

# Non-central limit theorem for non-linear functionals of vector valued Gaussian stationary random fields

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## Abstract

Here I prove non-central limit theorems for non-linear functionals of vector valued stationary random fields under appropriate conditions. They are the multivariate versions of the results in paper [7]. Previously A. M. Arcones formulated such a result in Theorem 6 of his paper [1]. But there are serious problems with his result. I explain these problems in the main text. In this paper I present the right formulation of the multivariate version of the non-central limit theorem in paper [7] together with its correct proof. To do this first the theory of the Gaussian stationary random fields described in the work [10] had to be generalized to the case of vector valued random fields. This was done in my work published in two subsequent papers [11] and [12]. Here I prove the multivariate version of the result about non-central limit theorems in paper [7] with their help.

*Keywords:* Non-central limit theorem, spectral measure of vector valued stationary random fields, random spectral measure in the multidimensional case, multiple Wiener–Itô integral, generalized vector valued stationary random fields, vague convergence.

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## 1. On the motivation for this research.

In this paper the following problem is considered.

Let us have a  $d$ -dimensional vector-valued Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , where  $\mathbb{Z}^\nu$  denotes the lattice points

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with integer coordinates in the  $\nu$ -dimensional Euclidean space  $\mathbb{R}^\nu$  and a function  $H(x_1, \dots, x_d)$  of  $d$  variables with arguments  $x_s \in \mathbb{R}^\nu$ ,  $1 \leq s \leq d$ . We define with their help the random variables  $Y(p) = H(X_1(p), \dots, X_d(p))$  for all  $p \in \mathbb{Z}^\nu$ . Let us introduce for all  $N = 1, 2, \dots$  the normalized sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p)$$

with an appropriate norming constant  $A_N > 0$ , where

$$B_N = \{p = (p^{(1)}, \dots, p^{(\nu)}) \in \mathbb{Z}^\nu: 0 < p^{(j)} \leq N \text{ for all } 1 \leq j \leq \nu\}. \quad (1.1)$$

(In this paper I shall denote the coordinates of a  $\nu$ -dimensional vector by superscript, because the subscript will be used for the indices of the elements in a set of  $\nu$ -dimensional vectors.) In this paper a non-Gaussian limit theorem is proved for these normalized sums  $S_N$  with an appropriate norming constant  $A_N$  if this vector-valued Gaussian stationary random field  $X(p)$ ,  $p \in \mathbb{Z}^\nu$ , and the function  $H(x_1, \dots, x_d)$  satisfy certain conditions. Paper [7] contains such limit theorems for non-linear functionals of scalar-valued stationary Gaussian random fields, and here their natural multivariate generalizations are presented.

A. M. Arcones formulated such a result in Theorem 6 of paper [1], but I found his discussion unsatisfactory. Here I briefly explain the main problems related to his proof.

The main result of Arcones about non-central limit theorems for non-linear functionals of vector-valued stationary random sequences is contained in Theorem 6 of [1] and in its extension formulated in Remark 7. Arcones presented the limit in this theorem as the distribution of a random variable defined in formula (3.9) of this paper. My main problems are related to this formula. It contains Wiener–Itô integrals whose definitions are missing. Such integrals were defined previously with respect to the random spectral measure of a scalar-valued stationary random field. So to formulate this result the Wiener–Itô integrals with respect to a vector-valued random spectral measure must be defined. Moreover, first the definition of the vector-valued spectral measures must be introduced.

There are also some other problems with the formulation of this result. One of them is that the Wiener–Itô integrals in formula (3.9) are taken in the cube  $[-\pi, \pi]^\nu$ , and not on the whole space  $\mathbb{R}^\nu$ . This is a more serious error, than it may seem at first sight. In a good proof of Theorem 6 it must

be clarified why the condition  $1 \leq r < \frac{1}{\alpha}$  had to be imposed in it. The reason for this condition is that the random variable defining the limit distribution in formula (3.9) must be meaningful. They are meaningful only if the Wiener–Itô integrals appearing in it exist, and this implies the condition that the square of the absolute value of the kernel functions in these Wiener–Itô integrals must be integrable. But the convergence or divergence of these integrals depends on the behavior of these kernel-functions in the neighbourhood of infinity. Thus the restriction of the integral to  $[-\pi, \pi]^\nu$  in formula (3.9) throws out the necessity of condition  $1 \leq r \leq \frac{1}{\alpha}$  from Theorem 6.

In short, in my opinion the proof of the multivariate generalization of the result in [7] must be started from the very beginning. First the basic results about the behavior of vector-valued stationary random fields must be worked out. This is missing from Arcones’ paper.

Let me remark that although Arcones’ proof of the non-central limit theorem for non-linear functionals of vector-valued Gaussian random fields was problematic, the proof of its counterpart about the central limit theorem for such linear functionals under appropriate conditions was correct. Moreover, in the study of this result he proved an important estimate in Lemma 1 of his paper which was applied also in this work.

This paper consists of five sections and two appendices. In Section 2 the basic notions and results of papers [11] and [12] are recalled. Section 3 contains the main results of this paper. In Section 4 the preparatory lemmas needed in the proof of the basic theorems are presented. Section 5 contains the proof of these results. In Appendix A the background of the limit theorems of this paper is discussed. In Appendix B I prove that not only the finite dimensional distributions of the stochastic processes considered in Theorem 3.4 converge, but their distribution also weakly converge to their limit in the Skorochod space on  $[0, 1]^\nu$ .

*Remark:* It was professor Herold Dehling who asked me to clarify the proof of Theorem 6 in Arcones’ paper [1]. The goal of this work together with the preliminary papers [11] and [12] was to answer Dehling’s question. It turned out that to settle this problem first the theory of vector-valued stationary Gaussian random fields has to be worked out. This theory is similar to the theory of scalar-valued Gaussian random fields, but there are also some essential differences between them. Hence the theory of vector-valued stationary Gaussian random fields cannot be considered as a simple generalization of the theory in the scalar-valued case. I am grateful to professor Dehling for

calling my attention to this problem.

## 2. On some properties of vector-valued Gaussian stationary random fields.

In this section I present the most important results of [11] and [12] needed in this paper. At this point I do not give their detailed formulation. I shall present them in a more detailed form when they appear in our investigation.

We are working with a  $d$ -dimensional vector-valued Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , where  $\mathbb{Z}^\nu$  denotes the lattice points with integer coordinates in the  $\nu$ -dimensional Euclidean space  $\mathbb{R}^\nu$  with expectation  $EX_j(0) = 0$  for all  $1 \leq j \leq d$ . The distribution of such random fields is determined by their covariance function  $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = EX_j(m)X_{j'}(m+p)$ ,  $1 \leq j, j' \leq d$ ,  $m, p \in \mathbb{Z}^\nu$ .

In a result of [11] it was shown that this covariance function  $r_{j,j'}(p)$ ,  $1 \leq j, j' \leq d$ , can be presented in the following way. For all  $1 \leq j, j' \leq d$  there is a complex measure  $G_{j,j'}$  on the torus  $[-\pi, \pi)^\nu$  with finite total variation such that  $r_{j,j'}(p) = \int e^{i(p,x)} G_{j,j'}(dx)$  for all  $p \in \mathbb{Z}^\nu$ , and  $G = (G_{j,j'})_{1 \leq j, j' \leq d}$  is an even, positive semidefinite matrix valued measure on the torus  $[-\pi, \pi)^\nu$ .  $G$  is called the spectral measure of the random field  $X(p)$ ,  $p \in \mathbb{Z}^\nu$ . (A  $d$ -dimensional matrix valued measure  $G = (G_{j,j'})_{1 \leq j, j' \leq d}$ , is called even if  $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$  for all  $1 \leq j, j' \leq d$  and measurable sets  $A$ .) For a more detailed discussion see Section 2 in [11].

In Section 3 of [11] I also defined a  $d$ -dimensional vector-valued random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to a  $d$ -dimensional matrix valued spectral measure  $G$  together with a random integral with respect to it in such a way that the random integrals  $X_j(p) = \int e^{i(p,x)} Z_{G,j}(dx)$ ,  $p \in \mathbb{Z}^\nu$ ,  $1 \leq j \leq d$ , define a  $d$ -dimensional Gaussian stationary random field with matrix valued spectral measure  $G$ . Besides, I gave the basic properties of a random spectral measure  $Z_G$  corresponding to a spectral measure  $G$ . These properties determine the distribution of the random spectral measure as a function of the spectral measure to which it corresponds.

Once these results are proved it is not difficult to generalize them to the case of vector-valued Gaussian stationary random fields defined on the lattice  $\frac{1}{K}\mathbb{Z}^\nu$  with some  $K > 0$ . We define the covariance function  $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = EX_j(m)X_{j'}(p+m)$ ,  $1 \leq j, j' \leq d$ ,  $p, m \in \frac{1}{K}\mathbb{Z}^\nu$  also in this case. There exists a spectral measure  $G = (G_{j,j'})_{1 \leq j, j' \leq d}$ , defined on the torus  $[-K\pi, K\pi)^\nu$  which is a  $d$ -dimensional matrix valued even measure,

and satisfies the identity  $r_{j,j'}(p) = \int e^{i(p,x)} G_{j,j'}(dx)$  for all  $p \in \frac{1}{K}\mathbb{Z}^\nu$  and  $1 \leq j, j' \leq d$ . There is also a vector-valued random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to this spectral measure  $G$  such that  $X_j(p) = \int e^{i(p,x)} Z_{G,j}(dx)$ ,  $1 \leq j \leq d$ ,  $p \in \frac{1}{K}\mathbb{Z}^\nu$ , is a vector-valued Gaussian stationary random field on  $\frac{1}{K}\mathbb{Z}^\nu$  with expectation zero and spectral measure  $G$ .

It is useful also to consider vector-valued Gaussian stationary random fields defined in the space  $\mathbb{R}^\nu$ . It turned out that it is even more useful to work with vector-valued generalized Gaussian stationary random fields.

$$X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi)) \text{ with parameter set } \varphi \in \mathcal{S}$$

(instead of  $\mathbb{Z}^\nu$  or  $\mathbb{R}^\nu$ ), where  $\mathcal{S}$  denotes the class of real valued functions in the  $\nu$ -dimensional Schwartz space. The definitions applied in the theory of generalized random fields were explained in Section 4 of [11] together with the notions needed to understand them.

In [11] generalized vector-valued Gaussian stationary random fields were also constructed, and their properties were explained. Results similar to those of Sections 2 and 3 in [10] about ordinary vector-valued Gaussian stationary random fields were proved for them. Generalized vector-valued stationary Gaussian random fields were constructed with the help of their spectral measure which were also defined.

The spectral measure of a generalized vector-valued stationary random field  $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$ ,  $\varphi \in \mathcal{S}$ , has properties similar to that of an ordinary vector-valued stationary random field, but there are some important differences between them. It is a  $d \times d$  even, positive definite matrix valued function  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  with  $G(A) = (G_{j,j'}(A))_{1 \leq j,j' \leq d}$ ,  $A \subset \mathbb{R}^\nu$ , defined on the bounded, measurable subsets of the  $\nu$ -dimensional Euclidean space  $\mathbb{R}^\nu$  whose restriction to the measurable subsets of any finite cube  $[-K, K]^\nu$  is a matrix valued measure with coordinates that are complex measures with finite total variation. On the other hand,  $\sup_A |G_{j,j}(A)|$ , where supremum is taken for all bounded, measurable sets  $A$  need not be finite. Only the weaker condition

$$\int (1 + |x|)^{-r} G_{j,j}(dx) < \infty \text{ for all } 1 \leq j \leq d \text{ with some number } r > 0, \quad (2.1)$$

is imposed. This property is called moderate increase at infinity. (It is enough to impose this condition only for the diagonal terms  $G_{j,j}$ . Actually, they imply together with the positive definiteness property of the spectral

matrix  $G$  sufficiently good estimates for the non-diagonal terms  $G_{j,j'}$ ,  $j \neq j'$ , too. See e.g. formula (3.2) in [11].)

The definition and construction of spectral measures of vector-valued generalized Gaussian stationary random fields was done in the following way.

Let us consider an even, positive definite matrix valued function  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  where for all  $1 \leq j, j' \leq d$  such a set function  $G_{j,j'}(A)$  is defined on the bounded and measurable sets  $A \subset \mathbb{R}^\nu$  which has moderate increase at infinity, and its restriction to a finite cube  $[-K, K]^\nu$  is a complex measure with finite total variation. If there exists a generalized vector-valued Gaussian stationary random field  $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$ ,  $\varphi \in \mathcal{S}$ , with the additional property  $EX_j(\varphi) = 0$ ,  $1 \leq j \leq d$ , for all  $\varphi \in \mathcal{S}$  such that the identity

$$EX_j(\varphi)X_{j'}(\psi) = \int \tilde{\varphi}(x)\overline{\tilde{\psi}(x)}G_{j,j'}(dx), \quad 1 \leq j, j' \leq d, \quad \text{for all } \varphi, \psi \in \mathcal{S}$$

holds, where  $\sim$  denotes Fourier transform, and overline means complex conjugate, then this set of matrix valued functions  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  is called the spectral measure of this generalized random field  $X(\varphi)$ ,  $\varphi \in \mathcal{S}$ .

For any set of matrix valued functions  $G(A) = (G_{j,j'})_{1 \leq j,j' \leq d}(A)$  with the above properties there exists a generalized vector-valued Gaussian stationary random field  $X(\varphi)$ ,  $\varphi \in \mathcal{S}$ , with expectation zero whose covariance function  $EX_j(\varphi)X_{j'}(\psi)$ ,  $\varphi, \psi \in \mathcal{S}$ , satisfies the above conditions. This means that a matrix valued function with the above properties is the spectral measure of a generalized, vector-valued Gaussian stationary random field. Moreover, the distribution of this random field is determined by its spectral measure.

Given the spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  of a generalized random field, such a vector-valued random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  can be constructed for which  $X_j(\varphi) = \int \tilde{\varphi}(x)Z_{G,j}(dx)$ ,  $1 \leq j \leq d$ ,  $\varphi \in \mathcal{S}$ , is a generalized Gaussian stationary random field with spectral measure  $G$  and  $EX_j(\varphi) = 0$  for all  $1 \leq j \leq d$  and  $\varphi \in \mathcal{S}$ . We say that such a random spectral measure is adapted to the generalized spectral measure  $G$ . The basic properties of the random spectral measures adapted to a generalized spectral measure also were proved. Their distribution is determined by the spectral measure to which they are adapted.

The introduction of the random spectral measures corresponding to the spectral measures of generalized Gaussian stationary random fields turned out to be useful for us. This class of random spectral measures is much larger than the class of random spectral measures corresponding to the spectral

measure of a classical vector-valued Gaussian random field. The limit in the limit theorems of this paper could be expressed by means of a sum of multiple Wiener–Itô integrals with respect to such a random spectral measure.

In the subsequent part of the works [11] and [12] my goal was to give a good representation of those random variables with finite second moment which are measurable with respect to the  $\sigma$ -algebra generated by the random variables of the underlying vector-valued random field and to present a useful formula for their shift transforms. Such results turned out to be very useful in the study of the limit theorems I was interested in. A good representation can be given with the help of multiple Wiener–Itô integrals with respect to vector-valued random spectral measures introduced in Section 5 of [11].

To define multiple Wiener–Itô integrals I considered the matrix valued spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  of a  $d$ -dimensional Gaussian stationary random field, (ordinary or generalized one), and took a random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to it. In Section 5 of [11] I defined for all  $n \geq 1$  and sequences of integers  $j_1, \dots, j_n$  with the property  $1 \leq j_s \leq d$  for all  $1 \leq s \leq n$  a set  $K_{n,j_1,\dots,j_n} = K_{n,j_1,\dots,j_n}(G_{j_1,j_1}, \dots, G_{j_n,j_n})$  of complex number valued functions with arguments in  $\mathbb{R}^{n\nu}$ . (In the terminology of this paper  $K_{n,j_1,\dots,j_n}$  is a subset of the class of complex valued functions  $f(x_1, \dots, x_n)$  of  $n$  variables with arguments  $x_s \in \mathbb{R}^\nu$ ,  $1 \leq s \leq n$ ). I defined the  $n$ -fold Wiener–Itô integral

$$I_n(f|j_1, \dots, j_n) = \int f(x_1, \dots, x_n) Z_{G,j_1}(dx_1) \dots Z_{G,j_n}(dx_n)$$

for the functions  $f \in K_{n,j_1,\dots,j_n}$ . (The definition of the set of functions  $K_{n,j_1,\dots,j_n}$  is recalled in Section 4 of this paper before the formulation of Proposition 4A.) Then I proved the most important properties of these random integrals.

In Section 6 of paper [11] I proved a technical result, called the diagram formula about the expression of the product of two multiple Wiener–Itô integrals as a sum of multiple Wiener–Itô integrals.

These results were exploited in paper [12]. Here I recalled the notion of Wick polynomials which turned out to be a useful tool in our investigations. Wick polynomials are natural multivariate generalizations of Hermite polynomials. Their definition together with their most important properties was recalled from [10] in Section 2 of [12]. Section 2 of [12] also contains an important formula about the expression of Wick polynomials by means of multiple Wiener–Itô integrals and another important formula about the calculation of

the shift transforms of a random variable presented in the form of a sum of multiple Wiener–Itô integrals. This made possible to reformulate our limit problems to limit problems about sums of multiple Wiener–Itô integrals. A result in Section 3 of [12] was proved in order to investigate such problems. It plays an important role in the investigation of this paper, hence I recalled it in Proposition 4A of this paper.

### 3. Formulation of the main results.

In this section I present the main results of this paper. I shall compare both the formulation of the conditions and the proof of the results with those appearing in the study of the analogous results in the scalar-valued case. But I shall refer to [10] instead of [7] in this comparison, because in that work the proofs are worked out in more detail.

I shall work with such random fields for which  $EX_j(p) = 0$  for all  $1 \leq j \leq d$  and  $p \in \mathbb{Z}^\nu$ . Besides this property I shall impose two kinds of conditions in this paper. The first of them deals with the covariance function  $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ ,  $1 \leq j, j' \leq d$ ,  $p \in \mathbb{Z}^\nu$ , of the vector-valued Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ , considered in this paper, the second one with the function  $H(x_1, \dots, x_d)$  which appears in the definition of the random sums whose limit behavior is investigated.

The following condition is imposed about the covariance function  $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ .

$$\lim_{T \rightarrow \infty} \sup_{p: p \in \mathbb{Z}^\nu, |p| \geq T} \frac{|r_{j,j'}(p) - a_{j,j'}(\frac{p}{|p|})|p|^{-\alpha}L(|p|)|}{|p|^{-\alpha}L(|p|)} = 0 \quad (3.1)$$

for all  $1 \leq j, j' \leq d$ , where  $0 < \alpha < \nu$ ,  $L(t)$ ,  $t \geq 1$ , is slowly varying at infinity, bounded in all finite intervals, and  $a_{j,j'}(t)$  is a real valued continuous function on the unit sphere  $\mathcal{S}_{\nu-1} = \{x: x \in \mathbb{R}^\nu, |x| = 1\}$ , which satisfies the identity  $a_{j,j'}(x) = a_{j',j}(-x)$  for all  $x \in \mathcal{S}_{\nu-1}$  and  $1 \leq j, j' \leq d$ .

I construct a vector-valued Gaussian stationary random field which satisfies relation (3.1).

*Example for a stationary random field with covariance function satisfying relation (3.1).* I shall construct a stationary random field whose covariance function satisfies (3.1). I will do this by defining the spectral measure of such



a random field. To do this I recall some results about the Fourier transform of generalized functions from the literature.

In the  $\nu$ -dimensional space  $\mathbb{R}^\nu$  the Fourier transform of the homogeneous function  $|x|^\lambda$  (as the Fourier transform of a generalized function) is  $C|p|^{-\lambda-\nu}$  with some coefficient  $C = C(\lambda, \nu) > 0$ . (See the list of Fourier transforms at the end of the book [8].) The value of this coefficient  $C(\lambda, \nu)$  is known, but it has no importance for us.

On the other hand, if  $u(x)$ ,  $x \in \mathbb{R}^\nu$ , is a sufficiently smooth function, concentrated in a compact domain, and  $u(0) = 1$ , then the Fourier transform of  $|x|^\lambda u(x)$  equals  $\int e^{i(x,p)} |x|^\lambda u(x) dx = C(\lambda, \nu) |p|^{-\lambda-\nu} (1 + \delta(p))$  with  $\sup_{|p| > T} \delta(p) \rightarrow 0$  as  $T \rightarrow \infty$ .

In the following construction the above property of the Fourier transform of  $|x|^\lambda u(x)$  will be exploited. Define some functions  $g_{j,j'}(x)$ ,  $1 \leq j, j' \leq d$ ,  $x \in [-\pi, \pi]^\nu$ , in the following way. Take a non-negative, smooth function  $u(x)$  concentrated in the cube  $[-\pi, \pi]^\nu$  such that  $u(-x) = u(x)$ , and  $u(0) = 1$ . Put  $g_{j,j}(x) = |x|^{\alpha-\nu} u(x)$ , for  $1 \leq j \leq d$ , and  $g_{j,j'}(x) = \varepsilon_{j,j'} |x|^{\alpha-\nu} u(x)$  for  $1 \leq j, j' \leq d$  if  $j \neq j'$  with a sufficiently small real valued coefficient  $\varepsilon_{j,j'}$  such that  $\varepsilon_{j,j'} = \varepsilon_{j',j}$ . (One could choose a complex valued coefficient  $\varepsilon_{j,j'}$  too, but this would demand a more complicated argument.) I claim that  $(g_{j,j'}(x))_{1 \leq j, j' \leq d}$ ,  $x \in [-\pi, \pi]^\nu$ , with the above defined functions  $g_{j,j'}(\cdot)$  is a spectral density function, and the covariance function  $r_{j,j'}(p)$ ,  $1 \leq j, j' \leq d$ ,  $p \in \mathbb{Z}^\nu$ , of a stationary random field with this spectral density satisfies relation (3.1) with  $|p|^{-\alpha}$ ,  $L(p) = 1$ ,  $a_{j,j} \left( \frac{p}{|p|} \right) = C(\alpha - \nu, \nu)$ , and  $a_{j,j'} \left( \frac{p}{|p|} \right) = \varepsilon_{j,j'} C(\alpha - \nu, \nu)$  for  $j \neq j'$ .

