_l$. Clearly

$$(3.19)\qquad R_{N,l,\bar{l}}(j_1, j_2) = \mathsf{P}(Y_n = j_1-j_2 \text{ for some } N_{\bar{l}}-N_l \leqq n \leqq N_{\bar{l}}).$$

It follows from standard results about concentration functions, cf. [3] Theorem 6.2, that

$$(3.20)\qquad \mathsf{P}(Y_n = x) = O(n^{-d/2}),$$

where $O(\cdot)$ is uniform in x. Relations (3.19) and (3.20) imply (3.14).

To prove (3.13) we introduce the quantities

$$G_{M,n}(x) = \sum_{r=M}^{n} \mathsf{P}(Y_r = x) \quad\text{and}\quad \tilde{G}(0) = \sum_{r=0}^{\infty} \mathsf{P}(Y_{r+M}-Y_M = 0),$$

where $M=N_{\bar{l}}-N_l$. We may assume that $t_{\bar{l}}>t_l$, since the case $t_l=t_{\bar{l}}$ was solved in Lemma 2. It can be seen similarly to the proof of (3.2) that because of (3.19)

$$(3.21)\qquad \frac{G_{N,N_{\bar{l}}}(j_1-j_2)}{\tilde{G}(0)} \leqq R_{N,l,\bar{l}}(j_1, j_2) \leqq \frac{G_{M,N_{\bar{l}}+k}(j_1-j_2)}{\tilde{G}(0)-\delta},$$

where k is chosen in such a way that $\sum_{r=0}^{k} \mathsf{P}(Y_{r+M}-Y_M=0)>\tilde{G}(0)-\delta$. Because of (3.21) it is enough to give a good asymptotics for $G_{M,N}(x)$ in order to prove (3.13).

If the random walks with transition probabilities p and q are strongly aperiodic then the local central limit theorem holds for $Y_{r+M}-Y_M$ with an error term $o(N^{-d/2})$. If the local central limit theorem fails to hold for $Y_{r+M}-Y_M$ then we make a reduction similar to the proof of Lemma 2.

Let us introduce the auxiliary random walks $(Y^\alpha)=(Y_n^\alpha)$ such that $Y_n^\alpha=Y_n$ for $n\leqq M$ and Y_n^α has transition probabilities

$$q^\alpha(x, y) = \alpha q(y-x)+(1-\alpha)\delta(x, y)$$

if $n>M$. Set

$$G_{M,n}^\alpha(x) = \sum_{r=M}^{n} \mathrm{P}(Y_r^\alpha = x).$$

Since (3.4) holds for all $x\in \mathbf{Z}_d$,

$$G_{M,n}^\alpha(x) = \sum_u \mathrm{P}(Y = u)\left[\sum_{r=M}^{n} \mathrm{P}(Y_r^\alpha - Y_M^\alpha = x-u)\right],$$

and a similar relation holds also for Y_n the relation

$$\alpha G_{M,p}^\alpha(x) \leqq G_{M,p}(x) \leqq \frac{N}{N-1} G_{M,M+\bar\alpha(p-M)}^\alpha(x) \tag{3.22}$$

holds true for all $p\geqq M$ and $x\in\mathbf{Z}_d$ with $\bar\alpha=(1+(1-\alpha)^{1/2})/\alpha$. For all $\alpha>0$ and $\delta>0$ the local central limit theorem holds for $\mathrm{P}(Y_{M+n}^\alpha-Y_M^\alpha=x)$ with an error term $o(N^{-d/2})$ if $n>\delta N$, where $o(\cdot)$ may depend on α and δ but not on x. (We must be a little careful, because we have not a local central limit theorem for $Y_{M+n}^\alpha-Y_M^\alpha$ with a good error term if n is relatively small. Relation (3.20) helps us to overcome the difficulties arising from this fact.)

