Limit theorems for non-linear functionals of stationary Gaussian random fields

Lectures, held by Péter Major

University of Bochum, 2017

1.   The subject of these lectures

The main subject of this series of talks is the study of limit theorems for normalized sums of elements in a stationary sequence of *dependent* random variables in such cases when the central limit theorem does not hold for them. Because of lack of time I cannot give a detailed proof of all results I discuus here. I concentrate instead on their content and the explanation of the picture behind them. I hope, this is interesting in itself, and it can give considerable help for those who are interested in a complete proof of the results. This can be found in my lecture note *Multiple Wiener–Itô Integrals.* Lecture Notes in Mathematics **849**, Revised (augmented) version, Springer Verlag, Berlin–Heidelberg–New York, (2014).

Let me describe this problem in more detail. Before discussing limit theorems for sums of dependent random variables let us recall some facts about limit theorems for i.i.d. random variables. There is a natural approach to the investigation of these limit theorems where first we try to find the possible limits by means of the study of an appropriately formulated fixed point problem in the space of distribution functions. This fixed point problem can be solved, and it shows that the possible limits are the normal and the so-called stable distributions whose Fourier transforms can be described explicitly.

We want to formulate a natural analogue of this fixed point problem which helps to find the possible limit in the case of limit theorems for normalized sums of dependent random variables. In this case we have to look for the distribution of an appropriate random process as the possible limit, because the one-dimensional distributions — unlike in the case of independent random variables — do not give sufficient information about the behaviour of the limit. This leads to the introduction of the notion of *renormalization* and *self-similar* fields. To define them first we have to introduce some additional notions.

I shall consider ν-dimensional stationary random fields. A ν-dimensional random field is a set of random variables ξ_n, $n \in \mathbb{Z}_\nu$, where $\mathbb{Z}_\nu$ denotes the ν-dimensional integer lattice. I shall call it a stationary random field if it satisfies the following definition.

Definition of Discrete (Strictly) Stationary Random Fields. *A set of random variables ξ_n, $n \in \mathbb{Z}_\nu$, is called a (strictly) stationary discrete random field if*

$$(\xi_{n_1}, \ldots, \xi_{n_k}) \stackrel{\Delta}{=} (\xi_{n_1+m}, \ldots, \xi_{n_k+m})$$

for all $k = 1, 2, \ldots$ and $n_1, \ldots, n_k$, $m \in \mathbb{Z}_\nu$, where $\stackrel{\Delta}{=}$ denotes equality in distribution.

Next I formulate the general limit problem we are interested in.

Given a discrete stationary random field ξ_n, $n \in \mathbb{Z}_\nu$, we define for all parameters $N = 1, 2, \ldots$ the new stationary random field

$$Z_n^N = A_N^{-1} \sum_{j \in B_n^N} \xi_j, \qquad N = 1, 2, \ldots, \quad n \in \mathbb{Z}_\nu, \tag{1.1}$$

where

$$B_n^N = \{ j \colon \ j \in \mathbb{Z}_\nu, \quad n^{(i)} N \le j^{(i)} < (n^{(i)} + 1)N, \ i = 1, 2, \ldots, \nu \},$$

and A_N, $A_N > 0$, is an appropriate norming constant. The superscript i denotes the i-the coordinate of a vector in this formula. We defined in formula (1.1) a new stationary random field Z_n^N, $n \in \mathbb{Z}_\nu$, for all indices N. We are interested in the question when the finite dimensional distributions of these random fields Z_n^N, $n \in \mathbb{Z}_\nu$, called the renormalizations of the original field ξ_n, $n \in \mathbb{Z}_\nu$, have a limit as $N \to \infty$, and we want to describe this limit if it exists. In particular, we would like to describe those random fields Z_n^*, $n \in \mathbb{Z}_\nu$, which appear as the limit of such random fields Z_n^N as $N \to \infty$. This problem, which is the natural counterpart of the fixed point problem leading to the description of the possible limits in the independent case suggested the introduction of the following notion.

Definition of Self-similar (Discrete) Random Fields. *A (discrete) random field ξ_n, $n \in \mathbb{Z}_\nu$, is called self-similar with self-similarity parameter α if the random fields Z_n^N defined in (1.1) with their help and the choice $A_N = N^\alpha$ satisfy the relation*

$$(\xi_{n_1}, \ldots, \xi_{n_k}) \stackrel{\Delta}{=} (Z_{n_1}^N, \ldots, Z_{n_k}^N) \tag{1.2}$$

for all $N = 1, 2, \ldots$ and $n_1, \ldots, n_k \in \mathbb{Z}_\nu$.

It is natural to expect that the self-similar random fields appear as the limit fields in the limit problem we are interested in now. We chose the norming

constant $A_N = N^\alpha$, because under some natural conditions we can satisfy formula (1.2) only with such a choice. We shall consider only such random fields in which the random variables ξ_n have a finite second moment. This excludes the random fields consisting of independent random variables with (non-normal) stable distribution from the classes of self-similar random fields we are interested in. With this restriction a self-similar random fields with parameter $\alpha \neq \frac{\nu}{2}$ must consist of strongly dependent random variables.