Indeed, relation (3.1) holds with such a choice, because  $r_{j,j'}(p)$  is the Fourier transform of  $g_{j,j'}(x)$ . We still have to check that  $(g_{j,j'}(x))_{1 \leq j, j' \leq d}$  is a spectral density matrix. The main point is to show that this matrix is positive definite. This property holds, since this matrix has the form  $|x|^{\alpha-\nu} u(x) (I + D(\varepsilon))$  with a small matrix  $D(\varepsilon)$ , where  $I$  denotes the identity matrix.

*Remark:* There is a natural generalization of the results of the present paper. One may consider such vector-valued stationary Gaussian random fields, where the partial sums of different coordinates have a limit with different normalization. They satisfy limit theorems similar to those of the present paper, but the different behavior of the different coordinates must be taken into consideration in the choice of the normalization.

Such more general models were considered in the paper [15] of Sanchez de Naranjo, who considered models whose covariance matrices satisfy a generalized version of relation (3.1). Namely, they satisfy the relation

$$r_{j,j'}(p) \sim |p|^{\alpha_{j,j'}} a_{j,j'} \left( \frac{p}{|p|} \right) L_{j,j'}(|p|)$$

with such an exponent  $\alpha_{j,j'}$  and slowly varying function  $L_{j,j'}(\cdot)$  which may depend on the indices  $j$  and  $j'$ . With a good choice of these quantities an interesting generalization of the results of the present paper can be obtained. Such results can be proved by means of a natural generalization of the arguments of the present paper, but since this would demand a lot of space and the introduction of many new quantities I omit their discussion here.

Next I explain the condition imposed on the function  $H(x_1, \dots, x_d)$  that appears in the limit theorems of this paper. In the scalar-valued case such stationary random fields were considered for which  $EX_0^2 = 1$ , First the special choice  $H(x) = H_k(x)$  was considered, where  $H_k(x)$  denotes the  $k$ -th Hermite polynomial with leading coefficient 1. Then it was shown that our limit problem with a function  $H(x)$  whose expansion by the Hermite polynomials has the form  $H(x) = \sum_{l=k}^{\infty} c_l H_l(x)$  with starting index  $k$  in the summation can be simply reduced to the special case when  $H(x) = c_k H_k(x)$ . Similar results will be proved for vector-valued models. In this case such vector-valued Gaussian stationary random fields  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , will be considered whose covariance functions satisfy besides condition (3.1) also the relation

$$\begin{aligned} EX_j^2(0) &= 1 \text{ for all } 1 \leq j \leq d, \text{ and } EX_j(0)X_{j'}(0) = 0 \\ &\text{if } j \neq j', \quad 1 \leq j, j' \leq d. \end{aligned} \quad (3.2)$$

First I show that this new condition does not mean a real restriction of our problem.

Let  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , be a vector-valued Gaussian stationary random field with expectation  $EX_j(p) = 0$ ,  $p \in \mathbb{Z}^\nu$ ,  $1 \leq j \leq d$ , and take the random variables  $X_1(0), \dots, X_d(0)$  in it. An appropriate number  $1 \leq d' \leq d$  can be chosen, and  $d'$  random variables  $X'_j(0) = \sum_{l=1}^d c_{j,l} X_l(0)$ ,  $1 \leq j \leq d'$ , can be defined with appropriate coefficients  $c_{j,l}$ ,  $1 \leq j \leq d'$ ,  $1 \leq l \leq d$ , with the following properties.  $EX'_j(0)X'_{j'}(0) = \delta_{j,j'}$ ,  $1 \leq j, j' \leq d'$ , where  $\delta_{j,j'} = 0$  if  $j \neq j'$ , and  $\delta_{j,j} = 1$ , and the random variables  $X_j(0)$ ,

$1 \leq j \leq d$ , can be expressed as the linear combinations of the random variables  $X'_l(0)$ ,  $1 \leq l \leq d'$ , i.e.  $X_j(0) = \sum_{l=1}^{d'} d_{j,l} X'_l(0)$  for all  $1 \leq j \leq d$  with appropriate coefficients  $d_{j,l}$ .

Let us define the vector-valued random field  $X'(p) = (X'_1(p), \dots, X'_{d'}(p))$  as  $X'_j(p) = \sum_{l=1}^d c_{j,l} X_l(p)$ ,  $1 \leq j \leq d'$ , with the same coefficients  $c_{j,l}$  as in the definition of  $X'_j(0)$  for all  $p \in \mathbb{Z}^\nu$ . Then it is not difficult to see that  $X'(p)$ ,  $p \in \mathbb{Z}^\nu$ , is a  $d'$ -dimensional Gaussian stationary random field whose elements have expectation zero, and it satisfies relation (3.2) (with parameter  $d'$  instead of  $d$ .) Moreover, if the covariance function of the original random field  $X(p)$  satisfied relation (3.1), then the covariance function of this new random field also satisfies this condition with appropriate new functions  $a'_{j,j'}(\frac{p}{|p|})$ . Besides, it is not difficult to find such a function  $H'(x_1, \dots, x_{d'})$  for which  $H'(X'_1(p), \dots, X'_{d'}(p)) = H(X_1(p), \dots, X_d(p))$  for all  $p \in \mathbb{Z}^\nu$ . This means that with the introduction of this new random field  $X'(p) = (X'_1(p), \dots, X'_{d'}(p))$  our problem can be reformulated in such a way that our vector-valued stationary Gaussian random field satisfies both relations (3.1) and (3.2). We shall work with such a new  $d'$ -dimensional random field  $X'(p)$  and function  $H'(x_1, \dots, x_{d'})$ , only the sign prime will be omitted everywhere.

First that case will be considered when we fix a positive integer  $k$ , and the function  $H(x_1, \dots, x_d)$  has the form

$$\begin{aligned} H(x_1, \dots, x_d) &= H^{(0)}(x_1, \dots, x_d) \\ &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} H_{k_1}(x_1) \cdots H_{k_d}(x_d) \end{aligned} \quad (3.3)$$

with the previously fixed number  $k$ , the coefficients  $c_{k_1, \dots, k_d}$  are real numbers, and  $H_{k_j}(\cdot)$  denotes the Hermite polynomial of order  $k_j$  with leading coefficient 1. The function  $H'(x_1, \dots, x_{d'})$  preserves this property of the function  $H(x_1, \dots, x_d)$  when the previously mentioned transformation is applied, only different coefficients  $c'_{k_1, \dots, k_d}$  appear in its expansion.

*Remark:* Although I shall not apply the observation of this remark, it may be worth mentioning that if  $(X_1, \dots, X_d)$  is a  $d$ -dimensional random vector with standard normal distribution then  $H(X_1, \dots, X_d)$  with a function  $H(x_1, \dots, x_d)$  having the form (3.3) is a Wick polynomial of order  $k$  of the random vector  $(X_1, \dots, X_d)$ . (See e.g. Corollary 2C in [12] or Corollary 2.3 in [10].) In general, one can say that Hermite polynomials play an important role in limit theorems for non-linear functionals of scalar-valued Gaussian

random fields. In the case of non-linear functionals of vector-valued Gaussian random fields, Wick polynomials take their role.

In scalar-valued models, i.e. in the case  $d = 1$  a non-central limit theorem was proved if  $H(x) = H_k(x)$ ,  $k \geq 2$ , and the covariance function  $r(n) = EX_0X_n$  satisfies condition (3.1) (with  $d = 1$ ) with  $0 < \alpha < \frac{\nu}{k}$ . This result was formulated in Theorem 8.2 of [10]. In this result the limit was described by means of a  $k$ -fold Wiener–Itô integral with respect to an appropriate random spectral measure. This random spectral measure corresponds to the spectral measure that appeared in Lemma 8.1 of [10] as the limit of a sequence of appropriately normalized versions of the spectral measure of a stationary random field  $X(p)$ ,  $p \in \mathbb{Z}^\nu$ , whose covariance function satisfies condition (3.1) with  $d = 1$ . Here I prove a multivariate version of Theorem 8.2 of [10] with the help of a multivariate version of Lemma 8.1 in [10] formulated below.

This generalization of Lemma 8.1 in [10] is a limit theorem for a sequence of appropriately rescaled versions of the coordinates  $G_{j,j'}$  of a spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  with some nice properties. In this limit theorem the vague convergence of complex measures is considered. Before the formulation of this result I recall the definition of this convergence from Section 3 of [12]. In this definition the notion of complex measures with locally finite total variation appears. I explain its meaning in a remark after the definition.

**Definition 3.1.** *Let  $G^{(N)}$ ,  $N = 1, 2, \dots$ , be a sequence of complex measures on  $\mathbb{R}^\nu$  with locally finite total variation. We say that this sequence  $G^{(N)}$  vaguely converges to a complex measure  $G^{(0)}$  on  $\mathbb{R}^\nu$  with locally finite total variation if*

$$\lim_{N \rightarrow \infty} \int f(x) G^{(N)}(dx) = \int f(x) G^{(0)}(dx)$$

*for all continuous functions  $f$  on  $\mathbb{R}^\nu$  with a bounded support.*

*Remark:* In the above definition the notion of complex measures with locally finite total variation appeared. This notion was introduced in Section 4 of [11] together with the notion of vector-valued Gaussian stationary generalized random fields and their matrix valued spectral measures. A complex measure with locally finite total variation is such a complex valued function on the bounded measurable subsets of  $\mathbb{R}^\nu$  whose restriction to the measurable subsets of a cube  $[-T, T]^\nu$  is a complex measure with finite total variation for all  $T > 0$ .

The above definition of vague convergence slightly differs from the classi-

cal one presented e.g. in Section 8 of [10] (before Lemma 8.1 of this paper), where the vague convergence of locally finite (non-negative) measures is considered. The locally finite measures were defined on all measurable subsets of  $\mathbb{R}^\nu$ . Here we deal with complex measures, because we also want to study the non-diagonal elements  $G_{j,j'}$ ,  $j \neq j'$ , of a matrix valued spectral measure, and they are complex (i.e. not necessary real valued) measures. A non-negative locally finite measure always can be extended to a measure on all measurable subsets of  $\mathbb{R}^\nu$ , while there are locally finite complex measures which do not have this property. This fact was taken into account in the introduction of the above definition.

The next Proposition 3.1 contains the multivariate version of Lemma 8.1 in [10].

**Proposition 3.1.** *Let  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  be the matrix valued spectral measure of a  $d$ -dimensional vector-valued stationary random field whose covariance function  $r_{j,j'}(p)$  satisfies relation (3.1) with some parameter  $0 < \alpha < \nu$  and slowly varying function  $L(\cdot)$ . Let us define the following rescaled versions of the coordinates  $G_{j,j'}$ ,  $1 \leq j, j' \leq d$ , of this matrix valued spectral measure:*

$$G_{j,j'}^{(N)}(A) = \frac{N^\alpha}{L(N)} G_{j,j'} \left( \frac{A}{N} \right), \quad A \in \mathcal{B}^\nu, \quad 1 \leq j, j' \leq d, \quad (3.4)$$

for all  $N = 1, 2, \dots$ , where  $\mathcal{B}^\nu$  denotes the  $\sigma$ -algebra of the Borel measurable sets on  $\mathbb{R}^\nu$ . Then  $G_{j,j'}^{(N)}$  is a complex measure with finite total variation concentrated in  $[-N\pi, N\pi]^\nu$ .

For all pairs  $1 \leq j, j' \leq d$  the sequence of complex measures  $G_{j,j'}^{(N)}$  defined in (3.4) tends vaguely to a complex measure  $G_{j,j'}^{(0)}$  on  $\mathbb{R}^\nu$  with locally finite total variation. These complex measures  $G_{j,j'}^{(0)}$ ,  $1 \leq j, j' \leq d$ , have the homogeneity property

$$G_{j,j'}^{(0)}(A) = t^{-\alpha} G_{j,j'}^{(0)}(tA) \quad \text{for all bounded, measurable sets } A \in \mathcal{B}^\nu, \\ 1 \leq j, j' \leq d, \quad \text{and } t > 0. \quad (3.5)$$

The complex measure  $G_{j,j'}^{(0)}$  is determined by the number  $0 < \alpha < \nu$  and functions  $a_{j,j}(\cdot)$ ,  $a_{j,j'}(\cdot)$ ,  $a_{j',j}(\cdot)$  and  $a_{j',j'}(\cdot)$  defined in formula (3.1) on the unit sphere  $S_{\nu-1}$ . This implies that for all spectral measures  $G$  that satisfy relation (3.1) with the same parameter  $\alpha$  and functions  $a_{j,j'}(\cdot)$ ,  $1 \leq j, j' \leq d$  the vague limit of the complex measures  $G_{j,j'}^{(N)}$  is the same for all  $1 \leq j, j' \leq d$ .

Finally, there exists a vector-valued generalized Gaussian stationary random field on  $\mathbb{R}^\nu$  whose matrix valued spectral measure is  $G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  with the complex measures  $(G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  defined in this Proposition.

*Remark:* In Proposition 3.1 such a spectral measure  $(G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  of a generalized Gaussian stationary random field was defined whose elements satisfy the homogeneity property formulated in (3.5). Such spectral measures of generalized random fields play an important role in the construction of the limit fields in our limit theorems. This fact will be discussed in Appendix A in more detail.

In the following Theorem 3.2 I formulate the multivariate version of Theorem 8.2 in [10]. In its formulation the result of Proposition 3.1 is applied where a matrix valued spectral measure  $G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  is constructed under some conditions which are imposed also in Theorem 3.2. Theorem 3.2 is a limit theorem where the limit is defined by means of a sum of multiple Wiener–Itô integrals with respect to a vector-valued random spectral measure that corresponds to the matrix valued spectral measure  $G^{(0)}$  constructed in Proposition 3.1. Let me remark that I formulated this result also in paper [11]. But in that work it was not proved. That work contained only a heuristic argument which indicated why it is natural to expect such a result. Its goal was to indicate the usefulness of the theory worked out in [11] and [12].

**Theorem 3.2.** *Fix an integer  $k \geq 1$ , and let  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , be a vector-valued Gaussian stationary random field whose covariance matrix  $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ ,  $1 \leq j, j' \leq d$ ,  $p \in \mathbb{Z}^\nu$ , satisfies both relation (3.1) with some number  $\alpha$  such that  $0 < \alpha < \frac{\nu}{k}$  and relation (3.2). Let  $H(x_1, \dots, x_d)$  be a function of the form given in (3.3) also with the previously fixed number  $k$ . Define the random variables  $Y(p) = H(X_1(p), \dots, X_d(p))$  for all  $p \in \mathbb{Z}^\nu$  together with their normalized partial sums*

$$S_N = \frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}} \sum_{p \in B_N} Y(p),$$

where the set  $B_N$  was defined in (1.1). These random variables  $S_N$ ,  $N = 1, 2, \dots$ , satisfy the following limit theorem.

Let  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  denote the spectral measure of the above considered vector-valued Gaussian stationary random field  $X(p)$ ,  $p \in \mathbb{Z}^\nu$ , and let

$G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  be the matrix valued spectral measure constructed with the help of the spectral measure  $G$  in Proposition 3.1, i.e. let  $G_{j,j'}^{(0)}$  be the vague limit of the sequence  $G_{j,j'}^{(N)}$  defined in (3.4) for all  $1 \leq j, j' \leq d$ . Let  $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$  be a vector-valued random spectral measure which corresponds to this matrix valued spectral measure  $G^{(0)}$ .

Then the sum of multiple Wiener–Itô integrals with the coefficients  $c_{k_1, \dots, k_d}$  appearing in (3.3)

$$S_0 = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int \prod_{l=1}^{\nu} \frac{e^{i(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} \quad (3.6)$$

$$Z_{G^{(0)}, j(1|k_1, \dots, k_d)}(dx_1) \dots Z_{G^{(0)}, j(k|k_1, \dots, k_d)}(dx_k)$$

exists, where the notation  $x_p = (x_p^{(1)}, \dots, x_p^{(\nu)}) \in \mathbb{R}^\nu$ ,  $p = 1, \dots, k$ , is applied, and the indices  $j(s|k_1, \dots, k_d)$ ,  $1 \leq s \leq k$ , are defined as  $j(s|k_1, \dots, k_d) = r$  if  $\sum_{u=1}^{r-1} k_u < s \leq \sum_{u=1}^r k_u$ ,  $1 \leq s \leq k$ ,  $1 \leq r \leq d$ . (For  $r-1 = 0$  the convention  $\sum_{u=1}^0 k_u = 0$  is applied in this definition.) The normalized sums  $S_N$  converge in distribution to the random variable  $S_0$  defined in (3.6) as  $N \rightarrow \infty$ .

The indexation of the terms  $Z_{G^{(0)}, j(s|k_1, \dots, k_d)}(dx_s)$  in formula (3.6) can be described in a simpler form. In the first  $k_1$  arguments  $x_1, \dots, x_{k_1}$ , i.e. for  $1 \leq s \leq k_1$   $Z_{G^{(0)},1}(dx_s)$ , is written, in the next  $k_2$  arguments, i.e. for  $k_1 + 1 \leq s \leq k_1 + k_2$   $Z_{G^{(0)},2}(dx_s)$  is written, and so on. In the last  $k_d$  arguments, i.e. when  $k_1 + \dots + k_{d-1} + 1 \leq s \leq k$ , ( $k = k_1 + \dots + k_d$ ),  $Z_{G^{(0)},d}(dx_s)$  is written.

In Theorem 3.2 the limit of  $A_N^{-1} \sum_{p \in B_N} H(X_1(p), \dots, X_d(p))$  is described if the expansion of the function  $H(x_1, \dots, x_d)$  is a linear combination of products of Hermite polynomials with different arguments, and all these products are polynomials of order  $k$ . The next Theorem 3.3 which is the multivariate version of Theorem 8.2' in [10] states that a similar result holds if the function  $H(x_1, \dots, x_d)$  is the linear combination of products of Hermite polynomials, but some of these products may be polynomials of order higher than  $k$ .

**Theorem 3.3.** *Let us consider a vector-valued Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , that satisfies the conditions of Theorem 3.2 and a function of the form*

$$H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d) + H^{(1)}(x_1, \dots, x_d),$$

where  $H^{(0)}(x_1, \dots, x_d)$  was defined in (3.3), and

$$H^{(1)}(x_1, \dots, x_d) = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d \geq k+1}} c_{k_1, \dots, k_d} H_{k_1}(x_1) \cdots H_{k_d}(x_d) \quad (3.7)$$

with real valued coefficients  $c_{k_1, \dots, k_d}$  such that

$$\sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d \geq k+1}} \frac{c_{k_1, \dots, k_d}^2}{k_1! \cdots k_d!} < \infty. \quad (3.8)$$

Define the random variables  $Y(p) = H(X_1(p), \dots, X_d(p))$  for all  $p \in \mathbb{Z}^\nu$  and their normalized partial sums

$$S_N = \frac{1}{N^{\nu-k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N} Y(p), \quad N = 1, 2, \dots,$$

with this function  $H(x_1, \dots, x_d)$ . The random variables  $S_N$  converge in distribution to the random variable  $S_0$  defined in formula (3.6) as  $N \rightarrow \infty$ .

Actually condition (3.8) in Theorem 3.3 means that

$$E [H^{(1)}(X_1(0), \dots, X_d(0))^2] < \infty.$$

Finally, I prove a finite-dimensional version of Theorem 3.3 together with a functional limit theorem version of this result. To formulate it, I define the parameter sets

$$\begin{aligned} B_N(t) &= B_N(t^{(1)}, \dots, t^{(\nu)}) \\ &= \{p = (p^{(1)}, \dots, p^{(\nu)}) \in \mathbb{Z}^\nu : 0 < p^{(l)} \leq N t^{(l)} \text{ for all } 1 \leq l \leq \nu\}. \end{aligned} \quad (3.9)$$

for all  $N = 1, 2, \dots$  and  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$ . With this notation, the following result holds.

**Theorem 3.4.** *Let us consider the same vector-valued Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , and function  $H(x_1, \dots, x_d)$  as in Theorem 3.3. Define the random variables  $Y(p) = H(X_1(p), \dots, X_d(p))$  for all  $p \in \mathbb{Z}^\nu$  together with the random fields*

$$S_N(t) = \frac{1}{N^{\nu-k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N(t)} Y(p) \quad (3.10)$$



with parameter set  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$ , where the set  $B_N(t)$  was defined in (3.9). The finite dimensional distributions of the random fields  $S_N(t)$  converge to that of the random field  $S_0(t)$ ,  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$ , defined by the formula

$$S_0(t) = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} \quad (3.11)$$

$$Z_{G^{(0)}, j(1|k_1, \dots, k_d)}(dx_1) \dots Z_{G^{(0)}, j(k|k_1, \dots, k_d)}(dx_k)$$

if the limit  $N \rightarrow \infty$  is taken. In particular, the expression  $S_0(t)$  defined in (3.11) as a linear combination of Wiener-Itô integrals is meaningful. Similarly to Theorem 3.2 the notation  $x_p = (x_p^{(1)}, \dots, x_p^{(\nu)})$ ,  $p = 1, \dots, k$ , is applied, and the indices  $j(s|k_1, \dots, k_d)$ ,  $1 \leq s \leq k$ , are defined in the same way as in formula (3.6).