Because of (3.20)

$$G_{M,p'}(x)-G_{M,p}(x) \leqq K(p'-p)N^{-(d-2)/2} \tag{3.23}$$

with some $K>0$ for all $p'>p\geqq M$ and $x\in\mathbf{Z}_d$. Because of (3.22)

$$G_{M,p'}(x)-G_{M,p}(x) \leqq 2K(p'-p)N^{-(d-2)/2} \tag{3.23'}$$

for all $p'>p\geqq M$ and $x\in\mathbf{Z}_d$ if $\alpha>1/2$. Relations (3.22), (3.23) and (3.23′) imply that given any $\varepsilon>0$ a $\delta=\delta(\varepsilon)>0$ and an $\eta=\eta(\delta,\varepsilon)>0$ can be chosen in such a way that

$$|G_{M,N_l}(x)-G_{M+[\delta N],N_l}^\alpha(x)| \leqq \varepsilon\big(N^{-(d-2)/2}+G_{M+[\delta N],N_l}^\alpha(x)\big) \tag{3.24}$$

if $0<1-\alpha<\eta$ and

$$G_{M+[\delta N],N_l}^\alpha = G_{M,N_l}-G_{M,M+[\delta N]}.$$

The variable Y_n^α has almost the same variance as Y_n if $1-\alpha$ is sufficiently small,

$$G_{M+[\delta N],N_l}^\alpha(x) = \sum_{n=M+[\delta N]}^{N_l} \mathrm{P}(Y_n^\alpha = x),$$

and a normal approximation can be made in the last sum. Hence relations (3.24) and (3.21) imply (3.13).

Now we turn to the investigation of the limiting behaviour of the vector $F_{\lambda,N_l,r}(\varphi^{(l)})$, $l=1, 2, \ldots, k$, in the case when the initial distribution of ξ_0 in the voter model has the equilibrium state μ_λ. Let us consider a voter model starting with the initial distribution defined in (1.1). Fix some integers $0 \leqq N_1 < \ldots < N_k$. Observe that the joint distributions of the fields $\xi_{N+N_1}, \ldots, \xi_{N+N_k}$ tend to that of the fields $\bar{\xi}_{N_1}, \ldots, \bar{\xi}_{N_k}$ as $N \to \infty$, where $(\bar{\xi}_n)$ denotes a voter model with initial distribution μ_λ and with the same transition probabilities as (ξ_n). Given an $A>0$ and $r>0$ we define the set $\bar{Q}$, and the distribution $\mathscr{P}(N_1, \ldots, N_k)$ over the partitions of $\bar{Q}$ as before. We introduce the notation $\mathscr{P}^N(N_1, \ldots, N_k) = \mathscr{P}(N_1+N, \ldots, N_k+N)$. Let us also define the following random partition of $\bar{Q}: j \in Q_l(Ar)$ and $j' \in Q_{l'}(Ar)$ belong to the same element of the partition if and only if the random walks $X(j, l)$ and $X(j', l')$ hit each other. Let $\mathscr{P}^\infty(N_1, \ldots, N_k)$ denote the distribution of these partitions. It is not difficult to see that $\mathscr{P}^N(N_1, \ldots, N_k) \to \mathscr{P}^\infty(N_1, \ldots, N_k)$ as $N \to \infty$. These relations imply that formula (3.12) remains valid if the partitions $\{B_i\}$ of $\bar{Q}$ are $\mathscr{P}^\infty(N_1, \ldots, N_k)$ distributed, and ξ_0 is μ_λ distributed. Introduce the notation

$$R^\infty_{N,l,\bar{l}}(j_1, j_2) = \mathrm{P}(X_n(j_1, l) = X_n(j_2, \bar{l}) \text{ for some } n).$$