The description of (stationary) self-similar random fields is a very hard problem, and we have only partial results. The description of the (stationary) Gaussian self-similar random fields and of their (Gaussian) domain of attraction is a relatively simple problem, because in this case only the correlation function of the elements of the random fields has to be studied. This problem is essentially solved. We want to find non-Gaussian self-similar random fields and to present such interesting, non-trivial limit theorems where they appear as the limit. To find such random fields we shall introduce the notion of random fields subordinated to a stationary Gaussian random field. We shall work out a method to work with such subordinated random fields, and we shall be able to construct non-trivial self-similar random fields and to prove interesting limit theorems. To introduce the notion of random fields subordinated to a Gaussian random field first we have to define the shift transformation determined by a stationary Gaussian random field. This will be done in the next consideration.

Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian random field. First we define the shift transformations T_m, $m \in \mathbb{Z}_\nu$, over this field by the formula $T_m X_n = X_{n+m}$, for all $n, m \in \mathbb{Z}_\nu$. Then we can extend this shift transformation by means of some results in measure theory for all such random variables $\xi(\omega)$, which are measurable with respect to the σ-algebra $\mathscr{B}(X_n(\omega), n \in \mathbb{Z}_\nu)$. Indeed, by some results of measure theory we can write such a random variable in the form $\xi(\omega) = f(X_n(\omega), n \in \mathbb{Z}_\nu)$ with some Borel measurable function $f(x_n, n \in \mathbb{Z}_\nu)$ on the product space $R^{\mathbb{Z}_\nu}$, and we can define with the help of this representation the shift T_m of the random variable $\xi(\omega)$ by the formula $T_m \xi(\omega) = f(X_{n+m}(\omega), n \in \mathbb{Z}_\nu)$. It must be still explained that although the function f is not unique in the representation of the random variable $\xi(\omega)$, nevertheless the above definition of $T_m \xi(\omega)$ is meaningful. We shall identify two random variables if they are equal with probability 1. To see that we gave a correct definition of the shift transformation we have to check that if $f_1(X_n(\omega), n \in \mathbb{Z}_\nu) = f_2(X_n(\omega), n \in \mathbb{Z}_\nu)$ for two functions f_1 and f_2 with probability 1, then also $f_1(X_{n+m}(\omega), n \in \mathbb{Z}_\nu) = f_2(X_{n+m}(\omega), n \in \mathbb{Z}_\nu)$ with probability 1 because of the stationarity of the random field $X_n(\omega)$, $n \in \mathbb{Z}_\nu$.

In such a way we have defined the shift transformation for all random variables measurable with respect to the σ-algebra generated by the random field X_n, $n \in \mathbb{Z}_\nu$. But since we shall work only with random variables of finite second moment we shall consider the action of the shift transformation only for a smaller class of

random variables, for the elements of the Hilbert space $\mathscr{H}$ introduced below.

Let $\mathscr{H}$ denote the *real* Hilbert space consisting of the square integrable random variables measurable with respect to the σ-algebra $\mathscr{B} = \mathscr{B}(X_n, \ n \in \mathbb{Z}_\nu)$. The scalar product in $\mathscr{H}$ is defined as $(\xi, \eta) = E\xi\eta$, $\xi, \eta \in \mathscr{H}$. We define the shift transformations T_m, $m \in \mathbb{Z}_\nu$, for the elements of $\mathscr{H}$ in the way as we have done before in the general case. It is not difficult to check that the shift transformations T_m, $m \in \mathbb{Z}_\nu$, map the elements of $\mathscr{H}$ to another element of $\mathscr{H}$, they are unitary, i.e. norm preserving, invertible linear transformations. Moreover, they constitute a unitary group in $\mathscr{H}$, i.e. $T_{n+m} = T_n T_m$ for all $n, m \in \mathbb{Z}_\nu$, and $T_0 = \mathrm{Id}$. Now we introduce the following

Definition of Subordinated Random Fields. *Given a stationary Gaussian random field X_n, $n \in \mathbb{Z}_\nu$, we define the Hilbert space $\mathscr{H}$ and the shift transformations T_m, $m \in \mathbb{Z}_\nu$, over $\mathscr{H}$ as before. A discrete stationary field ξ_n is called a random field subordinated to X_n if $\xi_n \in \mathscr{H}$, and $T_n \xi_m = \xi_{n+m}$ for all $n, m \in \mathbb{Z}_\nu$.*