A referee proposed to show that also a strengthened form of Theorem 3.4 formulated in the next Corollary holds. I shall present the proof of this result in Appendix B. I shall omit some technical details of the proof, and in the case  $\nu > 1$  I shall apply a result whose formulation I did not find in the literature. I chose such an approach, because a detailed proof would demand the elaboration of many complicated technical details which are not related to the subject of this paper.

**Corollary 3.5.** *Let us consider the restriction of the random fields  $S_N(t)$  defined in (3.10) to the unit cube  $[0, 1]^\nu$ , i.e. the random fields  $S_N(t)$  defined on the parameter set  $t = (t^{(1)}, \dots, t^{(\nu)})$ ,  $0 \leq t^{(l)} \leq 1$ ,  $1 \leq l \leq \nu$ . The distributions of these random fields converge weakly to the distribution of the random field  $S_0(t)$ ,  $t \in [0, 1]^\nu$  in the Skorochod space on  $[0, 1]^\nu$  as  $N \rightarrow \infty$ . Moreover, the trajectories of  $S_0(\cdot)$  are continuous functions on  $[0, 1]^\nu$ .*

The limit random field  $S_0(t)$ ,  $t \in [0, \infty)$ , defined in (3.11) has the following two important properties. It is a self-similar random field with stationary increments. I recall the definition of these properties, and show that the random field  $S_0(t)$  has them. Appendix A contains a more detailed discussion about the construction of self-similar random fields with stationary increments.

First I recall the definition of the self-similarity, and show that  $S_0(t)$ ,  $t \in [0, \infty)^\nu$ , is a self-similar random field.

**Definition of the self-similarity property.** A vector-valued random field  $S(t) = (S_1(t), \dots, S_m(t))$ ,  $t \in [0, \infty)^\nu$ , is called self-similar with self-similarity parameter  $\beta$  if  $S(ut) \stackrel{\Delta}{=} u^\beta S(t)$  for all  $u > 0$ , where  $\stackrel{\Delta}{=}$  means that the finite dimensional distributions of the two random fields agree.

The limit random field  $S_0(t)$ ,  $t \in [0, \infty)^\nu$ , defined in (3.11) is self-similar with parameter  $\nu - k\alpha/2$ . This self-similarity property can be proved by exploiting that by formula (3.5) in Proposition 3.1 the spectral measure  $G^{(0)}$  has the homogeneity property  $G^{(0)}(uA) = u^\alpha G^{(0)}(A)$  for all  $u > 0$  and bounded, measurable sets  $A \subset \mathbb{R}^\nu$ . This implies that

$$(Z_{G^{(0)},1}(uA_1), \dots, Z_{G^{(0)},d}(uA_d)) \stackrel{\Delta}{=} (u^{\alpha/2} Z_{G^{(0)},1}(A_1), \dots, u^{\alpha/2} Z_{G^{(0)},d}(A_d))$$

for all  $u > 0$  and bounded, measurable sets  $A_1 \in \mathbb{R}^\nu, \dots, A_d \in \mathbb{R}^\nu$ . We will also apply the fact that the kernel functions

$$f_t(x_1, \dots, x_k) = \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} = \tilde{\chi}_t(x_1 + \dots + x_k),$$

in the Wiener–Itô integrals in (1.13), (with the notation  $t = (t^{(1)}, \dots, t^{(\nu)})$  and  $x_j = (x_j^{(1)}, \dots, x_j^{(\nu)})$ ,  $1 \leq j \leq k$ ), have the property

$$f_{ut}(x_1, \dots, x_k) = u^\nu f_t(ux_1, \dots, ux_k)$$

for all  $u > 0$ ,  $t \in [0, \infty)^\nu$ ,  $x_j \in \mathbb{R}^\nu$ ,  $1 \leq j \leq d$ . (I remark that  $\tilde{\chi}_t(\cdot)$  denotes the Fourier transform of the indicator function of the rectangle  $[0, t] \subset \mathbb{R}^\nu$ .)

The self-similarity property of the random field  $S_0(t)$ ,  $t \in [0, \infty)^\nu$ , can be proved with the help of the above observations.

I also recall the definition of the property that a  $\nu$ -dimensional (possibly vector-valued) random field  $X(t)$ ,  $t \in [0, \infty)^\nu$ , has stationary increments. To formulate this notion first I introduce the definition of the increment of a random field defined on  $[0, \infty)^\nu$  on a  $\nu$ -dimensional rectangle.

Let us consider for all  $s = (s^{(1)}, \dots, s^{(\nu)}) \in [0, \infty)^\nu$  and  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$  the rectangle  $(s, s+t] = \bigotimes_{l=1}^{\nu} (s^{(l)}, s^{(l)} + t^{(l)}]$ , and let  $I((s, s+t])$  denote the indicator function of this rectangle.

To define the increment  $X((s, s+t])$  of the random field  $X(t)$  in the rectangle  $(s, s+t]$  let us make the following observation. Given two points

$s = (s^{(1)}, \dots, s^{(\nu)}) \in [0, \infty)^\nu$ ,  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$ , and a set  $V \subset \{1, \dots, \nu\}$  we define the point  $B(s, t, V) \in [0, \infty)^\nu$  as

$$B(s, t, V) = (u^{(1)}, \dots, u^{(\nu)}) \text{ with } \begin{cases} u^{(l)} = s^{(l)} + t^{(l)} \text{ for } l \in V, \\ u^{(l)} = s^{(l)} \text{ for } l \in \{1, \dots, \nu\} \setminus V. \end{cases}$$

Put  $\eta(V) = \nu - |V|$ , where  $|V|$  denotes the cardinality of the set  $V$ . Then the identity  $I((s, s+t]) = \sum_{V: V \subset \{1, \dots, \nu\}} (-1)^{\eta(V)} I((0, B(s, t, V)])$  holds.

Because of the above identity we define the increment  $X((s, s+t])$  in the rectangle  $(s, s+t]$ ,  $s, t \in [0, \infty)^\nu$ , of the random field  $X(t)$ ,  $t \in [0, \infty)^\nu$ , in the following way. We define  $X((0, t]) = X(t)$  in the special case  $s = 0$ , and in the general case,  $s, t \in [0, \infty)^\nu$ ,

$$X((s, s+t]) = \sum_{V: V \subset \{1, \dots, \nu\}} (-1)^{\eta(V)} X(B(s, t, V)).$$

Finally, we call a random field  $X(t)$ ,  $t \in [0, \infty)^\nu$ , a random field with stationary increments if for all  $s \in [0, \infty)^\nu$  the random field  $X^{(s)}(t) = X((s, s+t])$ ,  $t \in [0, \infty)^\nu$ , has the same distribution as the random field  $X(t)$ ,  $t \in [0, \infty)^\nu$ .

I briefly explain why the random field  $S_0(t)$ ,  $t \in [0, \infty)^\nu$ , defined in (3.11) has stationary increments.

By applying the formula expressing the increment  $X((s, s+t])$  for  $X(t) = S_0(t)$  and relation (3.11) we get the following expression for the random variable  $S_0^{(s)}(t) = S_0((s, s+t])$ :

$$\begin{aligned} S_0^{(s)}(t) = & \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} \\ & e^{is^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} Z_{G^{(0)}, j(1|k_1, \dots, k_d)}(dx_1) \dots Z_{G^{(0)}, j(k|k_1, \dots, k_d)}(dx_k). \end{aligned}$$

The kernel-functions in the stochastic integrals in this formula can be rewritten as

$$\begin{aligned} & \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} e^{is^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} \\ & = e^{i(s, x_1 + \dots + x_k)} \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})}. \end{aligned}$$

Besides, let us define  $Z_{G_s^{(0)}} = (Z_{G_s^{(0)},1}, \dots, Z_{G_s^{(0)},d})$  as

$$(Z_{G_s^{(0)},1}(dx), \dots, Z_{G_s^{(0)},d}(dx)) = (e^{i(s,x)} Z_{G^{(0)},1}(dx), \dots, e^{i(s,x)} Z_{G^{(0)},d}(dx)),$$

which means that  $Z_{G_s^{(0)},j}(A) = \int_A e^{i(s,x)} Z_{G^{(0)},j}(dx)$  for all  $1 \leq j \leq d$  and measurable, bounded sets  $A \subset \mathbb{R}^\nu$ . Then  $Z_{G_s^{(0)}}$  is a vector-valued random spectral measure corresponding to the same matrix valued spectral measure  $G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  as  $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ . I claim that the following identity also holds:

$$\begin{aligned} & \int e^{i(s,x_1+\dots+x_k)} h(x_1, \dots, x_k) Z_{G^{(0)},j_1}(dx_1) \dots Z_{G^{(0)},j_k}(dx_k) \\ &= \int h(x_1, \dots, x_k) Z_{G_s^{(0)},j_1}(dx_1) \dots Z_{G_s^{(0)},j_k}(dx_k) \end{aligned} \quad (3.12)$$

for all sets of indices  $(j_1, \dots, j_k)$  with  $1 \leq j_s \leq d$  for  $1 \leq s \leq k$  if the function  $h(x_1, \dots, x_k)$  belongs to that set of functions for which the above integrals exist. Hence,

$$\begin{aligned} S_0^{(s)}(t) &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int e^{i(s,x_1+\dots+x_k)} \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} \\ &\quad Z_{G^{(0)},j(1|k_1, \dots, k_d)}(dx_1) \dots Z_{G^{(0)},j(k|k_1, \dots, k_d)}(dx_k) \\ &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} \\ &\quad Z_{G_s^{(0)},j(1|k_1, \dots, k_d)}(dx_1) \dots Z_{G_s^{(0)},j(k|k_1, \dots, k_d)}(dx_k). \end{aligned} \quad (3.13)$$

This identity together with formula (3.11) and the fact that  $Z_{G_s^{(0)}}$  is a random spectral measure corresponding to the same spectral measure  $G^{(0)}$  as  $Z_{G^{(0)}}$  imply that the random field  $S_0(t)$ ,  $t \in [0, \infty)$ , has stationary increments.

It remained to prove (3.12). It holds for  $k = 1$ , since it holds in its special case when  $h(x)$  is the indicator function of a bounded measurable set  $A$ , and then because of the properties of the (one-fold) stochastic integrals this identity can be extended to general functions  $h(x)$ .

In the special case when the function  $h(\cdot)$  in (3.12) has the product form  $h(x_1, \dots, x_k) = h_1(x_1) \dots h_k(x_k)$  this identity can be proved with the help of

the multiple version of Itô's formula (see Theorem 2.2 in [12]). By applying this formula for both sides of (3.12) they can be written as  $:U_1 \cdots U_k:$  and  $:V_1 \cdots V_k:$  respectively with  $U_l = \int e^{i(s_l, x_l)} h(x_l) Z_{G^{(0)}, j_l}(dx_l)$  and  $V_l = \int h(x_l) Z_{G_s^{(0)}, j_l}(dx_l)$ ,  $1 \leq l \leq k$ , where  $: \cdots :$  denotes Wick polynomial. By the already proved case  $k = 1$  of (3.12)  $U_l = V_l$  for all  $1 \leq l \leq k$ , hence the two above defined Wick polynomials are equal, and the identity (3.12) holds in this case.

Then identity (3.12) also holds if  $h(x_1, \dots, x_k)$  is a finite linear combination of products of the previous form. Such linear combinations are dense in  $L_2$ -norm in the space of functions for which the above Wiener-Itô integrals exist. Hence the properties of the Wiener-Itô integrals enable us to prove formula (3.12) in the general case by means of a limiting procedure.

#### 4. Preparatory results for the proof of the main theorems.

This section contains the proof of Proposition 3.1 and the elaboration of a method that helps in proving the theorems of this paper. In the application of this method the normalized random sums  $S_N$  appearing in the formulation of Theorem 3.2 are rewritten in the form of a sum of multiple Wiener-Itô integrals with respect to a vector-valued random spectral measure. Then Proposition 3.1 of paper [12] is recalled, and it is shown how the sums of Wiener-Itô integrals expressing the random sums  $S_N$  can be investigated with its help.

*Proof of Proposition 3.1.* Proposition 3.1 is proved by means of an adaptation of the proof of Lemma 8.1 in [10]. The same argument works, only some steps of the proof must be modified in a natural way. I do not work out all details, I only briefly remark what kind of modifications are needed.

The diagonal elements  $G_{j,j}$ ,  $1 \leq j \leq d$ , of the matrix valued spectral measure  $G$  are spectral measures. Hence Lemma 8.1 of [10] implies that for any  $1 \leq j \leq d$  the measures  $G_{j,j}^{(N)}$  converge vaguely to a locally finite measure  $G_{j,j}^{(0)}$  determined by the function  $a_{j,j}(\cdot)$  and the number  $\alpha$  which appears in relation (3.5).

For the non-diagonal elements  $G_{j,j'}$ ,  $j \neq j'$ , this argument cannot be applied, because  $G_{j,j'}$  is a complex measure with finite total variation which may be not a (positive) measure. In this case it can be exploited that  $G$  is a

positive semidefinite matrix valued measure. Hence the  $2 \times 2$  matrix

$$G(A|j, j') = \begin{pmatrix} G_{j,j}(A) & G_{j,j'}(A) \\ G_{j',j}(A) & G_{j',j'}(A) \end{pmatrix}$$

is positive semidefinite for all pairs  $1 \leq j, j' \leq d$ ,  $j \neq j'$ , and measurable sets  $A \subset \mathbb{R}^\nu$ . This implies that the quadratic forms

$$(1, 1)G(A|j, j')(1, 1)^T = G_{j,j}(A) + G_{j',j'}(A) + G_{j,j'}(A) + G_{j',j}(A)$$

and

$$(1, i)G(A|j, j')(1, -i)^T = G_{j,j}(A) + G_{j',j'}(A) - i[G_{j,j'}(A) - G_{j',j}(A)]$$

are non-negative numbers for all measurable sets  $A \subset \mathbb{R}^\nu$ . Therefore the set-functions  $R_{j,j'}(\cdot)$  and  $S_{j,j'}(\cdot)$  defined as  $R_{j,j'}(A) = G_{j,j}(A) + G_{j',j'}(A) + G_{j,j'}(A) + G_{j',j}(A)$  and  $S_{j,j'}(A) = G_{j,j}(A) + G_{j',j'}(A) - i[G_{j,j'}(A) - G_{j',j}(A)]$  for all measurable sets  $A \in [-\pi, \pi)^\nu$  are finite measures. Their Fourier transforms equal  $r_{j,j'}^{(1)}(p) = \int e^{i(p,x)} R_{j,j'}(dx) = r_{j,j}(p) + r_{j',j'}(p) + r_{j,j'}(p) + r_{j',j}(p)$  and  $r_{j,j'}^{(2)}(p) = \int e^{i(p,x)} S_{j,j'}(dx) = r_{j,j}(p) + r_{j',j'}(p) + i[r_{j,j'}(p) - r_{j',j}(p)]$ ,  $p \in \mathbb{Z}^\nu$ . These Fourier transforms satisfy the following relation, similar to formula (3.1).

$$\lim_{T \rightarrow \infty} \sup_{p: p \in \mathbb{Z}^\nu, |p| \geq T} \frac{|r_{j,j'}^{(s)}(p) - a_{j,j'}^{(s)}(\frac{p}{|p|})| p|^{-\alpha} L(|p|)}{|p|^{-\alpha} L(|p|)} = 0 \quad (4.1)$$

both for  $s = 1$  and  $s = 2$  with some functions  $a_{j,j'}^{(s)}(\cdot)$  which can be expressed by means of the functions  $a_{j,j}(\cdot)$ ,  $a_{j,j'}(\cdot)$ ,  $a_{j',j}(\cdot)$  and  $a_{j',j'}(\cdot)$ . The only difference from formula (3.1) is that the continuous function,  $a_{j,j'}^{(2)}(\cdot)$  may be complex valued. (I also remark that the symmetry property  $a_{j,j'}(u) = a_{j',j}(-u)$  yields that  $a_{j,j'}^{(2)}(-u) = \overline{a_{j',j}^{(2)}(u)}$ . On the other hand,  $a_{j,j'}^{(1)}(\cdot)$  is a real valued function, for which  $a_{j,j'}^{(1)}(-u) = a_{j,j}^{(1)}(u)$ . These relations correspond to the fact that  $r_{j,j'}^{(s)}(p)$ ,  $s = 1, 2$  are Fourier transforms of real valued measures.)

As  $R_{j,j'}(\cdot)$  and  $S_{j,j'}(\cdot)$  are real valued positive measures on the torus  $[-\pi, \pi)^\nu$  (a property which may not hold for  $G_{j,j'}(\cdot)$ ), and their Fourier transforms satisfy relation (4.1) which is similar to (3.1) we can adopt the proof of Lemma 8.1 in [10] to show that they have properties similar to those

of the diagonal elements  $G_{j,j}(\cdot)$ . The main difference is that the function  $a_{j,j}(\cdot)$  must be replaced by  $a_{j,j'}^{(1)}(\cdot)$  and  $a_{j,j'}^{(2)}(\cdot)$  in the formulas describing these properties.

More explicitly, define the measures  $R_{j,j'}^{(N)}(\cdot)$ , and  $S_{j,j'}^{(N)}(\cdot)$  as

$$R_{j,j'}^{(N)}(A) = \frac{N^\alpha}{L(N)} R_{j,j'} \left( \frac{A}{N} \right), \quad S_{j,j'}^{(N)}(A) = \frac{N^\alpha}{L(N)} S_{j,j'} \left( \frac{A}{N} \right)$$

for all measurable sets  $A \subset [-N\pi, N\pi]^\nu$  and  $N = 1, 2, \dots$ . I claim that these measures converge vaguely to some locally finite measures  $R_{j,j'}^{(0)}(\cdot)$  and  $S_{j,j'}^{(0)}(\cdot)$  with some homogeneity property on  $\mathbb{R}^\nu$ .

In the adaptation of the proof of Lemma 8.1 in [10] let us introduce the measures  $\mu_N^{(1)}$  and  $\mu_N^{(2)}$ ,  $N = 1, 2, \dots$  as

$$\mu_N^{(1)}(A) = \int_A |K_N(x)|^2 R_{j,j'}^{(N)}(dx), \quad A \in \mathcal{B}^\nu, \quad N = 1, 2, \dots$$

and

$$\mu_N^{(2)}(A) = \int_A |K_N(x)|^2 S_{j,j'}^{(N)}(dx), \quad A \in \mathcal{B}^\nu, \quad N = 1, 2, \dots$$

with the function  $K_N(\cdot)$  defined as

$$K_N(x) = \frac{1}{N} \sum_{p \in B_N} e^{i(p,x/N)} = \prod_{j=1}^\nu \frac{e^{ix^{(j)}} - 1}{N(e^{ix^{(j)}/N} - 1)}, \quad N = 1, 2, \dots$$

I claim that both for  $s = 1$  and  $s = 2$  the sequence of measures  $\mu_N^{(s)}$  converge weakly to a measure  $\mu_0^{(s)}$  as  $N \rightarrow \infty$  whose Fourier transform depends on the function  $a_{j,j'}^{(s)}(\cdot)$  and parameter  $\alpha$  appearing in formula (4.1).

This can be proved by calculating the Fourier transforms of the measures  $\mu_N^{(s)}$  and by showing that they have a limit which is a continuous function. More precisely, Lemma 8.4 in [10] must be applied, which yields a modified version of this method. The reason for this modification is that we can calculate the Fourier transforms of the measures  $\mu_N^{(s)}$ ,  $s = 1, 2$ , only in the points  $\frac{p}{N}$ ,  $p \in \mathbb{Z}^\nu$ . On the other hand, the measures  $\mu_N^{(s)}$  are concentrated in the cube  $[-N\pi, N\pi]^\nu$ . Lemma 8.4 in [10] provides such a version of the characteristic function method which works in such cases.

The calculations needed to prove the above properties of the measures  $\mu_N^{(s)}$ ,  $s = 1, 2$ , are carried out in the proof of Theorem 8.2 in [10]. Actually, a more general result is proved there. I omit the details.

Let us define the continuous function  $K_0(x) = \prod_{j=1}^{\nu} \frac{e^{ix(j)} - 1}{ix(j)}$  on  $\mathbb{R}^{\nu}$ . In all compact subsets of  $\mathbb{R}^{\nu}$  the functions  $K_N(x)$  converge to  $K_0(x)$  in the supremum norm as  $N \rightarrow \infty$ . The proof of Lemma 8.1 in [10] shows on the basis of this property that the limit measures  $\mu_0^{(s)}$ ,  $s = 1, 2$ , have the following representation. There are measures  $H_{j,j'}^{(0)}$  and  $K_{j,j'}^{(0)}$  such that  $\mu_0^{(1)}(A) = \int_A |K_0(x)|^2 H_{j,j'}^{(0)}(dx)$ , and  $\mu_0^{(2)}(A) = \int_A |K_0(x)|^2 K_{j,j'}^{(0)}(dx)$  for all measurable sets  $A \subset \mathbb{R}^{\nu}$ . Moreover,  $H_{j,j'}^{(0)}$  and  $K_{j,j'}^{(0)}$  are locally finite measures, and they are the vague limits of the sequences of measures  $R_{j,j'}^{(N)}$  and  $S_{j,j'}^{(N)}$  respectively. The measures  $H_{j,j'}^{(0)}$  and  $K_{j,j'}^{(0)}$  are determined by the limit measures  $\mu_0^{(1)}$  and  $\mu_0^{(2)}$ , hence also by the parameter  $\alpha$  and functions  $a_{j,j'}^{(1)}(\cdot)$  and  $a_{j,j'}^{(2)}(\cdot)$  in (4.1). The argument of the proof in Lemma 8.1 of [10] enables us to show that (3.5) holds if the complex measure  $G_{j,j}^{(0)}$  is replaced by the measure  $H_{j,j'}^{(0)}$  or  $K_{j,j'}^{(0)}$  in it.