Fix some real numbers $0 \leqq t_1 < t_2 < \ldots < t_k$ and let $N_l = [t_l N]$, $l=1, 2, \ldots, k$. The following relation, analogue to (3.13), holds true. For all $\varepsilon > 0$, $\bar{l} \geqq l$

$$R_{N,l,\bar{l}}(j_1, j_2) = \begin{cases} C \displaystyle\int_{(N_{\bar{l}}-N_l)m}^{\infty} \exp \frac{|j_1-j_2|^2}{2u} u^{-d/2}\, du + o(N^{-d-2/2}) \\ \qquad \text{if } j_1-j_2 \in \mathscr{G}, \quad j_1-j_2 > \varepsilon N^{1/2} \\ 0 \qquad \text{if } j_1-j_2 \notin \mathscr{G} \end{cases}$$

where $o(\cdot)$ is uniform in $|j_1-j_2|$. Now a straightforward modification of the proof of Theorem 3 leads to the following

THEOREM 4. *Let (ξ_n) be a voter model with initial distribution μ_λ and with the same transition probabilities as before. Let $0 \leqq t_1 < \ldots < t_k$, $r^2 \sim N$, $N_l = [t_l N]$, $\varphi^{(l)} \in \mathscr{S}$, $l=1, 2, \ldots, k$, and $h(N, r) = r^{-(d+2)/2}$. Then the joint distributions of $F_{\lambda,N_l,r}(\varphi^{(l)})$, $l=1, 2, \ldots, k$, tend to a normal law with expectation zero and covariance $\bar{B}_{t_l, t_{\bar{l}}}(\varphi^{(l)}, \varphi^{(\bar{l})})$, $l, \bar{l} = 1, 2, \ldots, k$, where*

$$\bar{B}_{s,t}(\varphi, \psi) = \bar{C}_\lambda \int_{R_d} \int_{R_d} \varphi(x)\psi(y) G(t-s, \infty, |x-y|)\, dx\, dy$$

if $s \leqq t$ and $\varphi, \psi \in \mathscr{S}$.

REMARK 1. Given a set of random variables $X(t, \varphi)$, $\varphi \in \mathscr{S}$, $t \in T$, $T=[0, \infty]$ or $T=(-\infty, \infty)$, it is called a generalized field valued process if $X(t, \varphi)$ is a generalized field for all fixed t. Define the σ-algebras $\mathscr{F}_t = \mathscr{F}(X(u, \varphi), u \geqq t, \varphi \in \mathscr{S})$. A generalized field valued process is called Gaussian if the random vectors $(X(t_1, \varphi_1), \ldots, X(t_k, \varphi_k))$ are Gaussian for all $\varphi_l \in \mathscr{S}$, $t_l \in T$, $l=1, 2, \ldots, k$, $k=1, 2, \ldots$. It is called Markovian if

$$\mathrm{P}(X(t, \varphi) < x | \mathscr{F}_s) = \mathrm{P}(X(t, \varphi) < x | X(s, \psi), \psi \in \mathscr{S})$$

for all $x\in R_1, s\leqq t$, and $\varphi\in\mathcal{S}$. It is called stationary if the joint distribution of $X(t_1+u, \varphi_1), \dots, X(t_k+u, \varphi_k)$ agrees with that of $X(t_1, \varphi_1), \dots, X(t_k, \varphi_k)$ for all $u\in T$, $t_l\in T$, $\varphi_l\in\mathcal{S}$, $l=1, 2, \dots, k$, $k=1, 2, \dots$. A stationary Gaussian and Markovian generalized field valued process is called a generalized Ornstein-Uhlenbeck process.

In Theorem 3 we have got a Gaussian generalized field valued process $X(t, \varphi)$ with expectation zero and covariance function

$$\mathsf{E}X(s, \varphi)X(t, \psi) = B_{s,t}(\varphi, \psi)$$

as the limit. We show that it is Markovian. Because of the Gaussian property it is sufficient to show that given any $\varphi\in\mathcal{S}$ and real numbers $0\leqq s\leqq t$, there exists a $\hat{\varphi}_{s,t}\in\mathcal{S}$ such that

$$\mathsf{E}X(t, \varphi)X(u, \psi) = \mathsf{E}X(s, \hat{\varphi}_{s,t})X(u, \psi) \tag{3.25}$$

for all $0\leqq u\leqq s$ and $\psi\in\mathcal{S}$. Relation (3.25) holds if the function $\hat{\varphi}_{s,t}$ is defined as

$$\hat{\varphi}_{s,t}(x) = \hat{\varphi}_{t-s}(x) = \int_{R_d} [2\pi m(t-s)]^{-d/2} \exp\left(-\frac{|x-z|^2}{2m(t-s)}\right)\varphi(z)\,dz.$$