One of the main task of this series of talks is to work out a good method that enables us to study the fields subordinated to a stationary Gaussian random field together with the shift transformation acting on it. This enables us both to find non-trivial self-similar fields and to prove interesting limit theorems. This program will consist of several steps. First we study the underlying Gaussian fields. There is a classical result in analysis that enables us to describe the correlation function of this Gaussian field as the Fourier transform of a so-called spectral measure. We show that a so-called random spectral measure can be constructed, and a natural random integral can be defined with respect to it in such a way that the elements of the Gaussian random field can be expressed in a form that can be interpreted as the random Fourier transform of the random spectral measure. Then we introduce a multiple random integral, called multiple Wiener–Itô random integral with respect to the random spectral measure that enables us to express all elements of the above defined Hilbert space $\mathscr{H}$ as the sum of multiple Wiener–Itô integrals of different multiplicity. Moreover, this representation is unique, and the action of the shift transformation on $\mathscr{H}$ can be calculated in a simple way with its help. Then we make a most important step in our investigation, we prove the so-called diagram formula that enables us to rewrite the product of Wiener–Itô integrals as the sum of Wiener–itô integrals of different multiplicity. These results together with some basic facts about Hermite polynomials make possible to work out a technique that enables us to construct non-trivial self-similar fields, and to prove non-trivial (non-central) limit theorems.

In the above discussion I dealt with discrete time stationary random fields. However it is useful to handle discrete time Gaussian self-similar random fields together with their continuous time versions, since they yield — because of their stronger symmetry properties — an essential help also in the study of discrete

time stationary random fields. However, in the study of continuous time stationary fields serious additional technical difficulties appear, because, as it turned out, it is more useful to work with generalized and not with classical continuous time random fields. The study of generalized random fields demands some additional work. In particular, we have to present the appropriate notions and results needed in their study. Besides, we have to explain why we want to work with generalized random fields, why are the classical random fields inappropriate for us.

To give some feeling why the elaboration of the above theory is useful for us let us consider some limit problems which we can handle with its help.

Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian field with expectation $EX_n = 0$ and some correlation function $r(n) = EX_m X_{n+m}$, $n, m \in \mathbb{Z}_\nu$, and take the limit problem introduced in formula (1.1) with appropriate norming constants A_N if the random variables X_j play the role of the random variables ξ_j. This is a relatively simple problem, because it is enough to check whether the correlation functions $r_N(n) = EZ_m^N Z_{m+n}^N$ have a limit as $N \to \infty$ with an appropriate norming constants A_N. Next we are considering the following harder problem. Let us consider the previous Gaussian random field X_n, $n \in \mathbb{Z}_\nu$, and define with the help of some real valued function $f(x)$, $x \in R$ such that $Ef(X_n) = 0$ and $Ef^2(X_n) < \infty$ a new stationary random field $\xi_n = f(X_n)$, $n \in \mathbb{Z}_\nu$. Now we are interested in whether the random field Z_n^N, defined in formula (1.1) with the help of this new random field ξ_n have a limit with an appropriate norming constants A_N as $N \to \infty$. This is a considerably harder problem, and at the first sight we may have no idea how to handle it.

But let us observe that this random field ξ_n, $n \in \mathbb{Z}_\nu$, is subordinated to the Gaussian random field X_n, $n \in \mathbb{Z}_\nu$, hence we can apply the theory worked out to study this field. This theory enables us to present the normalized random sums Z_N^n in such a form that indicates what kind of limit these random variables may have, and how to choose the norming constants A_N to get a limit. Some (natural) calculation shows that the limit theorems suggested by the formulas obtained by our theory really hold.

2.   Random spectral measures

In the first step of our study we give a useful representation of the elements of the stationary Gaussian random field we are working with by means of a random integral. To get it first we apply the corollary of a classical result of analysis, — called Bochner's theorem —, about a good representation of the so-called positive definite functions by means of Fourier transform. This enables us to describe the correlation function of a stationary Gaussian random field as the Fourier transform of a finite measure which is called the spectral measure of this field in the literature. Because of lack of time I omit the discussion of the underlying Bochner

theorem, I will formulate only the result about the description of the correlation function of a stationary random field as the Fourier transform of the spectral measure of this field, because we need only this result. Then I show that a so-called random spectral measure can be constructed with the help of this result, and the elements of our stationary Gaussian random field can be represented as a random integral with respect to this random spectral measure. In an informal way this statement can be interpreted so that while the correlation function of a stationary Gaussian random field can be represented as the Fourier transform of its spectral measure, the random variables of this field can be represented as random Fourier transforms of the random spectral measure.

The introduction of the random spectral measure turned out to be very useful. The representation of our random variables by means of a random integral with respect to it enables us to work well with the shift transformations of our random field. Moreover, multiple Wiener–Itô integrals can be defined with respect to the random spectral measure, and they have an important role in our considerations. All elements of the Hilbert space $\mathscr{H}$ consisting of those random variables with finite second moment which are measurable with respect to the σ-algebra generated by the random variables of the stationary Gaussian random field we are working with can be written as the sum of multiple Wiener–Itô integrals of different multiplicity. This representation is very useful, and it will be the main tool in our investigation.

First I formulate the result about the description of the correlation function of discrete time stationary random fields.