Since the complex measure  $G_{j,j'}$  can be expressed as a linear combination of the measures  $G_{j,j}$ ,  $G_{j',j'}$ ,  $R_{j,j'}$  and  $S_{j,j'}$  the properties proved for them imply the statements formulated about the behavior of  $G_{j,j'}$  in Proposition 3.1.

To complete the proof of Proposition 3.1 we still have to show that  $(G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  is the spectral measure of a generalized vector-valued stationary Gaussian random field. By Theorem 4.1 of [11]  $(G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  is the spectral measure of a vector-valued generalized Gaussian stationary random field if it is a positive definite matrix valued even measure on  $\mathbb{R}^{\nu}$  whose distribution is moderately increasing at infinity, i.e. it satisfies relation (2.1). It follows from Lemma 3.2 in [12] and the already proved part of Proposition 3.1 that this system is a positive semidefinite matrix valued even measure on  $\mathbb{R}^{\nu}$ . The validity of relation (2.1) follows from the fact that  $G_{j,j}^{(0)}$  has locally finite total variation, and it satisfies relation (3.5).  $\square$

Now I turn to the representation of the normalized random sums in the form of a sum of multiple Wiener–Itô integrals. To do this let us first consider the random variable  $Y(0) = H(X_1(0), \dots, X_d(0))$  defined with the help of the function  $H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d)$  introduced in (3.3) and a vector-valued Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^{\nu}$ , with covariance function  $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ ,  $1 \leq j, j' \leq d$ ,  $p \in \mathbb{Z}^{\nu}$ , which satisfies relation (3.1) with some parameter  $0 < \alpha < \frac{\nu}{k}$  together with its shifts  $Y(p) = H(X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^{\nu}$ , and express them as a sum of Wiener–Itô integrals.



This will be done with the help of the results in [11] and [12].

Let  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  be the matrix valued spectral measure of the stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ , and let us consider that vector-valued random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to this spectral measure for which  $X_j(p) = \int e^{i(p,x)} Z_{G,j}(dx)$  for all  $p \in \mathbb{Z}^\nu$  and  $1 \leq j \leq d$ . By the results of [11] there exists such a vector-valued random spectral measure.

The random variable  $Y(0) = H(X_1(0), \dots, X_d(0))$  will be rewritten in the form of a sum of Wiener–Itô integrals with the help of the multiple version of Itô’s formula presented in Theorem 2.2 of [12], more precisely by the corollary of this result. As  $Y(0)$  is a Wick polynomial of the (independent) random variables  $X_j(0)$  with standard Gaussian distribution, and  $X_j(0) = \int Z_{G,j}(dy)$ ,  $1 \leq j \leq d$ , this formula yields the desired expression for  $Y(0)$ . Let me remark that by Lemma 8B of [10] condition (3.1) implies that the diagonal elements  $G_{j,j}$ ,  $1 \leq j \leq d$ , of the matrix valued spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  are non-atomic. Hence the multiple Wiener–Itô integrals with respect to the coordinates of the vector-valued random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  whose sum expresses  $Y(0)$  in the next formula are meaningful.

The above results yield the identity

$$\begin{aligned} Y(0) &= H(X_1(0), \dots, X_d(0)) = H^{(0)}(X_1(0), \dots, X_d(0)) \\ &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} : c_{k_1, \dots, k_d} X_1(0)^{k_1} \dots X_d(0)^{k_d} : \\ &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int \prod_{j=1}^d \left( \prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G,j}(dy_s) \right), \end{aligned}$$

where for  $j = 1$  we define  $\prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G,j}(dy_s) = \prod_{s=1}^{k_1} Z_{G,1}(dy_s)$ , and if  $k_j =$

0 for some  $1 \leq j \leq d$ , then we drop the term  $\prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G,j}(dy_s)$  from this expression. (Here  $:P(X_1(0), \dots, X_d(0)):$  denotes the Wick polynomial corresponding to  $P(X_1(0), \dots, X_d(0))$ , where  $P(x_1, \dots, x_d)$  is a homogeneous polynomial.)

Since  $Y(p) = T_p Y(0)$  with the shift transformation  $T_p$  for all  $p \in \mathbb{Z}^\nu$ , the

previous identity and Proposition 2.4 in [12] yield the formula

$$\begin{aligned} Y(p) &= T_p Y(0) \\ &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \int e^{i(p, y_1 + \dots + y_k)} \prod_{j=1}^d \left( \prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G,j}(dy_s) \right) \end{aligned}$$

for all  $p \in \mathbb{Z}^\nu$ . By summing up this formula for all  $p \in B_N$  we get that

$$\begin{aligned} S_N &= \frac{1}{N^{\nu - k\alpha/2} L(N)^{k/2}} \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} c_{k_1, \dots, k_d} \\ &\quad \int \prod_{l=1}^\nu \frac{e^{i(N(y_1^{(l)} + \dots + y_k^{(l)}))} - 1}{e^{i((y_1^{(l)} + \dots + y_k^{(l)}))} - 1} \prod_{j=1}^d \left( \prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G,j}(dy_s) \right), \end{aligned}$$

where we write  $y = (y^{(1)}, \dots, y^{(\nu)})$  for all  $y \in [-\pi, \pi]^\nu$ .

The above sum of Wiener–Itô integrals can be rewritten with the change of variables  $x_s = Ny_s$ ,  $1 \leq s \leq k$ , in the form

$$\begin{aligned} S_N &= \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} \int c_{k_1, \dots, k_d} f^N(x_1 + \dots + x_k) \\ &\quad \prod_{j=1}^d \left( \prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G^{(N)},j}(dx_s) \right), \quad (4.2) \end{aligned}$$

where

$$f^N(x) = \prod_{l=1}^\nu \frac{e^{ix^{(l)}} - 1}{N(e^{ix^{(l)}/N} - 1)} \quad (4.3)$$

is a function on  $[-N\pi, N\pi]^\nu$ , and  $Z_{G^{(N)},j}(A) = \frac{N^{\alpha/2}}{L(N)^{1/2}} Z_{G,j}(\frac{A}{N})$  for all measurable sets  $A \subset [-N\pi, N\pi]^\nu$  and  $j = 1, \dots, d$ . In formula (4.3) the notation  $x = (x^{(1)}, \dots, x^{(\nu)})$  is applied for all  $x \in \mathbb{R}^\nu$ . Let us observe that  $(Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$  is a vector-valued random spectral measure on the torus  $[-N\pi, N\pi]^\nu$  which corresponds to the matrix valued spectral measure  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j, j' \leq d}$  on the torus  $[-N\pi, N\pi]^\nu$ , defined by the formula

$$G_{j,j'}^{(N)}(A) = \frac{N^\alpha}{L(N)} \left( \frac{A}{N} \right) \quad \text{on the measurable sets } A \subset [-N\pi, N\pi]^\nu$$

for all  $1 \leq j, j' \leq d$ , where  $G = (G_{j,j'})_{1 \leq j, j' \leq d}$  is the matrix valued spectral measure of the original vector-valued stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$ ,  $p \in \mathbb{Z}^\nu$ .

In formulas (4.2) and (4.3) the normalized random sum  $S_N$  investigated in Theorem 3.2 is written in the form of a sum of  $k$ -fold multiple Wiener–Itô integrals. Let us observe that the kernel functions  $c_{k_1, \dots, k_d} f^N(x_1 + \dots + x_k)$  of these Wiener–Itô integrals satisfy the relation

$$\lim_{N \rightarrow \infty} c_{k_1, \dots, k_d} f^N(x_1 + \dots + x_k) = c_{k_1, \dots, k_d} f^0(x_1 + \dots + x_k) \quad (4.4)$$

for all indices  $k_1, \dots, k_d$  such that  $k_j \geq 0$ ,  $1 \leq j \leq d$ , and  $k_1 + \dots + k_d = k$  with the function

$$f^0(x) = \prod_{l=1}^{\nu} \frac{e^{ix^{(l)}} - 1}{ix^{(l)}} \quad (4.5)$$

defined on  $\mathbb{R}^\nu$ , and this convergence is uniform in all bounded subsets of  $\mathbb{R}^{k\nu}$ .

On the other hand, Proposition 3.1 states that the elements of the matrix valued spectral measures  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j, j' \leq d}$  vaguely converge to the elements of a matrix valued spectral measure  $G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j, j' \leq d}$  on  $\mathbb{R}^\nu$ . In (4.2) we integrate with respect to a vector-valued random spectral measure corresponding to the matrix valued spectral measure  $(G_{j,j'}^{(N)})_{1 \leq j, j' \leq d}$  of a generalized vector-valued Gaussian stationary random field. Hence it is natural to expect that the random variables  $S_N$  converge in distribution to the random variable

$$S_0 = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} \int c_{k_1, \dots, k_d} f^0(x_1 + \dots + x_k) \prod_{j=1}^d \left( \prod_{s=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G^{(0)}, j}(dx_s) \right), \quad (4.6)$$

where  $(Z_{G^{(0)}, 1}, \dots, Z_{G^{(0)}, d})$  is a vector-valued random spectral measure on  $\mathbb{R}^\nu$  corresponding to the matrix valued spectral measure  $(G_{j,j'}^{(0)})_{1 \leq j, j' \leq d}$ . This is actually the statement of Theorem 3.2 with a slightly different notation.

I shall formulate such a result in the following Proposition 4A which helps to justify the above heuristic argument. It states that this argument yields a correct result if some additional conditions are also satisfied. Theorem 3.2

will be proved with the help of this Proposition 4A which is a reformulation of Proposition 3.1 in [12].

Before the presentation of Proposition 4A I recall from Section 5 of [11] the definition of that class of functions which can be chosen for the kernel function of a multiple Wiener–Itô integral with respect to a vector-valued random spectral measure. This class of functions appears in the formulation of Proposition 4A.

Let us consider the matrix valued spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  with non-atomic measures  $G_{j,j}$ ,  $1 \leq j \leq d$ , of a vector-valued Gaussian stationary random field. (We can consider the spectral measure both of an ordinary or of a generalized random field.) In [11] I have defined a real Hilbert space  $\mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k})$  depending on the diagonal elements  $G_{1,1}, \dots, G_{d,d}$  of the spectral measure  $G$  and on a sequence of integers  $(j_1, \dots, j_k)$  of length  $k$  such that  $1 \leq j_s \leq d$  for all  $1 \leq s \leq k$ . This Hilbert space has the property that the  $k$ -fold Wiener–Itô integral

$$\int f(x_1, \dots, x_k) Z_{G,j_1}(dx_1) \dots Z_{G,j_k}(dx_k)$$

with respect to a vector-valued random spectral measure

$$Z_G = (Z_{G,1}, \dots, Z_{G,d})$$

corresponding to the matrix valued spectral measure  $G$  is defined for the kernel functions

$$f(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k}).$$

(In papers [11] and [12] I worked with Wiener–Itô integrals of order  $n$ , while here I work with Wiener–Itô integrals of order  $k$ . Hence I use a slightly different notation here.)

We have  $f \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k})$  for a complex number valued function  $f(x_1, \dots, x_k)$  with arguments  $x_s \in \mathbb{R}^\nu$ ,  $1 \leq s \leq k$ , if it satisfies the following conditions (a) and (b):

- (a)  $f(-x_1, \dots, -x_k) = \overline{f(x_1, \dots, x_k)}$  for all  $(x_1, \dots, x_k) \in \mathbb{R}^{k\nu}$ ,
- (b)  $\|f\|^2 = \int |f(x_1, \dots, x_k)|^2 G_{j_1,j_1}(dx_1) \dots G_{j_k,j_k}(dx_k) < \infty$ .

The scalar product in  $\mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1},\dots,G_{j_k,j_k})$  is defined in the usual way. If  $f, g \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1},\dots,G_{j_k,j_k})$ , then

$$\langle f, g \rangle = \int f(x_1, \dots, x_k) \overline{g(x_1, \dots, x_k)} G_{j_1,j_1}(dx_1) \dots G_{j_k,j_k}(dx_k).$$

In the formulation of Proposition 4A we take for all  $N = 1, 2, \dots$  a matrix valued non-atomic spectral measure  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j,j' \leq d}$  on the torus  $[-A_N\pi, A_N\pi]^\nu$  with a parameter  $A_N$  such that  $A_N \rightarrow \infty$  as  $N \rightarrow \infty$ . We also take some functions

$$h_{j_1,\dots,j_k}^N(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}^{(N)}, \dots, G_{j_k,j_k}^{(N)})$$

on the torus  $[-A_N\pi, A_N\pi]^\nu$  for all sets of indices  $(j_1, \dots, j_k)$  such that  $1 \leq j_s \leq d$ ,  $1 \leq s \leq k$ , and  $N = 1, 2, \dots$ . Besides, we fix for all  $N = 1, 2, \dots$  a vector-valued random spectral measure  $Z_{G^{(N)}} = (Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$  on the torus  $[-A_N\pi, A_N\pi]^\nu$  corresponding to the matrix valued spectral measure  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j,j' \leq d}$  and we define with the help of these quantities the sums of  $k$ -fold Wiener–Itô integrals

$$S_N = \sum_{\substack{(j_1,\dots,j_k) \\ 1 \leq j_s \leq d, \text{ for all } 1 \leq s \leq k}} \int h_{j_1,\dots,j_k}^N(x_1, \dots, x_k) Z_{G^{(N)},j_1}(dx_1) \dots Z_{G^{(N)},j_k}(dx_k), \quad (4.7)$$

$N = 1, 2, \dots$ . We want to find some good conditions under which these random variables  $S_N$  converge in distribution to a random variable  $S_0$ , expressed similarly as a sum of  $k$ -fold multiple Wiener–Itô integrals.

This will be done with the help of the following Proposition 4A which agrees with Proposition 3.1 in paper [12].

**Proposition 4A.** *Let us consider for all  $N = 1, 2, \dots$  the sum of  $k$ -fold Wiener–Itô integrals  $S_N$  defined in formula (4.7) with the help of a vector-valued random spectral measure  $Z_{G^{(N)}} = (Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$  corresponding to some non-atomic matrix valued spectral measure  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j,j' \leq d}$  defined on a torus  $[-A_N, A_N]^\nu$  such that  $A_N \rightarrow \infty$  as  $N \rightarrow \infty$  and functions*

$$h_{j_1,\dots,j_k}^N(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}^{(N)}, \dots, G_{j_k,j_k}^{(N)}).$$

*Let the coordinates  $G_{j,j'}^{(N)}$ ,  $1 \leq j, j' \leq d$ , of the matrix valued spectral measures  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j,j' \leq d}$  converge vaguely to the coordinates  $G_{j,j'}^{(0)}$  of a*

non-atomic matrix valued spectral measure  $G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$  on  $\mathbb{R}^\nu$  for all  $1 \leq j, j' \leq d$  as  $N \rightarrow \infty$ , and let  $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$  be a vector-valued random spectral measure on  $\mathbb{R}^\nu$  corresponding to the matrix valued spectral measure  $G^{(0)} = (G_{j,j'}^{(0)})_{1 \leq j,j' \leq d}$ . Let us also have some functions  $h_{j_1, \dots, j_k}^0(x_1, \dots, x_k)$  on  $\mathbb{R}^{k\nu}$  for all  $1 \leq j_s \leq d, 1 \leq s \leq k$ , such that these functions and matrix valued spectral measures satisfy the following conditions (a) and (b).

- (a) The functions  $h_{j_1, \dots, j_k}^0(x_1, \dots, x_k)$  are continuous on  $\mathbb{R}^{k\nu}$  for all  $1 \leq j_s \leq d, 1 \leq s \leq k$ , and for all  $T > 0$  and indices  $1 \leq j_s \leq d, 1 \leq s \leq k$ , the functions  $h_{j_1, \dots, j_k}^N(x_1, \dots, x_k)$  converge uniformly to the function  $h_{j_1, \dots, j_k}^0(x_1, \dots, x_k)$  on the cube  $[-T, T]^{k\nu}$  as  $N \rightarrow \infty$ .
- (b) For all  $\varepsilon > 0$  there is some  $T_0 = T_0(\varepsilon) > 0$  such that

$$\int_{\mathbb{R}^{k\nu} \setminus [-T, T]^{k\nu}} |h_{j_1, \dots, j_k}^N(x_1, \dots, x_k)|^2 G_{j_1, j_1}^{(N)}(dx_1) \dots G_{j_k, j_k}^{(N)}(dx_k) < \varepsilon^2 \quad (4.8)$$

for all  $1 \leq j_s \leq d, 1 \leq s \leq k$ , and  $N = 1, 2, \dots$  if  $T > T_0$ .

Then

$$h_{j_1, \dots, j_k}^0(x_1, \dots, x_k) \in \mathcal{K}_{k, j_1, \dots, j_k} = \mathcal{K}_{k, j_1, \dots, j_k}(G_{j_1, j_1}^{(0)}, \dots, G_{j_k, j_k}^{(0)}),$$

inequality (4.8) holds also for  $N = 0$ , the sum of random integrals

$$S_0 = \sum_{\substack{(j_1, \dots, j_k) \\ 1 \leq j_s \leq d, \text{ for all } 1 \leq s \leq k}} \int h_{j_1, \dots, j_k}^0(x_1, \dots, x_k) Z_{G^{(0)}, j_1}(dx_1) \dots Z_{G^{(0)}, j_k}(dx_k) \quad (4.9)$$

exists, and the random variables  $S_N$  defined in (4.7) converge to  $S_0$  in distribution as  $N \rightarrow \infty$ .

(In the formulation of Proposition 4A I took the natural identification of the torus  $[-A_N, A_N]^\nu$  with the cube  $[-A_N, A_N]^\nu$  in the space  $\mathbb{R}^\nu$ . Thus, I considered the functions  $h_{j_1, \dots, j_k}(\cdot)$  as functions on  $\mathbb{R}^{k\nu}$  which disappear outside  $[-A_N, A_N]^{k\nu}$ , and the complex measures  $G_{j,j'}^{(N)}$  as complex measures on  $\mathbb{R}^\nu$ , concentrated on  $[-A_N, A_N]^\nu$ . In such a way the vague convergence mentioned in the formulation of Proposition 4A is meaningful.)

In the proof of Theorem 3.2 we want to show with the help of Proposition 4A that the sequence of random variables  $S_N$ ,  $N = 1, 2, \dots$ , defined in (4.2) converge to the random variable  $S_0$  defined in (4.6) as  $N \rightarrow \infty$ . To do this we rewrite these formulas with a different indexation in such a way that the indices in the definition of the random variables  $S_N$  and  $S_0$  fit to the indices in the definition of the random variable of the random variables  $S_N$  and  $S_0$  appearing in the formulation of Proposition 4A. These random variables were defined in formulas (4.7) and (4.9).

In formulas (4.2) and (4.6) summation is taken for terms with indices  $(k_1, \dots, k_d)$  such that  $k_s \geq 0$ ,  $1 \leq s \leq d$  and  $k_1 + \dots + k_d = k$ , while in the corresponding expressions in formulas (4.7) and (4.9) in Proposition 4A it is taken for terms with indices  $(j_1, \dots, j_k)$  such that  $1 \leq j_s \leq d$ ,  $1 \leq s \leq k$ .

An important difference between the indexation in the two cases is that in (4.2) and (4.6) only a special subset of the indices in formulas (4.7) and (4.9) appear. Namely, if  $s < s'$ , and we compare the indices  $j$  and  $j'$  in the terms  $Z_{G^{(N)},j}(dx_s)$  and  $Z_{G^{(N)},j'}(dx_{s'})$  belonging to these indices  $s$  and  $s'$  in formula (4.2) or (4.6), then we find that  $j \leq j'$ . Hence such a reindexation of the indices in (4.2) and (4.6) will be made, where the set  $\mathcal{J}$  of the new indices is only a subset of the indices  $(j_1, \dots, j_k)$  appearing in formulas (4.7) and (4.9). Summation will be taken only for the elements of  $\mathcal{J}$  in these formulas.

More explicitly, the terms in the sums in (4.2) and (4.6) will be reindexed with such indices  $(j_1, \dots, j_k)$  for which the relation  $1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq d$  holds. This is a subset of the set of indices  $(j_1, \dots, j_k)$  appearing in formulas (4.7) and (4.9). To carry out the desired reindexation a one to one map will be defined between the sets

$$\mathcal{J} = \{(j_1, \dots, j_k): 1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq d\}$$

and

$$\mathcal{K} = \{(k_1, \dots, k_d): k_s \geq 0 \text{ for all } 1 \leq s \leq d, k_1 + \dots + k_d = k\}.$$

Put

$$\begin{aligned} \text{for all } (j_1, \dots, j_k) \in \mathcal{J} \quad k_s(j_1, \dots, j_k) &= \text{the number of such elements } j_p \\ &\text{for which } j_p = s, \text{ for all } 1 \leq s \leq d. \end{aligned} \quad (4.10)$$

This is a one to one map from  $\mathcal{J}$  to  $\mathcal{K}$  whose inverse is

$$\begin{aligned} \text{for all } (k_1, \dots, k_d) \in \mathcal{K} \quad j_s(k_1, \dots, k_d) &= \min p: k_1 + \dots + k_p \geq s, \\ &\text{for all } 1 \leq s \leq k. \end{aligned} \quad (4.11)$$

We shall apply these maps.