It is enough to check (3.25) in the special case $m=1$. Elementary calculation shows that

$$\int_{R_d} [2\pi(t-s)]^{-d/2} \exp\left(-\frac{|x-z|^2}{2(t-s)}\right)\left[\int_{s-u}^{s+u} v^{-d/2} \exp\left(-\frac{|x-y|^2}{2v}\right)dv\right]dx =$$

$$= \int_{t-u}^{t+u} v^{-d/2} \exp\left(-\frac{|y-z|^2}{2v}\right)dv$$

for all $0<u\leqq s<t$ and $y, z\in R_d$. Expressing the right-hand side of (3.25) as a multiple integral, and changing the order of integration we can get (3.25) with the help of the last identity.

Let us define the generalized field valued process

$$\bar{X}(t, \varphi) = \exp\left(-\frac{\alpha+2}{2}t\right)X\left(\frac{1}{m}e^{2t}, \varphi_t\right), \quad -\infty<t<\infty, \quad \varphi\in\mathcal{S},$$

where $\varphi_t(x)=\varphi(xe^{-t})$. Clearly, $\bar{X}(t, \varphi)$ is also a Gaussian and Markovian generalized field valued process. Moreover, some calculation shows that

$$\mathsf{E}\bar{X}(s, \varphi)\bar{X}(t, \psi) =$$

$$= \exp\left(-\frac{d+2}{2}(t-s)\right)\int_{R_d}\int_{R_d} \varphi(xe^{t-s})\psi(y)\left[\int_{1-e^{2(s-t)}}^{1+e^{2(s-t)}} \exp\left(-\frac{|x-y|^2}{2u}\right)u^{-d/2}\,du\right]dx\,dy$$

if $s\leqq t$, hence $\bar{X}(t, \varphi)$ is stationary, i.e. $\bar{X}(t, \varphi)$ is a generalized Ornstein—Uhlenbeck process.

It can be seen similarly that the limiting distributions appearing in Theorem 4 determine also a generalized Ornstein-Uhlenbeck process. The function $\hat{\varphi}_{s,t}$ can be calculated in the same way as in the former case.

Remark 1 was motivated by some results of HOLLEY and STROOCK [6]. They considered the continuous time version of the voter model, and proved a result analogous to Theorem 3. They proved directly that the limiting field is a generalized Ornstein—Uhlenbeck process. Our method works also in the continuous time case.

REMARK 2. We could have chosen the test functions φ also as indicator functions of unit cubes or as finite linear combination of such functions. Such a choice of the test functions leads to the discrete renormalization version of our results.

REMARK 3. The condition that the distribution p is concentrated on a finite set can be weakened. All the theorems of this paper remain valid if we assume only that p has a covariance matrix. Theorems 1 and 2 remain valid also if condition (ii) is violated. On the other hand the random variables $F_{\lambda, N_l, r}(\varphi^{(l)})$ in Theorems 3 and 4 tend in this case to independent normal random variables for different l. If condition (iii) is violated a non-isotropic limit appears.

REMARK 4. For $d=1$ or 2 the limit distribution μ_λ is a linear combination of μ_0 and μ_1; $\mu_0(\xi(i)=0)=1$, $\mu_1(\xi(i)=1)=1$ for all $i\in \mathbf{Z}_d$. Hence in this case there is no interesting counterpart of Theorem 1. On the other hand it can be asked whether some non-trivial result can be obtained for space-time renormalization if $r=r(T)$ is appropriately chosen. The most interesting case is when $r^2\sim\alpha T$, $\alpha>0$. If we want to investigate this question with the method of the present paper then the most important step we have to do is to give an asymptotic for $R_N(j_1, j_2)$.

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