Theorem 2A about the spectral representation of the correlation function of a discrete stationary random field. *Let X_n, $n \in \mathbb{Z}_v$, be a discrete (Gaussian) stationary random field with expectation $EX_n = 0$, $n \in \mathbb{Z}_n$. There exists a unique finite measure G on $[-\pi, \pi)^v$ such that the correlation function $r(n) = EX_0 X_n = EX_k X_{k+n}$, $n \in \mathbb{Z}_v$, $k \in \mathbb{Z}_v$, can be written in the form*

$$r(n) = \int e^{i(n,x)} G(dx), \tag{2.1}$$

where $(\cdot, \cdot)$ denotes scalar product. Further, $G(A) = G(-A)$ for all $A \in [-\pi, \pi)^v$.

We can identify $[-\pi, \pi)^v$ with the torus $R^v / 2\pi \mathbb{Z}_v$. Thus e.g. $-(-\pi, \ldots, -\pi) = (-\pi, \ldots, -\pi)$.

We want to formulate the continuous time version of the above result about the representation of the correlation function as the Fourier transform of a so-called spectral measure. There exists such a theorem, and it is very similar to the discrete time result. The only difference is that in the new case we have to consider a spectral measure G on R^v and not on $[-\pi, \pi)^v$ as in formula (2.1), and we have to assume that the correlation function $r(t) = EX_0 X_t$ is continuous. We

could work with such a result. Nevertheless, we shall follow a different approach. We shall work instead of classical continuous time stationary random fields with (stationary) generalized random fields, and formulate the results for them. This demands the introduction of some new notions and some extra explanation.

First we have to understand why the classical continuous time stationary random fields are not good for us, why do we want to work with generalized random fields instead. R. L. Dobrushin gave the following informal explanation for this. The trajectory of a continuous time random field can be very bad. It can be so bad, that it simply does not exist. In such cases we cannot consider our random field as a really existing field, but there may be a possibility to consider it as a generalized random field. As we shall later see, there are very important generalized random fields that cannot be obtained by means of a classical random field. Moreover, they play a useful role also in the study of discrete time random fields.

The following heuristic argument may explain the definition of generalized random fields. Let us have a classical continuous time random field $X(t)$ with parameters $t \in R^\nu$ in the ν-dimensional Euclidean space, and a linear topological space $\mathscr{F}$ of functions on R^ν with some nice properties. In nice cases we can define the integral $X(\varphi) = \int_{R^\nu} \varphi(s)X(s)\,ds$ for all functions $\varphi \in \mathscr{F}$, and if the space of functions $\mathscr{F}$ is sufficiently rich, then the random variables $X(\varphi)$ determine the values of the random variables $X(t)$ from which we obtained it. The random variables $X(\varphi)$ have the property $X(a\varphi + b\psi) = aX(\varphi) + bX(\psi)$, and if φ and ψ are two functions of $\mathscr{F}$ which are close to each other, then it is natural to expect that $X(\varphi)$ and $X(\psi)$ are also close to each other in some sense.

This means that we can correspond to a classical random field $X(t)$ a class of random linear functionals $X(\varphi)$, $\varphi \in \mathscr{F}$, indexed by the elements of $\mathscr{F}$, which have some nice properties. We will call a class of continuous random functionals indexed by the elements of a nice linear topological space $\mathscr{F}$ a generalized random field. We can correspond to each classical random field a generalized random field which determines it, but not all generalized random fields can be obtained in such a way. Now I introduce the precise definition of generalized random fields together with some additional notions we shall apply in our considerations.

2.1. Generalized random fields and some related notions

First I introduce the linear topological space we shall be working with in the definition of generalized random fields. There are several good choices for it. I shall use the so-called Schwartz space $\mathscr{S}$, because we can work with it very well.

We define the Schwartz space $\mathscr{S}$ together with its version $\mathscr{S}^c$ consisting of complex valued functions. The space $\mathscr{S}^c = (\mathscr{S}_\nu)^c$ consists of those complex valued functions of ν variables which decrease at infinity, together with their derivatives, faster than any polynomial degree. More explicitly, $\varphi \in \mathscr{S}^c$ for a complex

valued function φ defined on R^v if

$$\left| x_1^{k_1} \cdots x_v^{k_v} \frac{\partial^{q_1 + \cdots + q_v}}{\partial x_1^{q_1} \ldots \partial x_v^{q_v}} \varphi(x_1, \ldots, x_v) \right| \leq C(k_1, \ldots, k_v, q_1, \ldots, q_v)$$

for all point $x = (x_1, \ldots, x_v) \in R^v$ and vectors $(k_1, \ldots, k_v)$, $(q_1, \ldots, q_v)$ with non-negative integer coordinates with some constant $C(k_1, \ldots, k_v, q_1, \ldots, q_v)$ which may depend on the function φ. The elements of the space $\mathscr{S}$ are defined similarly, with the only difference that they are real valued functions.

To complete the definition of the spaces $\mathscr{S}$ and $\mathscr{S}^c$ we still have to define the topology in them. We introduce the following topology in these spaces.