With the help of this correspondence between the sets  $\mathcal{J}$  and  $\mathcal{K}$  the random sums  $S_N$  in (4.2) can be rewritten in a form where summation is taken for the sequences  $(j_1, \dots, j_k) \in \mathcal{J}$  instead of the sequences  $(k_1, \dots, k_d) \in \mathcal{K}$ , and  $k_s(j_1, \dots, j_k)$  is written instead of  $k_s$ ,  $1 \leq s \leq d$ .

The expression  $S_N$  defined in (4.2) can be rewritten as

$$S_N = \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_1 \leq \dots \leq j_k \leq d}} \int c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^N(x_1 + \dots + x_k) \prod_{s=1}^k Z_{G^{(N)}, j_s}(dx_s) \quad (4.12)$$

for all  $N = 1, 2, \dots$  with the functions  $f^N(x)$  defined in (4.3) and the indices  $k_s(j_1, \dots, j_k)$ ,  $1 \leq s \leq d$ , defined in (4.10).

To show why formula (4.12) holds let us rewrite formula (4.2) (with the help of the one to one map we defined between the sets  $\mathcal{K}$  and  $\mathcal{J}$ ) in the form

$$S_N = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} \int c_{k_1, \dots, k_d} f^N(x_1 + \dots + x_k) \prod_{s=1}^k Z_{G^{(N)}, j_s(k_1, \dots, k_d)}(dx_s). \quad (4.13)$$

To understand why formula (4.13) holds we have to show that in (4.13) the term  $Z_{G^{(N)}, u}(dx_s)$  with  $u = j_s(k_1, \dots, k_d)$  had to be chosen. It can be seen from (4.2) that this number  $u$  must be chosen in such a way that  $k_1 + \dots + k_{u-1} + 1 \leq s \leq k_1 + \dots + k_u$ . Then a comparison of this condition with the definition of the mapping from  $\mathcal{K}$  to  $\mathcal{J}$  in (4.11) shows that  $u = j_s(k_1, \dots, k_d)$ .

Then if we rewrite the formula at the right-hand side of (4.12) by replacing the arguments  $(k_1, \dots, k_d) \in \mathcal{K}$  by the arguments  $(j_1, \dots, j_k) \in \mathcal{J}$  with the help of the transformation we defined from  $\mathcal{J}$  to  $\mathcal{K}$ , and then we exploit that the transformation we defined from  $\mathcal{K}$  to  $\mathcal{J}$  is its inverse transformation, we get that formula (4.13) implies (4.12).

Relation (4.12) can be rewritten in the form

$$S_N = \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_1 \leq \dots \leq j_k \leq d}} \int h_{j_1, \dots, j_k}^N(x_1, \dots, x_k) Z_{G^{(N)}, j_1}(dx_1) \dots Z_{G^{(N)}, j_k}(dx_k) \quad (4.14)$$

with

$$h_{j_1, \dots, j_k}^N(x_1, \dots, x_k) = c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^N(x_1 + \dots + x_k), \quad (4.15)$$



where the indices  $k_s(j_1, \dots, j_k)$ ,  $1 \leq s \leq d$ , are defined in (4.10). Similarly, the random sum  $S_0$  in (4.6) can be rewritten in the form

$$S_0 = \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_1 \leq \dots \leq j_k \leq d}} \int c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^0(x_1 + \dots + x_k) \prod_{s=1}^k Z_{G^{(0)}, j_s}(dx_s)$$

with the function  $f^0(x)$  defined in (4.5) or in the following equivalent form.

$$S_0 = \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_1 \leq \dots \leq j_k \leq d}} \int h_{j_1, \dots, j_k}^0(x_1, \dots, x_k) Z_{G^{(0)}, j_1}(dx_1) \dots Z_{G^{(0)}, j_k}(dx_k) \quad (4.16)$$

with

$$h_{j_1, \dots, j_k}^0(x_1, \dots, x_k) = c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^0(x_1 + \dots + x_k). \quad (4.17)$$

## 5. Proof of the main theorems.

Theorem 3.2 will be proved by means of the application of Proposition 4A for the sequences  $S_N$  defined in (4.14), (4.3), (4.15) and (4.10) for  $N = 1, 2, \dots$ , and in (4.16), (4.5), (4.17) and (4.10) for  $N = 0$ . To do this we have to show that under the conditions of Theorem 3.2 the conditions of Proposition 4A are also satisfied with such a choice. Then the application of Proposition 4A implies Theorem 3.2. (I would remark that the random variable  $S_0$  defined in formula (3.6) as the limit in Theorem 3.2 agrees with the random variable  $S_0$  defined in (4.6), which is the same as the limit we get in the application of Proposition 2A with the above written choice. Only it is written there in a different form.)

To check the conditions of Proposition 4A let us first observe that it follows from Proposition 3.1 that the (non-atomic) elements  $G_{j, j'}^{(N)}$  of the spectral measures  $G^{(N)}$  vaguely converge to the (non-atomic) complex measures  $G_{j, j'}^{(0)}$  of a spectral measure  $G^{(0)}$  as  $N \rightarrow \infty$  for all  $1 \leq j, j' \leq d$ . It is also clear that the functions  $h_{j_1, \dots, j_k}^N(x_1, \dots, x_k)$  defined in (4.15) for all  $1 \leq j_1 \leq \dots \leq j_k \leq d$  and  $N = 1, 2, \dots$  satisfy the condition  $h_{j_1, \dots, j_k}^N(x_1, \dots, x_k) \in \mathcal{K}_{k, j_1, \dots, j_k}(G_{j_1, j_1}^{(N)}, \dots, G_{j_k, j_k}^{(N)})$ .

It follows from (4.4), (4.15) and (4.17) that condition (a) of Proposition 4A holds with the functions and measures chosen in the proof of Theorem 3.2. We still have to prove relation (4.8) in condition (b) of Proposition 4A. This will be done with the help of the following Proposition 5.1. (Actually in Proposition 5.1 we prove a result slightly sharper than we need.)

**Proposition 5.1.** *Let us fix an integer  $k \geq 1$ , and let  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  be the matrix valued spectral measure of a vector-valued stationary random field  $X(p) = (X_1(p), \dots, X_p(d))$ ,  $p \in \mathbb{Z}^\nu$ , defined on the torus  $[-\pi, \pi)^\nu$  with such correlation function  $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ ,  $1 \leq j, j' \leq d$ ,  $p \in \mathbb{Z}^\nu$ , that satisfies relation (3.1) with some  $0 < \alpha < \frac{\nu}{k}$ . For all  $N = 1, 2, \dots$  let us consider the measures  $(G_{j,j}^{(N)})_{1 \leq j \leq d}$  defined in formula (3.4) together with the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  defined for all sets of indices  $j_1, \dots, j_k$  such that  $1 \leq j_s \leq d$ ,  $1 \leq s \leq k$ , on  $\mathbb{R}^{k\nu}$  by the formula*

$$\mu_{j_1, \dots, j_k}^{(N)}(A) = \int_A |h_N(x_1, \dots, x_k)|^2 G_{j_1, j_1}^{(N)}(dx_1) \dots G_{j_k, j_k}^{(N)}(dx_k), \quad A \in \mathcal{B}^{k\nu}, \quad (5.1)$$

with

$$h_N(x_1, \dots, x_k) = f^N(x_1 + \dots + x_k) = \prod_{l=1}^{\nu} \frac{e^{i((x_1^{(l)} + \dots + x_k^{(l)}) - 1)} - 1}{N(e^{i((x_1^{(l)} + \dots + x_k^{(l)})/N - 1)} - 1)}, \quad (5.2)$$

where we use the notation  $x = (x^{(1)}, \dots, x^{(\nu)})$  for a vector  $x \in \mathbb{R}^\nu$ . These measures  $\mu_{j_1, \dots, j_k}^{(N)}$  converge weakly to a finite measure  $\mu_{j_1, \dots, j_k}^{(0)}$  on  $\mathbb{R}^{k\nu}$ .

*Proof of Theorem 3.2 with the help of Proposition 5.1.* As we have seen to prove Theorem 3.2 it is enough to check that the measures  $G^{(N)} = (G_{j,j'}^{(N)})_{1 \leq j,j' \leq d}$  and functions  $h_{j_1, \dots, j_k}^N$  defined before satisfy the conditions of Proposition 4A, since this enables us to apply this result. Moreover, we have proved the validity of all of these conditions except formula (4.8) in condition (b) of Proposition 4A. But the validity of this condition follows from Proposition 5.1, since this result implies that the measures  $\mu_{j_1, \dots, j_k}^{(N)}$ ,  $N = 1, 2, \dots$ , defined in (5.1) and (5.2) are uniformly tight. This fact together with the definition of the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  and the identity  $h_{j_1, \dots, j_k}^N(x_1, \dots, x_k) = c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} h_N(x_1, \dots, x_k)$  imply that relation (4.8) holds. Theorem 3.2 is proved.  $\square$

It remains to prove Proposition 5.1.

*Proof of Proposition 5.1.* Most calculations needed in the proof of Proposition 5.1 were actually carried out in the proof of Theorem 8.2 of [10]. Only some slight modifications are needed in the proof. In some steps I shall refer to the corresponding part in the proof of Theorem 8.2 in [10] and omit the details of the calculation.

I compute for all  $N = 1, 2, \dots$  the Fourier transform

$$\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k) = \int e^{i((t_1, x_1) + \dots + (t_k, x_k))} \mu_{j_1, \dots, j_k}^{(N)}(dx_1, \dots, dx_k)$$

of the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  defined in (5.1) and give a good asymptotic formula for it. More precisely I do this only for such coordinates  $(t_1, \dots, t_k)$  of the function  $\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k)$  which have the form  $t_l = \frac{p_l}{N}$  with some  $p_l \in \mathbb{Z}^\nu$ ,  $l = 1, \dots, k$ . But as it is explained at the end of this proof, even such a result is sufficient for us. In the calculation of the formula expressing  $\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k)$  I exploit that the function  $h_N(x_1, \dots, x_k)$  defined in (5.2) can be written in the form

$$h_N(x_1, \dots, x_k) = \frac{1}{N^\nu} \sum_{u \in B_N} \exp \left\{ i \frac{1}{N} (u, x_1 + \dots + x_k) \right\}.$$

Hence, and because of the definition of the spectral measures  $(G_{j,j}^{(N)}(\cdot))_{1 \leq j, j' \leq d}$  in (3.4)

$$\begin{aligned} \varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k) &= \frac{1}{N^{2\nu}} \int \exp \left\{ i \frac{1}{N} ((p_1, x_1) + \dots + (p_k, x_k)) \right\} \\ &\quad \sum_{u \in B_N} \sum_{v \in B_N} \exp \left\{ i \left( \frac{u-v}{N}, x_1 + \dots + x_k \right) \right\} G_{j_1, j_1}^{(N)}(dx_1) \dots G_{j_k, j_k}^{(N)}(dx_k) \\ &= \frac{1}{N^{2\nu}} \sum_{u \in B_N} \sum_{v \in B_N} \left( \prod_{s=1}^k \int \exp \left\{ i \left( \frac{u-v+p_s}{N}, x_s \right) \right\} G_{j_s, j_s}^{(N)}(dx_s) \right) \\ &= \frac{1}{N^{2\nu-k\alpha} L(N)^k} \sum_{u \in B_N} \sum_{v \in B_N} r_{j_1, j_1}(u-v+p_1) \dots r_{j_k, j_k}(u-v+p_k) \end{aligned}$$

if  $t_l = \frac{p_l}{N}$  with some  $p_l \in \mathbb{Z}^\nu$ ,  $1 \leq l \leq k$ . This identity can be rewritten by taking the summation at the right-hand side of the last formula first for such pairs  $(u, v)$  for which  $u-v = y$  with a fixed point  $y \in \mathbb{Z}^\nu$  and then for the lattice points  $y \in \mathbb{Z}^\nu$ . By working with  $x = \frac{y}{N}$  instead of  $y$  we get that

$$\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k) = \int_{[-1,1]^\nu} f_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k, x) \lambda_N(dx)$$

with

$$\begin{aligned} f_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k, x) &= \left( 1 - \frac{|x^{(1)} N|}{N} \right) \dots \left( 1 - \frac{|x^{(\nu)} N|}{N} \right) \frac{r_{j_1, j_1}(N(x+t_1))}{N^{-\alpha} L(N)} \dots \frac{r_{j_k, j_k}(N(x+t_k))}{N^{-\alpha} L(N)}, \end{aligned}$$

where  $\lambda_N$  is the measure concentrated in the points of the form  $x = \frac{p}{N}$  with such points  $p = (p_1, \dots, p_\nu) \in \mathbb{Z}^\nu$  for which  $-N < p_l < N$  for all  $1 \leq l \leq \nu$ , and  $\lambda_N(x) = N^{-\nu}$  for each point  $x$  with this property. (Here such a calculation is applied which is similar to that in the proof of Theorem 8.2 of [10] when formula (8.20) of that work was rewritten in another form.)

Let us extend the definition of  $\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k)$  to all  $(t_1, \dots, t_k) \in \mathbb{R}^{k\nu}$  by defining it as

$$\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k) = \varphi_{j_1, \dots, j_d}^{(N)}\left(\frac{p_1}{N}, \dots, \frac{p_k}{N}\right), \quad t_l \in \mathbb{R}^\nu \text{ for all } 1 \leq l \leq k,$$

where  $p_l = p_l(t_l)$  is defined as the integer part  $[t_l N]$  of  $t_l N$ ,  $1 \leq l \leq k$ , i.e.  $p_l \in \mathbb{Z}^\nu$ , and  $p_l^{(s)} \leq t_l^{(s)} N < p_l^{(s)} + 1$  if  $t_l^{(s)} > 0$ , and  $p_l^{(s)} - 1 < t_l^{(s)} N \leq p_l^{(s)}$  if  $t_l^{(s)} \leq 0$ ,  $1 \leq s \leq \nu$ .

Let us also extend the definition of the function  $f_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k, x)$  to  $(t_1, \dots, t_k, x) \in \mathbb{R}^{k\nu} \times [-1, 1]^\nu$  by means of the formula

$$\begin{aligned} f_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k, x) \\ = \left(1 - \frac{|q^{(1)}|}{N}\right) \dots \left(1 - \frac{|q^{(\nu)}|}{N}\right) \frac{r_{j_1, j_1}(q + p_1)}{N^{-\alpha} L(N)} \dots \frac{r_{j_k, j_k}(q + p_k)}{N^{-\alpha} L(N)} \end{aligned}$$

for  $t_l \in \mathbb{R}^\nu$ ,  $1 \leq l \leq k$ , and  $x \in [-1, 1]^\nu$ , where  $p_l = p_l(t)$  is defined as before, and  $q = q(x)$  is defined as  $q = (q^{(1)}, \dots, q^{(\nu)}) \in \mathbb{Z}^\nu$  with  $q_l \in \mathbb{Z}^\nu$ , and  $q^{(s)} \leq x_l^{(s)} < q^{(s)} + 1$ .

We have

$$\varphi_{j_1, \dots, j_d}^{(N)}(t_1, \dots, t_k) = \int_{[-1, 1]^\nu} f_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k, x) dx \quad (5.3)$$

for the functions  $\varphi^{(N)}(\cdot)$  and  $f^{(N)}(\cdot)$  with this extended domain of definition, where  $dx$  denotes integration with respect to the Lebesgue measure.

It follows from relation (3.1) and the fact that  $\frac{q}{N}$  is very close to  $x$ , and  $\frac{p_l}{N}$  is very close to  $t_l$ , for all  $1 \leq l \leq k$  if  $N$  is large that for all parameters  $t_1, \dots, t_k$  and  $\varepsilon > 0$

$$f_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k, x) \rightarrow f_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k, x)$$

holds uniformly with the limit function

$$\begin{aligned} f_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k, x) \\ = (1 - |x^{(1)}|) \dots (1 - |x^{(\nu)}|) \frac{a_{j_1, j_1}\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha} \dots \frac{a_{j_k, j_k}\left(\frac{x+t_k}{|x+t_k|}\right)}{|x+t_k|^\alpha} \end{aligned}$$

on the set  $x \in [-1, 1]^\nu \setminus \bigcup_{l=1}^k \{x: |x + t_l| < \varepsilon\}$ .

Some additional calculation shows that for small  $\varepsilon > 0$  integration on the domain

$$\begin{aligned} [-1, 1]^\nu \setminus \left( [-1, 1]^\nu \setminus \bigcup_{l=1}^k \{x: |x + t_l| < \varepsilon\} \right) \\ = [-1, 1]^\nu \cap \left( \bigcup_{l=1}^k \{x: |x + t_l| < \varepsilon\} \right) \end{aligned}$$

gives a negligible contribution to the integral in formula (5.3) (with parameters  $j_1, \dots, j_k$  and  $t_1, \dots, t_k$ ), or to the integral that we get if the kernel function  $f_{j_1, \dots, j_k}^{(N)}$  is replaced by  $f_{j_1, \dots, j_k}^{(0)}$  in the integral in (5.3). Hence the relation

$$\varphi_{j_1, \dots, j_k}^{(N)}(t_1, \dots, t_k) \rightarrow \varphi_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k) = \int_{[-1, 1]^\nu} f_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k, x) dx \quad (5.4)$$

holds for all  $(t_1, \dots, t_k) \in \mathbb{R}^{k\nu}$  as  $N \rightarrow \infty$ , and  $\varphi_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k)$  is a continuous function. This calculation was carried out in that part of the proof of Theorem 8.2 in [10] which followed the discussion of Lemma 8.4. Hence here I omit it.

By a classical result of probability theory if the Fourier transforms of a sequence of finite measures on  $\mathbb{R}^{k\nu}$  converge to a function continuous at the origin, then the limit function is also the Fourier transform of a finite measure on  $\mathbb{R}^{k\nu}$ , and the sequence of probability measures whose Fourier transforms were taken converge to this measure. In the proof of Proposition 5.1 this result cannot be applied, because we have a control on the Fourier transform of  $\mu_{j_1, \dots, j_k}^{(N)}$  only in points of the form  $(t_1, \dots, t_k)$  with  $t_l = \frac{p_l}{N}$  and  $p_l \in \mathbb{Z}^\nu$ ,  $1 \leq l \leq k$ . But the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  have the additional property that they are concentrated in the cube  $[-N\pi, N\pi]^{k\nu}$ . Lemma 8.4 of [10] can be applied, and it shows that relation (5.4) and the continuity of the limit function  $\varphi_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k)$  together with the above mentioned concentration property of the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  imply the weak convergence of the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  to a finite measure  $\mu_{j_1, \dots, j_k}^{(0)}$ . This result also implies that this finite measure  $\mu_{j_1, \dots, j_k}^{(0)}$  has the Fourier transform  $\varphi_{j_1, \dots, j_k}^{(0)}(t_1, \dots, t_k)$ .  $\square$

To prove Theorem 3.3 with the help of Theorem 3.2 it is enough to show that if a function  $H^{(1)}(\cdot)$  satisfies (3.7) and (3.8), and the Gaussian stationary random field  $X(p) = (X_1(p), \dots, X_d(p))$  satisfies (3.1) and (3.2), then

$$\frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}} \sum_{p \in B_N} H^{(1)}(X_1(p), \dots, X_d(p)) \Rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad (5.5)$$

where  $\Rightarrow$  denotes convergence in probability. I shall prove that even the second moments of the normalized sums in (5.5) tend to zero as  $N \rightarrow \infty$ . The following Lemma 5A which agrees with Lemma 1 of [1] (only with a slightly different notation) helps in the proof of this statement.

**Lemma 5A.** *Let  $X = (X_1, \dots, X_d)$  and  $Y = (Y_1, \dots, Y_d)$  be two Gaussian random vectors with expectation zero such that  $EX_j X_{j'} = EY_j Y_{j'} = \delta_{j,j'}$ ,  $1 \leq j, j' \leq d$ , and let  $r_{j,j'} = EX_j Y_{j'}$ ,  $1 \leq j, j' \leq d$ . Take a number  $k \geq 1$  and a function  $H^{(1)}(x_1, \dots, x_d)$  that satisfies relations (3.7) and (3.8). Assume that*

$$\psi = \max \left( \left( \sup_{1 \leq j \leq d} \sum_{j'=1}^d |r_{j,j'}| \right), \left( \sup_{1 \leq j' \leq d} \sum_{j=1}^d |r_{j,j'}| \right) \right) \leq 1.$$

Then

$$|EH^{(1)}(X_1, \dots, X_d)H^{(1)}(Y_1, \dots, Y_d)| \leq \psi^{k+1} E [H^{(1)}(X_1, \dots, X_d)]^2.$$

*Proof of Theorem 3.3.* It follows from relations (3.1), (3.2) and Lemma 5A together with the inequality  $E [H^{(1)}(X_1(0), \dots, X_d(0))]^2 < \infty$  which holds because of (3.8) that for two elements  $X(p) = (X_1(p), \dots, X_d(p))$  and  $X(q) = (X_1(q), \dots, X_d(q))$ ,  $p, q \in \mathbb{Z}^\nu$ , of our vector-valued Gaussian stationary random field there exists some threshold index  $n_0 \geq 1$  and constant  $0 < C < \infty$  such that

$$\begin{aligned} & |EH^{(1)}(X_1(p), \dots, X_d(p))H^{(1)}(X_1(q), \dots, X_d(q))| \\ & \leq C|p - q|^{-(k+1)\alpha} L(|p - q|)^{k+1} \end{aligned}$$

if  $|p - q| \geq n_0$ . On the other hand,

$$\begin{aligned} & |EH^{(1)}(X_1(p), \dots, X_d(p))H^{(1)}(X_1(q), \dots, X_d(q))| \\ & \leq E [H^{(1)}(X_1(0), \dots, X_d(0))]^2 \leq C_1 \end{aligned}$$

for all  $p, q \in \mathbb{Z}^\nu$  with some  $C_1 < \infty$  by the Schwarz inequality and relation (3.8). Hence we get by summing up the above two inequalities for all  $q \in B_N$  with a fixed  $p \in B_N$ , and applying the first inequality if  $|p - q| > n_0$  and the second one if  $|p - q| \leq n_0$  that

$$\left| EH^{(1)}(X_1(p), \dots, X_d(p)) \left( \sum_{q \in B_N} H^{(1)}(X_1(q), \dots, X_d(q)) \right) \right| \leq C_2(1 + N^{\nu+\varepsilon-(k+1)\alpha})$$

for all  $p \in B_N$  and  $\varepsilon > 0$  with an appropriate  $C_2 = C_2(\varepsilon) > 0$ . Since  $\nu - k\alpha > 0$  we get by summing up the last inequality for all  $p \in B_N$  that

$$\frac{1}{N^{2\nu-k\alpha}L(N)^k} E \left[ \sum_{p \in B_N} H^{(1)}(X_1(p), \dots, X_d(p)) \right]^2 \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Indeed, it can be seen that for all  $\varepsilon > 0$  the expression in the last formula can be bounded from above by  $C(\varepsilon)N^{-\delta+\varepsilon}$  with  $\delta = \min(\nu - k\alpha, \alpha) > 0$  and a constant  $C(\varepsilon) > 0$  depending only on  $\varepsilon$ . This implies formula (5.5). Formula (5.5) together with Theorem 3.2 yields Theorem 3.3.  $\square$

*Proof of Theorem 3.4.* The proof of Theorem 3.4 is very similar to that of Theorems 3.2 and 3.3. Hence I only briefly explain it.

It is enough to show that for any positive integer  $K$ , parameters  $t_1, \dots, t_K$ ,  $t_p \in [0, \infty)^\nu$ ,  $1 \leq p \leq K$  and real constants  $C_1, \dots, C_K$  the linear combinations  $\sum_{p=1}^K C_p S_N(t_p)$  converge to  $\sum_{p=1}^K C_p S_0(t_p)$  in distribution as  $N \rightarrow \infty$ , since this implies that the random vectors  $(S_N(t_1), \dots, S_N(t_K))$  converge in distribution to the random vector  $(S_0(t_1), \dots, S_0(t_K))$  as  $N \rightarrow \infty$ . Moreover, similarly to the proof of Theorem 3.3 the proof of Theorem 3.4 can be reduced to the case  $H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d)$  with a function  $H^{(0)}(x_1, \dots, x_d)$  which satisfies relation (3.3).

In the first step of the proof the linear combinations  $\sum_{p=1}^K C_p S_N(t_p)$ ,  $N = 0, 1, 2, \dots$ , are written in the form of a sum of  $k$ -fold Wiener-Itô integrals with respect to the coordinates of an appropriate vector-valued random spectral measure. This can be done, first by writing the random variables  $S_N(t)$  for all  $t \in [0, 1\infty)^\nu$  in the desired form. The random variables  $S_0(t)$  are written in such a form in (3.11). In the case  $N = 1, 2, \dots$  the right expression of  $S_N(t)$  in the form of a sum of Wiener-Itô integrals can be found similarly to

the method applied in the proof of Theorem 3.2. We can write, similarly to the proof of formulas (4.2) and (4.3)

$$S_N(t) = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} \int c_{k_1, \dots, k_d} f^N(t, x_1 + \dots + x_k) \\ \prod_{j=1}^d \left( \prod_{l=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G^{(N)}, j}(dx_l) \right)$$

with

$$f^N(t, x) = \prod_{l=1}^{\nu} \frac{\exp \left\{ i \frac{[t^{(l)}N]}{N} x^{(l)} \right\} - 1}{N (\exp \{ i \frac{1}{N} x^{(l)} \} - 1)},$$

where  $x = (x^{(1)}, \dots, x^{(\nu)}) \in \mathbb{R}^{\nu}$ ,  $t = (t^{(1)}, \dots, t^{(\nu)}) \in \mathbb{R}^{\nu}$ , the number  $[t^{(l)}N]$  in the definition of the function  $f^N(t, x_1, \dots, x_k)$  is the smallest integer which is not smaller than  $t^{(l)}N$ , and  $Z_{G^{(N)}, j}$  agrees with the spectral measure that appeared in formula (4.2).

It is not difficult to see that, similarly to relations (4.4) and (4.5) the limit relation

$$\lim_{N \rightarrow \infty} f^N(t, x_1 + \dots + x_k) = f^0(t, x_1 + \dots + x_k)$$

holds with the function

$$f^0(t, x) = \prod_{l=1}^{\nu} \frac{e^{it^{(l)}x^{(l)}} - 1}{ix^{(l)}}$$

for all  $(x_1, \dots, x_k) \in \mathbb{R}^{k\nu}$ , and for a fixed parameter  $t \in \mathbb{R}^{\nu}$  this convergence is uniform in all bounded subsets of  $\mathbb{R}^{k\nu}$ .

With the help of the above considerations the proof of Theorem 3.4 can be reduced, similarly to the proof of Theorem 3.2 to the following statement.

Fix some number  $K$ , real constants  $C_1, \dots, C_K$  and points  $t_1, \dots, t_K$  with  $t_p \in [0, \infty)^{\nu}$ ,  $1 \leq p \leq K$  together with some constants  $c_{k_1, \dots, k_d}$  with parameters  $k_j \geq 0$ ,  $1 \leq j \leq d$ , and  $k_1 + \dots + k_d = k$  which agree with the coefficients



in the sum (3.3). Let us define with their help the random sums

$$S_N = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} \int \left( \sum_{p=1}^K C_p c_{k_1, \dots, k_d} f^N(t_p, x_1 + \dots + x_k) \right) \prod_{j=1}^d \left( \prod_{l=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G^{(N)}, j}(dx_l) \right) \quad (5.6)$$

with the above defined functions  $f^{(N)}(t, x_1, \dots, x_k)$  for all  $N = 1, 2, \dots$ , and

$$S_0 = \sum_{\substack{(k_1, \dots, k_d), k_j \geq 0, 1 \leq j \leq d, \\ k_1 + \dots + k_d = k}} \int \left( \sum_{p=1}^K C_p c_{k_1, \dots, k_d} f^0(t_p, x_1 + \dots + x_k) \right) \prod_{j=1}^d \left( \prod_{l=k_1 + \dots + k_{j-1} + 1}^{k_1 + \dots + k_j} Z_{G^{(0)}, j}(dx_l) \right) \quad (5.7)$$

with the previously defined function  $f^0(t, x_1, \dots, x_k)$ . The sequence of random variables  $S_N$  defined in (5.6) converge in distribution to  $S_0$  defined in (5.7) as  $N \rightarrow \infty$ .

This statement can be proved, similarly to Theorem 3.2 with the help of Proposition 4A. First the random variables  $S_N$ ,  $N = 1, 2, \dots$ , and  $S_0$  must be rewritten in a form in which Proposition 4A can be applied. They can be rewritten in the form of a sum of multiple Wiener–Itô integrals indexed by sequences of integers  $j_1, \dots, j_k$  such that  $1 \leq j_1 \leq \dots \leq j_k \leq d$ . The random variable  $S_N$  in (5.6) can be rewritten as

$$S_N = \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_1 \leq \dots \leq j_k \leq d}} \int \left( \sum_{p=1}^K C_p c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^N(t_p, x_1 + \dots + x_k) \right) Z_{G^{(N)}, j_1}(dx_1) \dots Z_{G^{(N)}, j_k}(dx_k) \quad (5.8)$$

for all  $N = 1, 2, \dots$ , and the random variable in (5.7) as

$$S_0 = \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_1 \leq \dots \leq j_k \leq d}} \int \left( \sum_{p=1}^K C_p c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^0(t_p, x_1 + \dots + x_k) \right) Z_{G^{(0)}, j_1}(dx_1) \dots Z_{G^{(0)}, j_k}(dx_k), \quad (5.9)$$

where the indices  $k_s(j_1, \dots, j_k)$ ,  $1 \leq s \leq d$ , are defined in (4.10).

The random integrals in formulas (5.8) and (5.9) have kernel functions of the form

$$\begin{aligned} h_{j_1, \dots, j_k}^N(x_1, \dots, x_k) &= h_{j_1, \dots, j_k, t_1, \dots, t_K}^N(x_1, \dots, x_k) \\ &= \sum_{p=1}^K C_p c_{k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k)} f^N(t_p, x_1 + \dots + x_k) \end{aligned} \quad (5.10)$$

for all  $N = 0, 1, 2, \dots$ . Let us define for all  $N = 0, 1, 2, \dots$  the measures  $\mu_{j_1, \dots, j_k}^{(N)}$  as

$$\begin{aligned} \mu_{j_1, \dots, j_k}^{(N)}(A) &= \mu_{j_1, \dots, j_k, t_1, \dots, t_K}^{(N)}(A) \\ &= \int_A |h_{j_1, \dots, j_k, t_1, \dots, t_K}^N(x_1, \dots, x_k)|^2 G_{j_1, j_1}^{(N)}(dx_1) \dots G_{j_k, j_k}^{(N)}(dx_k) \end{aligned} \quad (5.11)$$

where the integral is taken for all measurable sets  $A \in \mathcal{B}^{k\nu}$ .

We want to show with the help of Proposition 4A that the distributions of the random variables  $S_N$ ,  $N = 1, 2, \dots$ , defined in (5.8) converge weakly to the distribution of the random variable  $S_0$  defined in (5.9). (In particular, the sum of Wiener–Itô integrals in (5.9) is meaningful.) This implies Theorem 3.4.

To prove this convergence we have to show that the functions  $h_{j_1, \dots, j_k}^N$ ,  $N = 0, 1, 2, \dots$ , defined in (5.10) and the measures  $(G_{j,j}^{(N)})_{1 \leq j \leq d}$ ,  $N = 0, 1, 2, \dots$ , satisfy the conditions of Proposition 4A. The main point is to prove relation (4.8) in condition (b) of Proposition 4A. To prove this we show that the measures  $\mu_{j_1, \dots, j_k}^{(N)}$ ,  $N = 1, 2, \dots$ , defined in (5.11) are tight, i.e. for all  $\varepsilon > 0$  there exists a  $T = T(\varepsilon, j_1, \dots, j_k, t_1, \dots, t_K)$  such that

$$\mu_{j_1, \dots, j_k, t_1, \dots, t_K}^{(N)}(\mathbb{R}^{k\nu} \setminus [-T, T]^{k\nu}) < \varepsilon \quad \text{for all } N = 1, 2, \dots$$

Because of the Schwarz inequality and the definition of the functions  $h_{j_1, \dots, j_k, t_1, \dots, t_K}^N(x_1, \dots, x_k)$  the proof of this tightness property can be reduced to the justification of the subsequent inequality written in formula (5.12).

To formulate this inequality let us define for all  $t = (t^{*1}), \dots, t^{(\nu)} \in$

$[0, \infty)^\nu$ , and  $N = 1, 2, \dots$  the measure  $\mu_{N,t}$  on  $\mathbb{R}^{k\nu}$  by the formula

$$\begin{aligned}\mu_{N,t}(A) &= \int_A |f^N(t, x_1 + \dots + x_k)|^2 G_{j_1, j_1}^{(N)}(dx_1) \dots G_{j_k, j_k}^{(N)}(dx_k) \\ &= \int_A \left| \prod_{l=1}^\nu \frac{\exp \left\{ i \frac{[t^{(l)}N]}{N} (x_1^{(l)} + \dots + x_k^{(l)}) \right\} - 1}{N \left( \exp \left\{ i \frac{1}{N} (x_1^{(l)} + \dots + x_k^{(l)}) \right\} - 1 \right)} \right|^2 \\ &\quad G_{j_1, j_1}^{(N)}(dx_1) \dots G_{j_k, j_k}^{(N)}(dx_k)\end{aligned}$$

for all  $A \in \mathcal{B}^{k\nu}$ . The inequality

$$\mu_{N,t}(\mathbb{R}^{k\nu} \setminus [-T, T]^{k\nu}) < \varepsilon \quad (5.12)$$

holds for all  $N = 1, 2, \dots$ , if  $T \geq T_0(\varepsilon, t)$  with an appropriate threshold index  $T_0(\varepsilon, t) > 0$ .

I claim that the measures  $\mu_{N,t}$  converge weakly to a measure  $\mu_{0,t}$  on  $\mathbb{R}^{k\nu}$  as  $N \rightarrow \infty$ . This convergence implies the above inequality. This convergence can be proved similarly to Proposition 5.1. Namely, we can write

$$\begin{aligned}&\prod_{l=1}^\nu \frac{\exp \left\{ i \frac{[t^{(l)}N]}{N} (x_1^{(l)} + \dots + x_k^{(l)}) \right\} - 1}{N \left( \exp \left\{ i \frac{1}{N} (x_1^{(l)} + \dots + x_k^{(l)}) \right\} - 1 \right)} \\ &= \frac{1}{N^\nu} \sum_{u \in B_N(t)} \exp \left\{ i \frac{1}{N} (u, x_1 + \dots + x_k) \right\},\end{aligned}$$

where the set  $B_N(t)$  was defined in (3.9). With the help of this formula the Fourier transform of the measure  $\mu_{N,t}$  can be calculated in all points of the form  $u = (u_1, \dots, u_k)$ ,  $u_s = \frac{p_s}{N}$ ,  $p_s \in \mathbb{Z}^\nu$ ,  $1 \leq s \leq k$ . This can be done similarly to the corresponding calculation in Proposition 5.1. Then a good asymptotic formula can be proved for this Fourier transform with the help of relation (3.1), and this implies the above mentioned convergence. Here again the method of proof in Proposition 5.1 is applied. I omit the details.

This implies that condition (b) of Proposition 4A holds in our model. The proof of the remaining conditions is much simpler. Similarly to the proof of Theorem 3.2 it can be shown with the help of Proposition 3.1 that the spectral measures  $(G_{j, j'}^{(N)})_{1 \leq j, j' \leq d}$  satisfy the required convergence property. Finally, it is not difficult to check that the functions  $h_{j_1, \dots, j_k}^N$  defined in (5.10) satisfy condition (a) of Proposition 4A.  $\square$

## Appendix A. On the background of the limit theorems of this paper.

In Theorem 3.4 a limit theorem was proved for the normalized partial sums of the elements of a stationary random field. The limit was a self-similar random field with stationary increments. Similar results may hold in the general case. It is natural to expect that if the normalized partial sums of the random variables in a stationary random field have a limit, then this limit is a self-similar random field with stationary increments. Hence in the study of limit theorems for partial sums of a stationary random field it is useful first to look for self-similar random fields with stationary increments, and then to find a large class of stationary random fields, for which the limit of the (appropriately) normalized random sums belong to one of these self-similar random fields with stationary increments. Such an approach was followed in paper [7] where limit-theorems were proved with limits that belong to the class of self-similar random fields constructed before in paper [6]. In Appendix A I explain how the vector-valued versions of the self-similar random fields with stationary increments appearing in [6] can be constructed, and how limit theorems can be proved where these random fields appear as the limit.

The self-similar random fields with stationary increments will be constructed with the help of such spectral measures  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  which have the homogeneity property  $G_{j,j'}(tA) = t^\alpha G_{j,j'}(A)$ ,  $1 \leq j, j' \leq d$ , with some parameter  $\alpha > 0$  for all  $t > 0$  and measurable, bounded sets  $A \subset \mathbb{R}^\nu$ . This homogeneity property will imply that the random fields we construct with their help will be self-similar. On the other hand, the coordinates  $G_{j,j}$  of a homogeneous spectral measure  $G$  are not integrable functions. As a consequence,  $G$  cannot be the spectral measure of an ordinary stationary random field, and we have to work with the spectral measure of generalized, stationary random fields. Some properties of generalized random fields must be recalled in our construction. Next I explain its details. First I construct Gaussian, and then non-Gaussian self-similar random fields with stationary increments.

We consider a generalized stationary random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to the spectral measure  $G$  we are working with. The Gaussian generalized random field corresponding to this random spectral measure  $Z_G$  consists of the random vectors

$$Z_G(\varphi) = (Z_{G,1}(\varphi), \dots, Z_{G,d}(\varphi))$$

with coordinates  $Z_{G,j}(\varphi) = \int \tilde{\varphi}(x) Z_{G,j}(dx)$ ,  $1 \leq j \leq d$ , where the function  $\varphi(\cdot)$  is an element of the Schwartz space  $\mathcal{S}$ , and  $\tilde{\varphi}$  denotes its Fourier transform. On the other hand, the random vectors  $Z_G(\varphi)$  can be defined for a larger class of arguments. They can be defined for those real-valued function  $\varphi$  on  $\mathbb{R}^\nu$  for which the random integrals  $Z_{G,j}(\varphi) = \int \tilde{\varphi}(x) Z_{G,j}(dx)$  exist, i.e.  $\int |\tilde{\varphi}(x)|^2 G_{j,j}(dx) < \infty$  for all  $1 \leq j \leq d$ .

We are interested in those generalized spectral measures  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  for which the random vector  $Z_G(\varphi)$  exists for all those functions  $\varphi$  which are the indicator functions  $\varphi_{s,t} = I((s, s+t])$  of some rectangle  $(s, s+t] \subset \mathbb{R}^\nu$  with  $s = (s^{(1)}, \dots, s^{(\nu)}) \in \mathbb{R}^\nu$  and  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$ . These random vectors exist if the spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  satisfies the inequality

$$\int |I(\widetilde{(s, s+t]})(x)|^2 G_{j,j}(dx) = \int \left( \prod_{l=1}^{\nu} \frac{2(1 - \cos(t^{(l)} x^{(l)}))}{(x^{(l)})^2} \right) G_{j,j}(dx) < \infty \quad (\text{A.1})$$

for all  $t \in [0, \infty)^\nu$  and  $1 \leq j \leq d$ .

If condition (A.1) holds for a spectral measure  $G$ , then we take a random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to it, and define the random vectors

$$Z^{(G)}(t) = (Z_1^{(G)}(t), \dots, Z_d^{(G)}(t)), \quad 1 \leq j \leq d, \quad t \in [0, \infty)^\nu, \quad (\text{A.2})$$

with  $Z_j^{(G)}(t) = Z_{G,j}(I((0, t]))$ ,  $t \in [0, \infty)^\nu$ ,  $1 \leq j \leq d$ . These random vectors define a vector-valued Gaussian random field with stationary increments. The property of stationary increments follows from the argument at the end of Section 3. (See in particular formulas (3.12) and (3.13) in it.) Moreover, I claim that if the spectral measure  $G$  has the homogeneity property with some parameter  $\alpha$ , then this random field is self-similar with self-similarity parameter  $\frac{\nu}{2} - \alpha$ .

To see this self-similarity observe that because of the homogeneity property of  $G$  we get for any  $u > 0$  that the set of random vectors

$$Z_G^{(u)}(A) = (Z_{G,1}^{(u)}(A), \dots, Z_{G,d}^{(u)}(A))$$

defined by the formula  $Z_{G,j}^{(u)}(A) = u^{-\alpha/2} Z_{G,j}(uA)$ ,  $1 \leq j \leq d$ , for all bounded, measurable sets  $A \subset \mathbb{R}^\nu$  define a random spectral measure  $Z_G^{(u)} = (Z_{G,1}^{(u)}, \dots, Z_{G,d}^{(u)})$  which corresponds to the same spectral measure  $G$

as the random spectral measure  $Z_G$ . Hence

$$\begin{aligned}
Z_j^{(G)}(ut) &= Z_{G,j}(I((0, ut])) = \int \widetilde{I((0, ut])}(x) Z_{G,j}(dx) \\
&= u^\nu \int \widetilde{I((0, t])}(ux) Z_{G,j}(dx) \\
&= u^{\nu-\alpha/2} \int \widetilde{I((0, t])}(x) Z_{G,j}^{(u)}(dx) \triangleq u^{\nu-\alpha/2} Z_j^{(G)}(t)
\end{aligned}$$

for all  $t \in \mathbb{R}^\nu$ ,  $u > 0$  and  $1 \leq j \leq d$ , where  $\triangleq$  means that the expressions at the two sides of the equation have the same distribution. (In the above construction of the random field  $Z^{(G)}(t)$ ,  $t \in [0, \infty)^\nu$ , I adapted the arguments of [6] to this problem, but I omitted an important point from the considerations of that paper. I have not introduced a topology in the space of functions, and have not investigated the continuity of the functionals with respect to it.)

It can be proved similarly, that for any  $K > 0$ , vectors  $t_l \in \mathbb{R}^\nu$ , real valued coefficients  $c_l$ , indices  $1 \leq j_l \leq d$  for all  $1 \leq l \leq K$  and number  $u > 0$  the following relation holds. The random variables  $\sum_{l=1}^K c_l Z_{j_l}^{(G)}(t_l)$  and  $u^{\nu-\alpha/2} \sum_{l=1}^K c_l Z_{j_l}^{(G)}(ut_l)$  have the same distribution. This means that the random field  $Z^{(G)}(t)$ ,  $t \in [0, \infty)^\nu$ , defined in (A.2) with the help of a spectral measure  $G$  with the above properties is self-similar with self-similarity parameter  $\nu - \frac{\alpha}{2}$ .