Let a basis of neighbourhoods of the origin consist of the sets

$$U(k, q, \varepsilon) = \left\{ \varphi \colon \ \max_x (1 + |x|^2)^k |D^q \varphi(x)| < \varepsilon \right\}$$

with $k = 0, 1, 2, \ldots$, $q = (q_1, \ldots, q_v)$ with non-negative integer coordinates and $\varepsilon > 0$, where $|x|^2 = x_1^2 + \cdots + x_v^2$, and $D^q = \frac{\partial^{q_1 + \cdots + q_v}}{\partial x_1^{q_1} \ldots \partial x_v^{q_v}}$. A basis of neighbourhoods of an arbitrary function $\varphi \in \mathscr{S}^c$ (or $\varphi \in \mathscr{S}$) consists of sets of the form $\varphi + U(k, q, \varepsilon)$, where the class of sets $U(k, q, \varepsilon)$ is a basis of neighbourhood of the origin. Let me remark that a sequence of functions $\varphi_n \in \mathscr{S}^c$ (or $\varphi_n \in \mathscr{S}$) converges to a function φ in this topology if and only if

$$\lim_{n \to \infty} \sup_{x \in R^v} (1 + |x|^2)^k |D^q \varphi_n(x) - D^q \varphi(x)| = 0.$$

for all $k = 1, 2, \ldots$ and $q = (q_1, \ldots, q_v)$. The limit function φ is also in the space $\mathscr{S}^c$ (or in the space $\mathscr{S}$).

I shall define the generalized random fields and some related notions with the help of the notion of Schwartz spaces.

Definition of Generalized Random Fields. *We say that the set of random variables $X(\varphi)$, $\varphi \in \mathscr{S}$, is a generalized random field over the Schwartz space $\mathscr{S}$ of rapidly decreasing, smooth functions if:*

(a) $X(a_1\varphi_1 + a_2\varphi_2) = a_1 X(\varphi_1) + a_2 X(\varphi_2)$ *with probability 1 for all real numbers a_1 and a_2 and $\varphi_1 \in \mathscr{S}$, $\varphi_2 \in \mathscr{S}$. (The exceptional set of probability 0 where this identity does not hold may depend on a_1, a_2, φ_1 and φ_2.)*

(b) $X(\varphi_n) \Rightarrow X(\varphi)$ *stochastically if $\varphi_n \to \varphi$ in the topology of $\mathscr{S}$.*

We also introduce the following definitions.

Definition of Stationarity and Gaussian Property of a Generalized Random Field and the Notion of Convergence of Generalized Random Fields in Distribution. *The generalized random field $X = \{X(\varphi), \varphi \in \mathscr{S}\}$ is stationary if $X(\varphi) \stackrel{\Delta}{=} X(T_t\varphi)$ for all $\varphi \in \mathscr{S}$ and $t \in R^\nu$, where $T_t\varphi(x) = \varphi(x-t)$. It is Gaussian if $X(\varphi)$ is a Gaussian random variable for all $\varphi \in \mathscr{S}$. The relation $X_n \stackrel{\mathscr{D}}{\to} X_0$ as $n \to \infty$ holds for a sequence of generalized random fields X_n, $n = 0, 1, 2, \dots$, if $X_n(\varphi) \stackrel{\mathscr{D}}{\to} X_0(\varphi)$ for all $\varphi \in \mathscr{S}$, where $\stackrel{\mathscr{D}}{\to}$ denotes convergence in distribution.*

Next I formulate the version of our limit problem for generalized fields.

Given a stationary generalized random field X and a function $A(t) > 0$, $t > 0$, on the set of positive real numbers we define the (stationary) random fields X_t^A for all $t > 0$ by the formula

$$X_t^A(\varphi) = X(\varphi_t^A), \quad \varphi \in \mathscr{S}, \qquad \text{where } \varphi_t^A(x) = A(t)^{-1}\varphi\left(\frac{x}{t}\right). \qquad (2.2)$$

We are interested in the following

Question. *When does a generalized random field X^* exist such that $X_t^A \stackrel{\mathscr{D}}{\to} X^*$ as $t \to \infty$ (or as $t \to 0$)?*

In relation to this question we introduce the following

Definition of Self-similarity. *The stationary generalized random field X is self-similar with self-similarity parameter α if $X_t^A(\varphi) \stackrel{\Delta}{=} X(\varphi)$ for all $\varphi \in \mathscr{S}$ and $t > 0$ with the function $A(t) = t^\alpha$.*

To answer the above question one should first describe the generalized self-similar random fields.

We define analogously to the case of discrete random fields the notion of generalized subordinated random fields.

Let $X(\varphi)$, $\varphi \in \mathscr{S}$, be a generalized stationary Gaussian random field. The formula $T_t X(\varphi)) = X(T_t\varphi)$, $T_t\varphi(x) = \varphi(x-t)$, defines the shift transformation for all $t \in R^\nu$. Let $\mathscr{H}$ denote the real Hilbert space consisting of the $\mathscr{B} = \mathscr{B}(X(\varphi), \varphi \in \mathscr{S})$ measurable random variables with finite second moment. The shift transformation can be extended to a group of unitary transformations over $\mathscr{H}$ similarly to the discrete case. (This definition has the following idea. If a random variable $\xi \in \mathscr{H}$ has the form $\xi = F(X(\varphi_1), \dots, X(\varphi_s))$ with some functions $\varphi_1, \dots \varphi_s \in \mathscr{S}$, and a measurable function F of s variables, then we define $T_t\xi = F(X(T_t\varphi_1), \dots, X(T_t\varphi_s))$. A general random variable $\xi \in \mathscr{H}$ has a somewhat more complicated, but similar representation, and its shift can be defined in a similar way.) With the help of the notion of shift transformations in generalized fields we can introduce the following definition.

Definition of Generalized Random Fields Subordinated to a Generalized Stationary Gaussian Random Field. *Given a generalized stationary Gaussian random field $X(\varphi)$, $\varphi \in \mathscr{S}$, we define the Hilbert space $\mathscr{H}$ and the shift transformations T_t, $t \in R^\nu$, over $\mathscr{H}$ as above. A generalized stationary random field $\xi(\varphi)$, $\varphi \in \mathscr{S}$, is subordinated to the field $X(\varphi)$, $\varphi \in \mathscr{S}$, if $\xi(\varphi) \in \mathscr{H}$ and $T_t \xi(\varphi) = \xi(T_t \varphi)$ for all $\varphi \in \mathscr{S}$ and $t \in R^\nu$, and $E[\xi \varphi_n) - \xi(\varphi)]^2 \to 0$ if $\varphi_n \to \varphi$ in the topology of $\mathscr{S}$.*

Now we can formulate the analogue of Theorem 2A about the Fourier representation of the correlation function of a generalized field. Before doing it I recall an important property of the Fourier transform of the functions in the Schwartz spaces $\mathscr{S}$ and $\mathscr{S}^c$. Actually this property of the Schwartz spaces made useful their application in the definition of generalized fields.

The Fourier transform $f \to \tilde{f}$ is a bicontinuous map from $\mathscr{S}^c$ to $\mathscr{S}^c$. (This means that this transformation is invertible, and both the Fourier transform and its inverse are continuous maps from $\mathscr{S}^c$ to $\mathscr{S}^c$.) (The restriction of the Fourier transform to the space $\mathscr{S}$ of real valued functions is a bicontinuous map from $\mathscr{S}$ to the subspace of $\mathscr{S}^c$ consisting of those functions $f \in \mathscr{S}^c$ for which $f(-x) = \overline{f(x)}$ for all $x \in R^\nu$.) I omit the proof of this statement, I only remark that the smoothness properties of the functions in $\mathscr{S}^c$ imply the fast decrease of their Fourier transform at infinity, and their fast decrease at infinity imply the smoothness properties of their Fourier transform.

In a thorough analysis one also studies the properties of the elements of the space of generalized functions $\mathscr{S}'$ which are the continuous linear functionals over $\mathscr{S}$. But since they are needed only in such proofs which I omit in this discussion, I do not discuss these problems. Next I formulate the following result.

Theorem 2B about the spectral representation of the correlation function of a generalized stationary field. *Let $X(\varphi)$, $\varphi \in \mathscr{S}$, $EX(\varphi) = 0$ for $\varphi \in \mathscr{S}$, be a generalized Gaussian stationary random field over $\mathscr{S} = \mathscr{S}_\nu$. There exists a unique σ-finite measure G on R^ν such that*

$$EX(\varphi)X(\psi) = \int \tilde{\varphi}(x) \bar{\tilde{\psi}}(x) G(dx) \qquad \text{for all } \varphi, \psi \in \mathscr{S}, \qquad (2.3)$$

where ˜ denotes Fourier transform and ‾ complex conjugate. The measure G has the properties $G(A) = G(-A)$ for all $A \in \mathscr{B}^\nu$, and

$$\int (1 + |x|)^{-r} G(dx) < \infty \quad \text{with an appropriate } r > 0. \qquad (2.4)$$

Let me remark that while in Theorem 2A the spectral measure G had to be finite, in Theorem 2B it had to satisfy a much weaker condition (2.4). This indicates

that there are such generalized stationary random fields that cannot be obtained from non-generalized random fields. This difference between the properties of the spectral measures in Theorems 2A and 2B also have other interesting and important consequences about which I shall write at the end of this section.

The proof that the correlation function of a generalized field must satisfy Theorem 2B depends on some deep theorems about generalized functions, hence I omit it. On the other hand, I briefly show that in Theorem 2B actually we defined the correlation function of a stationary generalized Gaussian random field. Before doing this I present a short calculation that indicates that formula (2.3) can be considered as the natural analogue of formula (2.1) when we are working with a generalized field.