Let us introduce the random vector  $\mathbf{e} = (1, \dots, 1) \in \mathbb{Z}^\nu$ , and define the vector-valued discrete random field  $X(p) = (X^{(1)}(p), \dots, X^{(d)}(p))$ ,  $p \in \mathbb{Z}^\nu$ , with coordinates  $X^{(j)}(p) = Z_{G,j}(I((p - \mathbf{e}, p]))$ ,  $1 \leq j \leq d$ , where  $G$  is a homogeneous spectral measure with homogeneity parameter  $\alpha$  which satisfies relation (A.1). This is a Gaussian stationary random field, and  $\sum_{p \in B_N} X(p) = Z^{(G)}(N\mathbf{e})$ .

This relation together with the self-similarity property of the random field  $Z^{(G)}(t)$ ,  $t \in [0, \infty)^\nu$ , imply that for any real-valued coefficients  $c_j$ ,  $1 \leq j \leq \nu$ , the normalized random sums  $\frac{1}{N^{\nu-\alpha/2}} \sum_{p \in B_N} Y(p)$  with  $Y(p) = \sum_{j=1}^d c_j X^{(j)}(p)$ ,  $p \in \mathbb{Z}^\nu$ , have the same distribution for any  $N = 1, 2, \dots$ . This means that these random variables  $Y(p)$  satisfy Theorem 3.2 with  $k = 1$  and  $L(p) \equiv 1$ . It can

be seen similarly that these random variables  $Y(p)$  also satisfy Theorem 3.4 with  $k = 1$  and  $L(t) \equiv 1$ .

It remains to decide when a homogeneous spectral measure  $G$  satisfies condition (A.1). We get an example for it with the help of the spectral measure  $G$  of a stationary random field whose covariance function satisfies relation (3.1) with  $0 < \alpha < \nu$ . In Proposition 3.1 a homogeneous spectral measure  $G^{(0)}$  with homogeneity parameter  $\alpha$  was constructed with the help of this spectral measure  $G$ . Moreover, by Theorem 3.4 the random variables  $S_0(t)$  defined in formula (3.11) of Theorem 3.4 in the case  $k = 1$  with the help of the spectral measure  $G^{(0)}$  exist for all  $t \in [0, \infty)^\nu$ , and this implies that the above spectral measure  $G^{(0)}$  satisfies relation (A.1).

Actually, it is rather difficult to find a complete description of the homogeneous spectral measures satisfying relation (A.1). But there are homogeneous spectral measures which do not belong to the class of homogeneous spectral measures constructed in Proposition 3.1.

To find spectral measures that satisfy relation (A.1) observe that the inequality

$$\prod_{l=1}^{\nu} \frac{2(1 - \cos(t^{(l)}x^{(l)}))}{(x^{(l)})^2} \leq K_t \prod_{l=1}^{\nu} \frac{1}{1 + (x^{(l)})^2}$$

holds with some constant  $K_t < \infty$  depending on  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, \infty)^\nu$  for all  $x \in \mathbb{R}^\nu$ . Hence relation (A.1) holds for all  $t \in [0, \infty)$  and  $1 \leq j \leq d$  if

$$\int \left( \prod_{l=1}^{\nu} \frac{1}{1 + (x^{(l)})^2} \right) G_{j,j}(dx) < \infty \quad \text{for all } 1 \leq j \leq d. \quad (\text{A.3})$$

It can be shown that for any  $0 < \alpha < \nu + 1$  the homogeneous spectral measures  $G$  whose elements  $G_{j,j'}$ ,  $1 \leq j, j' \leq d$ , have a density function of the form  $g_{j,j'}(x) = |x|^{\alpha-\nu} a_{j,j'}\left(\frac{x}{|x|}\right)$  with a bounded function  $a_{j,j'}(\cdot)$  on the unit sphere of  $\mathbb{R}^\nu$  have homogeneity parameter  $\alpha$ , and they satisfy relation (A.3), hence relation (A.1), too. To see this observe that in the case of such a density function relation (A.3) means that  $\int \left( \prod_{l=1}^{\nu} \frac{1}{1 + (x^{(l)})^2} \right) |x|^{\alpha-\nu} dx < \infty$ .

If  $0 < \alpha < \nu$ , then this inequality holds since in this case  $|x|^{\alpha-\nu} \leq C \prod_{l=1}^{\nu} |x^{(l)}|^{(\alpha-\nu)\nu}$  with some constant  $C > 0$ . If  $\nu \leq \alpha < \nu + 1$ , then the

inequality  $|x|^{\alpha-\nu} < \left( \prod_{l=1}^{\nu} (|x^{(l)}|^2 + 1) \right)^{(\alpha-\nu)/2}$  provides this inequality.

The above example implies that there are homogeneous spectral measures with homogeneity parameter  $\nu \leq \alpha < \nu + 1$  which satisfy relation (A.1). As a consequence, there are (self-similar Gaussian) random fields with stationary increments which satisfy Theorem 3.4 for  $k = 1$  with norming constant  $N^{\nu - \frac{\alpha}{2}}$  also in the case  $\nu \leq \alpha < \nu + 1$ . In this case the exponent in the norming constant is  $\nu - \frac{\alpha}{2} < \frac{\nu}{2}$ . Thus we got limit theorems which do not appear among the results of this paper.

By a natural adaptation of the previous method a larger class of self-similar random fields with stationary increments can be constructed with the help of homogeneous spectral measures when multiple Wiener–Itô integrals are applied in the construction. The random fields appearing as the limit field in Theorem 3.4 also belong to this class of random fields.

In this construction we fix some integer  $k \geq 1$ , take a spectral measure  $(G_{j,j'}(\cdot))_{1 \leq j,j' \leq d}$  which has the homogeneity property with some parameter  $\alpha$ , and consider a random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to this spectral measure. We define with the help of this random spectral measure  $Z_G$  the random variables  $Z_{G,j_1, \dots, j_k}((s, s+t])$  for all rectangles  $(s, s+t]$  with  $s \in \mathbb{R}^\nu$ ,  $t \in [0, \infty)^\nu$ , and indices  $1 \leq j_u \leq d$  for all  $1 \leq u \leq k$  by means of the  $k$ -fold Wiener–Itô integral

$$\begin{aligned} Z_{G,j_1, \dots, j_k}((s, s+t]) &= \int I(\widetilde{(s, s+t]})(x_1 + \dots + x_k) \\ &\quad Z_{G,j_1}(dx_1) \dots Z_{G,j_k}(dx_k) \\ &= \int \left( \prod_{l=1}^{\nu} \frac{e^{i(s^{(l)}+t^{(l)})(x_1^{(l)}+\dots+x_k^{(l)})} - e^{is^{(l)}(x_1^{(l)}+\dots+x_k^{(l)})}}{i(x_1^{(l)} + \dots + x_k^{(l)})} \right) \\ &\quad Z_{G,j_1}(dx_1) \dots Z_{G,j_k}(dx_k), \end{aligned}$$

provided that this Wiener–Itô integral exists, i.e. if

$$\begin{aligned} K(j_1, \dots, j_k, t) &= \int \left| I(\widetilde{(s, s+t]})(x_1 + \dots + x_k) \right|^2 \\ &\quad G_{j_1}(dx_1) \dots G_{j_k}(dx_k) \\ &= \int \left( \prod_{l=1}^{\nu} \frac{2 \left( 1 - \cos(t^{(l)}(x_1^{(l)} + \dots + x_k^{(l)})) \right)}{\left( x_1^{(l)} + \dots + x_k^{(l)} \right)^2} \right) \\ &\quad G_{j_1}(dx_1) \dots G_{j_k}(dx_k) < \infty \quad (\text{A.4}) \end{aligned}$$



for all  $t \in [0, \infty)$  and  $1 \leq j_u \leq d$ ,  $1 \leq u \leq k$ .

For a spectral measure  $G$  of a generalized random field which satisfies relation (A.4) take a random spectral measure  $Z_G = (Z_{G,1}, \dots, Z_{G,d})$  corresponding to it, and define with its help the following vector-valued random field  $Z^{(G)}(t)$  for  $t \in [0, \infty)^\nu$  with values in  $\mathbb{R}^{d^k}$ :

$$Z^{(G)}(t) = \{Z_{j_1, \dots, j_k}^{(G)}(t), \quad 1 \leq j_u \leq d, \text{ for } 1 \leq u \leq k\}$$

with  $Z_{j_1, \dots, j_k}^{(G)}(t) = Z_{G, j_1, \dots, j_k}((0, t])$  for  $t \in [0, \infty)$ .

It can be seen similarly to the case  $k = 1$ , that  $Z^{(G)}(t)$  is a vector-valued random field on  $[0, \infty)^\nu$  with stationary increments, and if the spectral measure  $G$  has the homogeneity property with parameter  $\alpha$ , then  $Z^{(G)}(\cdot)$  is self-similar with parameter  $\nu - \frac{k\alpha}{2}$ .

Let us define (similarly to the definition of the discrete valued random field  $X(p)$ ,  $p \in \mathbb{Z}^\nu$ , in the case  $k = 1$ ), the discrete vector-valued random field  $S(p)$  in the case  $k \geq 2$  by the formula

$$S(p) = \{S^{(j_1, \dots, j_k)}(p), \quad 1 \leq j_u \leq d \text{ for all } 1 \leq u \leq k\}, \quad p \in \mathbb{Z}^\nu,$$

with  $S^{(j_1, \dots, j_k)}(p) = Z_{G, j_1, \dots, j_k}((p - \mathbf{e}, p])$ . Take arbitrary real-valued coefficients  $c_{j_1, \dots, j_k}$  with  $1 \leq j_u \leq d$  for all  $1 \leq u \leq k$ , and define with their help the random field  $Y(p)$ ,  $p \in \mathbb{Z}^\nu$ , by the formula

$$Y(p) = \sum_{\substack{(j_1, \dots, j_k): \\ \text{for } 1 \leq u \leq k \\ 1 \leq j_u \leq d}} c(j_1, \dots, j_k) S^{(j_1, \dots, j_k)}(p).$$

It can be seen similarly to the case  $k = 1$ , that if the spectral measure  $G$  is homogeneous with parameter  $\alpha$ , then  $Z(t)$  is self-similar with self-similarity parameter  $\nu - \frac{k\alpha}{2}$ , and the normalized partial sums made with the help of the stationary random field  $Y(p)$ ,  $p \in \mathbb{Z}^\nu$ , satisfy Theorem 3.4 with the norming constants  $N^{\nu - k\alpha/2}$ .

It remains to check when a homogeneous spectral measure  $G$  satisfies relation (A.4). Proposition 3.1 together with Theorem 3.4, more explicitly with its statement about the existence of the multiple Wiener–Itô integrals defined in (3.11) yield an example for such spectral measures. To get them, let us first fix some integer  $k \geq 1$ . If the covariance function of a stationary random field satisfies relation (3.1) with  $0 < \alpha < \frac{\nu}{k}$ , then by applying an appropriate vague limit procedure for the spectral measure of this random

field a spectral measure  $G^{(0)}$  is defined in Proposition 3.1, and it satisfies relation (3.5), i.e. it is homogeneous with parameter  $\alpha$ . Moreover, by the existence of the expression (3.11) it also satisfies relation (A.4) with this number  $k$ .

Homogeneous spectral measures that satisfy relation (A.4) with  $k$ -fold integrals for a fixed  $k$  can be defined also in the following way.

Let us consider a spectral measure  $G = (G_{j,j'})_{1 \leq j,j' \leq d}$  whose coordinates  $G_{j,j'}$ ,  $1 \leq j,j' \leq d$ , have density functions of the form  $g_{j,j'}(x) = |x|^{\alpha-\nu} a_{j,j'}\left(\frac{x}{|x|}\right)$ ,  $x \in \mathbb{R}^\nu$ , where the functions  $a_{j,j'}(\cdot)$  are bounded functions on the unit circle of  $\mathbb{R}^\nu$ , and let  $0 < \alpha < \frac{\nu}{k}$  with some number  $k$ . Then the spectral measure  $G$  is homogeneous with homogeneity parameter  $\alpha$ , and I claim that this spectral measure satisfies relation (A.4) if  $k$ -fold integrals are taken in it. I present a short explanation of this result, but without a full proof.

An estimate will be proved with the help of the following inequality. If  $\alpha < \nu$ , then  $|x|^{\alpha-\nu} \leq C \prod_{l=1}^{\nu} |x^{(l)}|^{-1+\alpha/\nu}$  with a constant  $C$  depending only on  $\alpha$  and  $\nu$ . It can be proved with the help of this inequality that

$$\begin{aligned} K(j_1, \dots, j_k, t) &\leq C \int \left( \prod_{l=1}^{\nu} \frac{1}{\left(1 + \left(x_1^{(l)} + \dots + x_k^{(l)}\right)^2\right)} \right) \\ &\quad |x_1|^{\alpha-\nu} \dots |x_k|^{\alpha-\nu} dx_1 \dots dx_k \\ &\leq C' \prod_{l=1}^{\nu} \left( \int \frac{|x_1^{(l)}|^{-1+\alpha/\nu} \dots |x_k^{(l)}|^{-1+\alpha/\nu}}{\left(1 + \left(x_1^{(l)} + \dots + x_k^{(l)}\right)^2\right)} dx_1^{(l)} \dots dx_k^{(l)} \right). \end{aligned}$$

It can be proved that the integrals at the right-hand side of this formula are finite if  $-1 + \frac{\alpha}{\nu} < -1 + \frac{1}{k}$  for the exponents in this expression, i.e. if  $\alpha < \frac{\nu}{k}$ . I omit the proof of this statement, but I remark that it implies that under the above conditions  $K(j_1, \dots, j_k, t) < \infty$ , hence relation (A.4) holds.

The goal of Appendix A was to explain the background of the results in this paper. First I concentrated on the explanation of the definition of the self-similar random fields which appear as the limit in our limit theorems. This was based on the theory of multiple Wiener–Itô integrals for several di-

mensional stationary Gaussian random fields, in particular on the properties of the random spectral measures of generalized Gaussian random fields.

The proof of the limit theorems consisted of two steps. In the first step the normalized random sums we wanted to study were rewritten as sums of Wiener–Itô integrals. This could be done with the help of the multivariate version of Itô’s formula formulated in Theorem 2.2 of [12]. Then it was useful to rewrite the sums of these random integrals in a more appropriate rescaled form which suggests what kind of limit theorems can be expected. These steps of the proof were done in Section 4 after the proof of Proposition 3.1.

These formulas together with the result of Proposition 3.1 gave a heuristic argument about the form of the limit theorems in this paper. But a rigorous proof demands a more careful analysis. This could be done with the help of Proposition 4A. In particular, the existence of the Wiener–Itô integrals appearing in the definition of the limit random field could be proved with the help of this proposition. Let me also remark that Proposition 4A could be applied for  $k$ -fold integrals only if formula (3.1) holds with  $0 < \alpha < \frac{\nu}{k}$ . This was the reason for this condition in Theorems 3.2–3.4 of this paper.

If the conditions of Theorems 3.2–3.4 hold, but with a parameter  $\alpha \geq \frac{\nu}{k}$ , then the random integrals defining the limit in these theorems do not exist. In such cases the central limit theorem holds with the classical normalization. This follows from the result of [3] in the scalar-valued and from its multivariate generalization in Theorem 4 of [1] in the vector-valued case. This problem is discussed in Appendix B of [12].

In this paper limit theorems were proved for non-linear functionals of stationary Gaussian random fields. I try to give a short overview about papers which deal with similar problems. The scalar-valued version of the results in this paper was proved in [7]. M. S. Taqqu proved similar results in paper [17]. Both papers contain non-central limit theorems for sequences of non-linear functionals of (scalar-valued) stationary Gaussian random fields. Taqqu’s result has no multivariate version, and I do not know how such a result can be proved.

Paper [3] contains a result which tells when the classical central limit theorem holds for sequences of non-linear functionals of stationary Gaussian random fields similar to those considered in [7]. The book [13] generalizes this result. A. M. Arcones wanted to generalize both the central limit theorem of [3] and the non-central limit theorem of [7] for non-linear functionals of vector-valued Gaussian random fields. He proved the central limit theorem

part of these results in Theorem 4 of his paper [1]. He claimed to have also proved the multivariate version of the non-central limit theorem in [7] in Theorem 6 of his paper. But I found the proof (and also the formulation) of this result problematic. The goal of the present paper was to formulate and prove the right multivariate version of the result in [7].

Sanchez de Naranjo also dealt with the problem of non-central limit theorems for non-linear functionals of vector-valued Gaussian random fields in paper [15]. His results seem to be correct, but the proof contains serious gaps. He formulated several results in the vector-valued case without proof. These results are plausible generalizations of their scalar-valued counterparts, but some of them cannot be proved by an adaptation of the proof in the scalar-valued case. Hence I consider these proofs incomplete.

Let me also remark that H. C. Ho and T. C. Sun proved an interesting result in [9]. They proved a result which can be considered as an interesting mixture of the central and non-central limit theorem for non-linear functionals of stationary Gaussian random fields. They considered a two-dimensional vector-valued stationary Gaussian random process  $(X_n, Y_n)$ ,  $n \in \mathbb{Z}$ , together with two non-linear functionals which are of such type as the non-linear functionals applied in [7]. They investigated the joint distribution of two random sequences which are obtained when linear functionals defined in the same way as in the earlier mentioned papers are applied both for the sequence  $X_n$ ,  $n \in \mathbb{Z}$ , and the sequence  $Y_n$ ,  $n \in \mathbb{Z}$ . In Theorem 1 of their paper they investigated the case when the first sequence satisfies a non-central and the second sequence satisfies a central limit theorem. They proved under some additional conditions that the joint distributions of these sequences also have a limit, and the two coordinates of this limit are independent. (The processes  $X_n$  and  $Y_n$  need not be independent.) I would remark that the usual proofs of the central and non-central limit theorem apply different methods. This indicates that the proof of this result in [9] demanded new ideas.

Donatas Surgailis proved results about similar problems, and they are also worth mentioning. He proved limit theorems for non-linear functionals of a new class of stationary stochastic processes which contains non-Gaussian processes, too. I would mention paper [16] from his works. Here he proved results whose proof demanded new ideas. The main point of his approach was the application of Appell polynomials. He showed that they have several properties similar to those of the Hermite polynomials, and he could prove new results with their help about models that could not be investigated by earlier methods.

Finally I briefly mention a field of research where similar limit theorems are proved with the help of essentially different arguments. This research deserves special attention because of its importance. This is the theory about the KPZ (Kardar–Parisi–Zhang) universality classes and the problems related to them. Many important problems can be studied with their help. On the other hand, the application of this theory demands hard analysis. An overview about it can be found in paper [4] together with a long list of literature.

## Appendix B. Proof of Corollary 3.5.

*Proof of Corollary 3.5.* We want to show that for all pairs  $\varepsilon > 0$  and  $\eta > 0$  there exists some  $\delta = \delta(\varepsilon, \eta) > 0$  and threshold index  $N_0 = N_0(\varepsilon, \eta)$  such that for all  $N \geq N_0$  the inequality

$$P \left( \sup_{\substack{(s,t): s,t \in [0,1]^\nu \\ |t-s| < \delta}} |S_N(t) - S_N(s)| > \varepsilon \right) < \eta \quad (\text{B.1})$$

holds for the random field  $S_N(t)$ ,  $t \in [0, 1]^\nu$ , defined in (3.9) and (3.10).

Inequality (B.1) means that the random fields  $S_N(t)$ ,  $t \in [0, 1]^\nu$ , introduced in Theorem 3.4 satisfy besides the limit theorem formulated in Theorem 3.4 also the tightness condition for probability measures in the space of continuous functions  $C([0, 1]^\nu, \mathcal{C})$ . It can be seen that these two properties together imply the desired functional limit theorem.

First I show that relation (B.1) can be replaced with the following set of simpler inequalities.

Define for all  $1 \leq j \leq \nu$  and  $\delta > 0$  the following set  $V(j, \delta)$  consisting of pairs of vectors  $(s, t)$ ,  $s \in [0, 1]^\nu$  and  $t \in [0, 1]^\nu$ .

$$V(j, \delta) = \{(s, t): s = (s^{(1)}, \dots, s^{(\nu)}), t = (s^{(1)}, \dots, s^{(j-1)}, t^{(j)}, s^{j+1}, \dots, s^{(\nu)}): \\ 0 \leq s^{(l)} \leq 1 \text{ for all } 1 \leq l \leq \nu, \text{ and } s^{(j)} \leq t^{(j)} \leq \min(1, s^{(j)} + \delta)\}.$$

Then

$$P \left( \sup_{(s,t) \in V(j,\delta)} |S_N(t) - S_N(s)| > \varepsilon \right) < \eta \quad \text{for all } 1 \leq j \leq \nu \quad (\text{B.2})$$

if  $\delta \leq \delta(\varepsilon, \eta)$  and  $N \geq N_0(\varepsilon, \eta)$  with some  $\delta(\varepsilon, \eta) > 0$  and  $N_0(\varepsilon, \eta)$ .

To see the possibility of such a reduction let us first observe that inequality (B.1) follows from its following formally weaker version.