Let us consider a continuous time Gaussian stationary field $X(t)$, $t \in R^\nu$, with correlation function $EX(s)X(t) = \int e^{i(s-t,x)}G(dx)$, and let us calculate the correlation function $EX(\varphi)X(\psi)$, $\varphi, \psi \in \mathscr{S}$, where $X(\varphi) = \int \varphi(t)X(t)\,dt$, and $X(\psi) = \int \psi(t)X(t)\,dt$. We have

$$
\begin{aligned}
EX(\varphi)X(\psi) &= E \int \varphi(s)X(s)\,ds \int \psi(t)X(t)\,dt \\
&= \int \int \varphi(s)\psi(t)EX(s)X(t)\,ds\,dt \\
&= \int \int \varphi(s)\psi(t) \left[\int e^{i(s-t,x)}G(dx) \right] ds\,dt \\
&= \int \left[\int e^{(i(s,x))}\varphi(s)\,ds \right] \left[\int e^{-i(t,x)}\psi(t)\,dt \right] G(dx) \\
&= \int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)G(dx).
\end{aligned}
$$

Next we show that formulas (2.3) and (2.4) in Theorem 2.1 define the correlation function of a generalized stationary Gaussian random field.

First we show that $EX(\varphi)X(\psi)$ defined in (2.3) is a real number for all $\varphi, \psi \in \mathscr{S}$. To show this we apply the change of variables $x \to -x$ in this formula, and we exploit that $G(A) = G(-A)$, and $\bar{\tilde{f}}(x) = \tilde{f}_-(x)$ with $f_-(x) = f(-x)$ for a real valued function f. This implies that $EX(\varphi)X(\psi) = \overline{EX(\varphi)X(\psi)}$ i.e. $EX(\varphi)X(\psi)$ is a real number.

By Kolmogorov's existence theorem a random process with prescribed finite dimensional distributions exists if these distributions are consistent. By this result to prove that there is a Gaussian random field $X(\varphi)$, $\varphi \in \mathscr{S}$, with expectation zero and correlation function $EX(\varphi)X(\psi)$ defined in (2.4) it is enough to show that for arbitrary finite set of functions $\varphi_1, \ldots \varphi_n \in \mathscr{S}$ the matrix $(d_{j,k})$, $1 \le j, k \le n$, $di_{j,k} = EX(\varphi_j)X(\varphi_k)$ is positive semidefinite. This is equivalent to the statement

that for any function ψ of the form $\psi(x) = c_1\varphi_1(x) + \cdots + c_n\varphi_n(x)$ with real numbers $c_1, \ldots, c_n$ the expression $EX(\psi)X(\psi)$ defined in (2.4) is non-negative. This fact can be simply checked.

We also have to show that a random field with such a distribution is a generalized field, i.e. it satisfies properties (a) and (b) given in the definition of generalized fields.

Property (a) holds, because, as it is not difficult to check with the help of formula (2.3),

$$E[a_1 X(\varphi_1) + a_2 X(\varphi_2) - X(\varphi(a_1\varphi_1 + a_2\varphi_2))]^2$$
$$= \int \left| a_1 \tilde{\varphi}_1(x) + a_2 \tilde{\varphi}_2(x) - (a_1\widetilde{\varphi_1 + a_2}\varphi_2)(x) \right|^2 G(dx) = 0.$$

It is not difficult to show that if $\varphi_n \to \varphi$ in the topology of the space $\mathscr{S}$, then $E[X(\varphi_n) - X(\varphi)]^2 = \int |\tilde{\varphi}_n(x) - \tilde{\varphi}(x)|^2 G(dx) \to 0$ as $n \to \infty$, hence property (b) also holds. (Here we exploit that the transformation $\varphi \to \tilde{\varphi}$ is bicontinuous in the space $\mathscr{S}$.)

It is clear that the Gaussian random field constructed in such a way is stationary.

Finally, I remark that some additional investigation shows that the correlation function $EX(\varphi)X(\psi)$ uniquely determines the spectral measure G in formula (2.4), since the class of functions $\mathscr{S}$ is sufficiently rich.

2.2. Construction of random spectral measures

We shall construct generalized spectral measures both for discrete valued and generalized stationary Gaussian random fields. The construction in the two cases is similar, but there is some difference between them. In both cases we construct an appropriate unitary operator I, and we define the random spectral measure with its help. Let me remark that I shall speak also about unitary operators between two different Hilbert spaces. Given two Hilbert spaces $\mathscr{H}_0$ and $\mathscr{H}_1$ I call a linear transformation $I\colon \mathscr{H}_0 \to \mathscr{H}_1$ unitary if it is norm preserving and invertible. We shall define the operator I in the case of discrete and generalized fields in a similar way. First we define them on a dense subspace of $\mathscr{H}_0$, and then we extend it to the whole space in a natural way.

First I define the Hilbert spaces $\mathscr{H}_0$ and $\mathscr{H}_1$, (more precisely its complexification $\mathscr{H}_1^c$ we shall work with) both in the discrete and generalized random field case.

Let us consider a stationary Gaussian random field (discrete or generalized one) with spectral measure G. We shall denote the space $L_2([-\pi, \pi)^\nu, \mathscr{B}^\nu, G)$ or $L_2(R^\nu, \mathscr{B}^\nu, G)$ simply by L_G^2. This will play the role of the Hilbert space $\mathscr{H}_0$. (The space $\mathscr{H}_0$ contains also complex valued functions.)