For all  $\varepsilon > 0$  and  $\eta > 0$  there exists some  $\delta = \delta(\varepsilon, \eta) > 0$  and  $N_0 = N_0(\varepsilon, \eta)$  such that

$$P \left( \sup_{\substack{(s,t): 0 \leq s^{(j)} < t^{(j)} \leq 1, \\ \text{for all } 1 \leq j \leq \nu, |t-s| < \delta}} |S_N(t) - S_N(s)| > \varepsilon \right) < \eta \quad (\text{B.3})$$

if  $N \geq N_0$ .

Indeed, for a pair of vectors  $(s, t)$ ,  $s \in [0, 1]^\nu$ ,  $t \in [0, 1]^\nu$ , define the vector

$$s^* = s^*(s, t) = (\min(s^{(1)}, t^{(1)}), \dots, \min(s^{(\nu)}, t^{(\nu)})),$$

and consider the pairs  $(s^*, s)$  and  $(s^*, t)$ . Let us apply relation (B.3) with parameters  $\delta$  and  $N_0$  corresponding to the parameters  $\frac{\varepsilon}{2}$  and  $\frac{\eta}{2}$ . If the pair  $(s, t)$  satisfies the conditions appearing in the supremum of (B.1) with these parameters, then the pairs  $(s^*, s)$  and  $(s^*, t)$  satisfy the conditions in the supremum of (B.3) with the same parameter  $\delta$ . Also the relation  $|S_N(t) - S_N(s)| \leq |S_N(s) - S_N(s^*)| + |S_N(t) - S_N(s^*)|$  holds. Hence inequality (B.3) with the above chosen  $\delta$  and  $N_0$  implies that

$$\begin{aligned} \sup_{\substack{(s,t): s, t \in [0,1]^\nu \\ |t-s| < \delta}} |S_N(t) - S_N(s)| &\leq \sup_{\substack{(s^*,s): 0 \leq (s^*)^{(j)} < s^{(j)} \leq 1, \\ \text{for all } 1 \leq j \leq \nu, |s-s^*| < \delta}} |S_N(s) - S_N(s^*)| \\ &+ \sup_{\substack{(s^*,t): 0 \leq (s^*)^{(j)} < t^{(j)} \leq 1, \\ \text{for all } 1 \leq j \leq \nu, |t-s^*| < \delta}} |S_N(t) - S_N(s^*)| \leq \varepsilon \end{aligned}$$

with probability more than  $1 - \eta$  if  $N > N_0$ . This means that relation (B.3) implies relation (B.1).

Relation (B.2) can be reduced to relation (B.3) in a similar way. To do this let us first define for a pair of vectors  $s = (s^{(1)}, \dots, s^{(\nu)}) \in [0, 1]^\nu$ ,  $t = (t^{(1)}, \dots, t^{(\nu)}) \in [0, 1]^\nu$  and number  $1 \leq j \leq \nu$  the vector  $s(j) = s(j, s, t) = (t^{(1)}, \dots, t^{(j-1)}, s^{(j)}, \dots, s^{(\nu)})$  (for  $j = 1$   $s(1) = (s^{(1)}, \dots, s^{(\nu)}) = s$ ), and consider the pairs of vectors  $(s(j), s(j+1))$ ,  $1 \leq j \leq \nu - 1$ . Observe that  $(s(j), s(j+1)) = (s(j, s, t), s(j+1, s, t)) \in V(j, \delta)$  if the pair  $(s, t)$  satisfies the relations  $|t - s| < \delta$  and  $0 \leq s^{(j)} < t^{(j)} \leq 1$  for all  $1 \leq j \leq \nu$ . Let us choose a  $\delta > 0$  and  $N_0$  in such a way that inequality (B.2) holds with parameters  $\frac{\varepsilon}{\nu}$  and  $\frac{\eta}{\nu}$  with this number  $\delta$  and  $N \geq N_0$ . Let us take those

pairs of vectors  $(s, t)$  which satisfy the conditions imposed in the supremum of formula (B.3) with this number  $\delta$ . We have seen that

$$\begin{aligned} \{ & (s(j, s, t), s(j+1, s, t)) : 0 \leq s^{(l)} < t^{(l)} \leq 1, \\ & \text{for all } 0 \leq l \leq \nu, |t - s| \leq \delta \} \subset V(j, \delta). \end{aligned}$$

The identity  $S_N(t) - S_N(s) = \sum_{j=1}^{\nu-1} [S_N(s(j+1, s, t)) - S_N(s(j, s, t))]$  also holds.

These relations imply that with our choice of  $\delta$

$$\sup_{\substack{(s,t): 0 \leq s^{(j)} < t^{(j)} \leq 1, \\ \text{for all } 1 \leq j \leq \nu, |t-s| < \delta}} |S_N(t) - S_N(s)| \leq \sum_{j=1}^{\nu-1} \sup_{(s,t) \in V(j, \delta)} |S_N(t) - S_N(s)| \leq \varepsilon$$

with probability more than  $1 - \eta$  if  $N > N_0$ . This means that (B.2) implies (B.3).

Next I present an inequality with the help of the random variables  $H(X_1(p), \dots, X_d(p))$  instead of  $S_N(t)$  which implies inequality (B.2). For this goal I introduce the following notations.

For all pairs of vectors  $r = (r^{(1)}, \dots, r^{(\nu)})$  and  $s = (s^{(1)}, \dots, s^{(\nu)})$  with integer coordinates such that  $0 \leq r^{(j)} < s^{(j)} \leq N$  for all  $1 \leq j \leq \nu$  let us define the rectangle  $D_N(r, s)$  by the formula

$$\begin{aligned} D_N(r, s) = \{ & p = (p^{(1)}, \dots, p^{(\nu)}) : p \in \mathbb{Z}^\nu, \\ & r^{(j)} < p^{(j)} \leq s^{(j)} \text{ for all } 1 \leq j \leq \nu \}, \end{aligned}$$

and introduce for all  $\delta > 0$  and  $1 \leq j \leq \nu$  a set  $\mathcal{D}_N(\delta, j)$  consisting of the above defined rectangles  $D_N(r, s)$  with some additional properties. We define

$$\begin{aligned} \mathcal{D}_N(\delta, j) = \{ & D_N(r, s) : r^{(l)} = 0 \text{ for } l \neq j, 0 \leq r^{(j)} \leq N, \\ & 0 < s^{(l)} \leq N \text{ for all } 1 \leq l \leq \nu, \text{ and } 0 < s^{(j)} - r^{(j)} \leq \delta N \}. \end{aligned}$$

Inequality (B.2) follows from the relation

$$\begin{aligned} P \left( \sup_{D_N(r,s) \in \mathcal{D}_N(\delta,j)} \left| \frac{\sum_{p \in D_N(r,s)} H(X_1(p), \dots, X_d(p))}{N^{\nu-k\alpha/2} L(N)^{k/2}} \right| > \varepsilon \right) \leq \eta \\ \text{for all } 1 \leq j \leq \nu \end{aligned} \quad (\text{B.4})$$

if  $\delta \leq \delta(\varepsilon, \eta)$  with some  $\delta(\varepsilon, \eta) > 0$ . Here, and also in the remaining part of the proof  $H(x_1, \dots, x_d)$  is a sum of the form

$$H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d) + H^{(1)}(x_1, \dots, x_d),$$

with functions  $H^{(0)}(\cdot)$  and  $H^{(1)}(\cdot)$  defined in formulas (3.3) and (3.8).

In formulas (B.4) and (B.2) very similar expressions are estimated. The main difference between them is that in (B.2) random variables of the form  $|S_N(t) - S_N(s)|$  are considered with arguments  $s, t \in [0, 1]^\nu$ , while in (B.4) random variables of the form  $|S_N(\frac{s}{N}) - S_N(\frac{r}{N})|$  with arguments  $\frac{s}{N}$  and  $\frac{r}{N}$ , where  $s$  and  $r$  are vectors with integer coordinates with values between 0 and  $N$ . This is a sort of discretization, and in the reduction of (B.4) to (B.2) it has to be shown that this discretization has a negligible effect in the case of a large sample size  $N$ .

This can be seen with the help of the following observation. If  $N$  is large, then because of the definition of the random field  $S_N(t)$  for all  $t \in [0, 1]^\nu$  there exists a vector  $r = (r^{(1)}, \dots, r^{(\nu)})$  with integer coordinates  $r^{(j)}$ ,  $0 \leq r^{(j)} \leq N$ , such that  $S_N(t) = S_N(\frac{r}{N})$ , and  $t$  and  $\frac{r}{N}$  are very close to each other.

Inequality (B.4) will be proved by means of a good estimate on the tail distribution of the random variables  $\sum_{p \in D_N(r, s)} H(X_1(p), \dots, X_d(p))$  for the rectangles  $D_N(r, s)$ . These expressions will be estimated by means of an argument similar to the proof of Theorem 3.3. To do this let us first remark that Lemma 1 of [1] implies the following result, too. The inequality in Lemma 5A holds also in the case when the function  $H^{(1)}(x_1, \dots, x_d)$  is replaced in it by  $H(x_1, \dots, x_d)$ , and the coefficient  $\psi^{k+1}$  in the upper bound is replaced by  $\psi^k$ .

This modified version of Lemma 5A yields that there exists a threshold index  $n_0$  and some constant  $C > 0$  such that if the parameters  $p$  and  $q$  of two elements  $X(p) = (X_1(p), \dots, X_d(p))$  and  $X(q) = (X_1(q), \dots, X_d(q))$  of our random field satisfy the inequality  $|p - q| \geq n_0$ , then

$$|EH(X_1(p), \dots, X_d(p))H(X_1(q), \dots, X_d(q))| \leq C|p - q|^{-k\alpha} L(|p - q|)^k.$$

Let us observe that for such pairs  $p$  and  $q$  also the inequality

$$\begin{aligned} & |p - q|^{-k\alpha} L(|p - q|)^k \\ & \leq C \prod_{\substack{j: 1 \leq j \leq \nu \\ p^{(j)} - q^{(j)} \neq 0}} |p^{(j)} - q^{(j)}|^{-k\alpha/\nu} (L(|p^{(j)} - q^{(j)}|)^{k/\nu} + I_{\{|p^{(j)} - q^{(j)}| < D\}}) \end{aligned}$$



holds with some  $C > 0$  and  $D > 0$ , where  $I_{\{x < D\}}$  denotes the indicator function of the set  $\{x: x < D\}$ . Hence the previous estimate has the consequence

$$\begin{aligned} & |EH(X_1(p), \dots, X_d(p))H(X_1(q), \dots, X_d(q))| \\ & \leq C \prod_{\substack{j: 1 \leq j \leq \nu \\ p^{(j)} - q^{(j)} \neq 0}} |p^{(j)} - q^{(j)}|^{-k\alpha/\nu} (L(|p^{(j)} - q^{(j)}|)^{k/\nu} + I_{\{|p^{(j)} - q^{(j)}| \leq D\}}) \end{aligned}$$

if  $|p - q| \geq n_0$ . This inequality is more appropriate for us than the previous one.

On the other hand, the inequality

$$\begin{aligned} & |EH(X_1(p), \dots, X_d(p))H(X_1(q), \dots, X_d(q))| \\ & \leq EH^2(X_1(0), \dots, X_d(0)) \leq C_1 \end{aligned}$$

also holds for all  $p, q \in \mathbb{Z}^\nu$  with some  $C_1 < \infty$  because of the Schwarz inequality and relation (3.8).

The last two inequalities imply that for any rectangular  $D_N(r, s) \subset B_N$  and  $p \in D_N(r, s)$

$$\begin{aligned} & \left| EH(X_1(p), \dots, X_d(p)) \left( \sum_{q \in D_N(r, s)} H(X_1(q), \dots, X_d(q)) \right) \right| \\ & \leq C_2 \prod_{j=1}^{\nu} (1 + (s^{(j)} - r^{(j)})^{1-k\alpha/\nu} L(s^{(j)} - r^{(j)})^{k/\nu}) \quad (\text{B.5}) \end{aligned}$$

with an appropriate constant  $C_2$ . Indeed, these inequalities imply that

$$\begin{aligned} & \left| EH(X_1(p), \dots, X_d(p)) \left( \sum_{q \in D_N(r, s)} H(X_1(q), \dots, X_d(q)) \right) \right| \\ & \leq C \prod_{j=1}^{\nu} \left( 1 + 2 \sum_{q^{(j)=1}^{s^{(j)}-r^{(j)}}} (q^{(j)})^{-k\alpha/\nu} (L(q^{(j)})^{k/\nu} + I_{\{q^{(j)} \leq D\}}) \right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{q^{(j)=1}^{s^{(j)}-r^{(j)}}} (q^{(j)})^{-k\alpha/\nu} (L(q^{(j)})^{k/\nu} + I_{\{q^{(j)} \leq D\}}) \\ & \leq C' (1 + (s^{(j)} - r^{(j)})^{1-k\alpha/\nu} L(s^{(j)} - r^{(j)})^{k/\nu}) \end{aligned}$$

with some  $C' > 0$ , since  $k\alpha/\nu < 1$  by the conditions of Theorem 3.4. These relations imply (B.5).

By summing up inequality (B.5) for all  $p \in D_N(r, s)$ , and applying an appropriate normalization we get that

$$\begin{aligned} & \frac{1}{N^{2\nu-k\alpha}L(N)^k} E \left[ \sum_{p \in D_N(r, s)} H(X_1(p), \dots, X_d(p)) \right]^2 \\ & \leq C_2 \prod_{j=1}^{\nu} \left( \frac{(s^{(j)} - r^{(j)}) + (s^{(j)} - r^{(j)})^{2-k\alpha/\nu} L(s^{(j)} - r^{(j)})^{k/\nu}}{N^{2-k\alpha/\nu} L(N)^{k/\nu}} \right). \end{aligned} \quad (\text{B.6})$$

I claim that if  $1 \leq s^{(j)} - r^{(j)} \leq N$  for some  $1 \leq j \leq \nu$  and  $\eta > 0$  chosen so small that  $\beta = \frac{k\alpha}{\nu} + \eta < 1$ , then

$$\frac{(s^{(j)} - r^{(j)}) + (s^{(j)} - r^{(j)})^{2-k\alpha/\nu} L(s^{(j)} - r^{(j)})^{k/\nu}}{N^{2-k\alpha/\nu} L(N)^{k/\nu}} \leq C \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-k\alpha/\nu-\eta} \quad (\text{B.7})$$

with some  $C = C(\eta) > 0$ .

Indeed,

$$\begin{aligned} \frac{(s^{(j)} - r^{(j)})}{N^{2-k\alpha/\nu} L(N)^{k/\nu}} &= \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-k\alpha/\nu-\eta} (s^{(j)} - r^{(j)})^{k\alpha/\nu+\eta-1} \frac{N^{-\eta}}{L(N)^{k/\nu}} \\ &\leq C \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-k\alpha/\nu-\eta} \end{aligned}$$

if  $\eta > 0$  is chosen so small that  $\frac{k\alpha}{\nu} + \eta < 1$ , and

$$\begin{aligned} \frac{(s^{(j)} - r^{(j)})^{2-k\alpha/\nu} L(s^{(j)} - r^{(j)})^{k/\nu}}{N^{2-k\alpha/\nu} L(N)^{k/\nu}} &= \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-k\alpha/\nu-\eta} \\ &\quad \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{\eta} \left( \frac{L(s^{(j)} - r^{(j)})}{L(N)} \right)^{k/\nu} \leq C \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-k\alpha/\nu-\eta}. \end{aligned}$$

These two inequalities imply (B.7).

Let us choose a sufficiently large number  $D > 0$  (whose value does not depend on  $N$  and  $\delta$ ), and introduce the quantity  $d_N(p) = \frac{D}{N}$  for all  $1 \leq p \leq N$ . With such a notation we can get the following inequality with the help of relations (B.6) and (B.7).

Take some rectangle  $D_N(r, s) \subset B_N$ . Then we have for any  $\lambda > 0$

$$\begin{aligned}
P \left( \left| \frac{\sum_{p \in D_N(r, s)} H(X_1(p), \dots, X_d(p))}{N^{\nu-k\alpha/2} L(N)^{k/2}} \right| > \lambda \right) \\
\leq \frac{1}{\lambda^2} \frac{E \left[ \sum_{p \in D_N(r, s)} H(X_1(p), \dots, X_d(p)) \right]^2}{N^{2\nu-k\alpha} L(N)^k} \\
\leq \frac{C_3}{\lambda^2} \prod_{j=1}^{\nu} \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-k\alpha/\nu-\eta} \leq \frac{1}{\lambda^2} \prod_{j=1}^{\nu} \left( \sum_{p^{(j)=r^{(j)}+1}^{s^{(j)}} d_N(p^{(j)}) \right)^{2-\beta}
\end{aligned} \tag{B.8}$$

with  $\beta = \frac{k\alpha}{\nu} + \eta < 1$ .

With the help of formula (B.8) one can get such a maximum-type inequality which implies formula (B.4).

In the case  $\nu = 1$  Theorem 10.2 of Billingsley's book [2] can be applied. In this case this result together with formula (B.8) imply that

$$\begin{aligned}
P \left( \sup_{(u,v): r < u < v \leq s} \left| \frac{\sum_{u < p \leq v} H(X_1(p), \dots, X_d(p))}{N^{\nu-k\alpha/2} L(N)^{k/2}} \right| > \lambda \right) \\
\leq \frac{K}{\lambda^2} \left( \sum_{r < p \leq s} d_N(p) \right)^{2-\beta} = \frac{D^{2-\beta} K n}{\lambda^2} \left( \frac{s-r}{N} \right)^{2-\beta}
\end{aligned}$$

for any pairs  $0 \leq r < s \leq N$  with some  $K > 0$ . In particular,

$$P \left( \sup_{(u,v): r < u < v \leq r+\delta} \left| \frac{\sum_{u < \frac{p}{N} \leq v} H(X_1(p), \dots, X_d(p))}{N^{\nu-k\alpha/2} L(N)^{k/2}} \right| > \lambda \right) \leq \frac{K'}{\lambda^2} \delta^{2-\beta}$$

for any interval  $[r, r + \delta] \subset [0, 1]$  with  $\delta > 0$ ,

Since the exponent of  $\delta$  in the last inequality equals  $2 - \beta > 1$  it is not difficult to see that this relation implies inequality (B.4) in the case  $\nu = 1$ .

Indeed, we get (B.4) by applying this inequality with the choice  $\lambda = \varepsilon$  for the intervals  $[k\delta, (k+2)\delta]$  for all  $0 \leq k < \frac{1}{\delta}$  with a sufficiently small  $\delta > 0$ . Then inequality (B.4) implies inequality (B.1), too.

There is a multivariate version of the inequality cited from Billingsley's book [2] also in the case  $\nu > 1$  which, together with formula (B.8) imply inequality (B.4) in the general case. This inequality implies for any  $\nu \geq 1$  that if inequality (B.8) holds for all rectangles  $D_N(r, s)$ , then

$$\begin{aligned} P \left( \sup_{\substack{D_N(u,v): \\ D_N(u,v) \subset D_N(r,s)}} \left| \frac{\sum_{p \in D_N(u,v)} H(X_1(p), \dots, X_d(p))}{N^{\nu-k\alpha/2} L(N)^{k/2}} \right| > \lambda \right) \\ \leq \frac{K}{\lambda^2} \prod_{j=1}^{\nu} \left( \sum_{r^{(j)} < p^{(j)} \leq s^{(j)}} d_N(p_j) \right)^{2-\beta} = \frac{D^{(2-\beta)\nu} K}{\lambda^2} \prod_{j=1}^{\nu} \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-\beta} \end{aligned} \quad (\text{B.9})$$

with some  $K > 0$ ,  $D > 0$  and  $\beta = \frac{k\alpha}{\nu} + \eta < 1$ . (Here we are working with the previously defined  $d_N(p) = \frac{D}{N}$ .)

Indeed, although I did not find this result in the literature there is such a generalized version of the inequality quoted from Billingsley's book which states that if inequality (B.8) holds, then it implies inequality (B.9), too. This can be proved for instance by means of induction with respect to the dimension  $\nu$  by exploiting that this result holds for  $\nu = 1$ . In the proof we have to exploit that the upper bound in (B.8) has a special product form.

Let us fix some parameter  $1 \leq j \leq \nu$ , a number  $0 < \delta \leq 1$  an integer  $0 \leq r \leq N$ , and define with their help, similarly to the definition of  $\mathcal{D}_N(\delta, j)$  the set of rectangles

$$\begin{aligned} \mathcal{D}_N(\delta, j, r) = \{ D_N(u, v) : & u^{(l)} = 0, \text{ and } 0 < v^{(l)} \leq N \text{ for } l \neq j, \\ & \text{and } r \leq u^{(j)} < v^{(j)} \leq r + N\delta \}. \end{aligned}$$

The following inequality is a special case of (B.9).

$$\begin{aligned} P \left( \sup_{D(u,v) \in \mathcal{D}_N(\delta, j, r)} \left| \frac{\sum_{p \in D_N(u,v)} H(X_1(p), \dots, X_d(p))}{N^{\nu-k\alpha/2} L(N)^{k/2}} \right| > \lambda \right) \\ \leq \frac{K'}{\lambda^2} \left( \frac{s^{(j)} - r^{(j)}}{N} \right)^{2-\beta}. \end{aligned}$$

Inequality (B.4), hence inequality (B.1) can be proved with the help of the last inequality in the same way as it was done for  $\nu = 1$ . Actually, in that proof a special case of this inequality was applied. Since, as it was mentioned at the start of the proof relation (B.1) together with Theorem 3.4 imply the desired weak convergence Corollary 3.5 is proved.  $\square$

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