Given a stationary Gaussian random field, either a discrete field, X_n, $n \in \mathbb{Z}_n$, or a generalized one $X(\varphi)$, $\varphi \in \mathscr{S}$, first we define a real Hilbert space $\mathscr{H}_1$ and then its complexification $\mathscr{H}_1^c$. The real Hilbert space $\mathscr{H}_1$ is that subspace of the Hilbert space of square integrable random variables (in the probability space we are working with) which is generated by the finite linear combination of the random variables X_n, $n \in \mathbb{Z}_n$, in the discrete field case, and by $X(\varphi)$, $\varphi \in \mathscr{S}$, in the generalized field case. We define its complexification $\mathscr{H}_1^c$ in the following way. The elements of $\mathscr{H}_1^c$ are of the form $X + iY$, $X, Y \in \mathscr{H}_1$, and the scalar product is defined in it as $(X_1 + iY_1, X_2 + iY_2) = EX_1X_2 + EY_1Y_2 + i(EY_1X_2 - EX_1Y_2)$. We are going to construct a unitary transformation I from L_G^2 to $\mathscr{H}_1^c$. We shall define the random spectral measure with the help of this transformation.

I recall that we also introduced the Schwartz space $\mathscr{S}^c$ consisting of the rapidly decreasing, smooth, *complex valued* functions with the usual topology of the Schwartz space. It can be proved with the help of some results in analysis that the set of finite trigonometric polynomials $\sum c_n e^{i(n,x)}$ are dense in L_G^2 in the discrete field, and the functions $\varphi \in \mathscr{S}^c$ are dense in L_G^2 in the generalized field case. We shall exploit this fact in our construction,

We define the mapping

$$I\left(\sum c_n e^{i(n,x)}\right) = \sum c_n X_n \tag{2.5}$$

in the discrete field case, where the sum is finite, and

$$I(\widetilde{\varphi + i\psi}) = X(\varphi) + iX(\psi), \quad \varphi, \psi \in \mathscr{S} \tag{2.6}$$

in the generalized field case.

Simple calculation with the help of Theorems 2A and 2B shows that

$$\begin{aligned}
\left\|\sum c_n e^{i(n,x)}\right\|_{L_G^2}^2 &= \sum\sum c_n \bar{c}_m \int e^{i(n-m,x)} G(dx) \\
&= \sum\sum c_n \bar{c}_m EX_nX_m = E\left|\sum c_n X_n\right|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|\widetilde{\varphi + i\psi}\|_{L_G^2}^2 &= \int [\tilde{\varphi}(x)\bar{\tilde{\varphi}}(x) - i\tilde{\varphi}(x)\bar{\tilde{\psi}}(x) + i\tilde{\psi}(x)\bar{\tilde{\varphi}}(x) + \tilde{\psi}(x)\bar{\tilde{\psi}}(x)] G(dx) \\
&= EX(\varphi)^2 - iEX(\varphi)X(\psi) + iEX(\psi)X(\varphi) + EX(\psi)^2 \\
&= E\left(|X(\varphi) + iX(\psi)|\right)^2.
\end{aligned}$$

This means that the mapping I from a linear subspace of L_G^2 to $\mathscr{H}_1^c$ is norm preserving. Besides, the subspace where I was defined is dense in L_G^2, Hence the mapping I can be uniquely extended to a norm preserving transformation from

L_G^2, to $\mathscr{H}_1^c$. Since the random variables X_n or $X(\varphi)$ are obtained as the image of some element from L_G^2 under this transformation, the range of I is the whole space $\mathscr{H}_1^c$, and I is a unitary transformation from L_G^2 i.e. from $\mathscr{H}_0$ to $\mathscr{H}_1^c$. A unitary transformation preserves not only the norm, but also the scalar product. Hence $\int f(x)\bar{g}(x)G(dx) = EI(f)\overline{I(g)}$ for all $f, g \in L_G^2$.

We shall define the random spectral measure $Z_G(A)$ related to our Gaussian stationary random field for those Borel measurable sets $A \in \mathscr{B}^\nu$ for which $G(A) < \infty$, and it is defined by the formula

$$Z_G(A) = I(\chi_A),$$

where χ_A denotes the indicator function of the set A. It is not difficult to see that

(i) The random variables $Z_G(A)$ are complex valued, jointly Gaussian random variables. (The random variables $\operatorname{Re} Z_G(A)$ and $\operatorname{Im} Z_G(A)$ with possibly different sets A are jointly Gaussian.)

(ii) $EZ_G(A) = 0$,

(iii) $EZ_G(A)\overline{Z_G(B)} = G(A \cap B)$,

(iv) $\sum_{j=1}^{n} Z_G(A_j) = Z_G\left(\bigcup_{j=1}^{n} A_j\right)$ if $A_1, \ldots, A_n$ are disjoint sets.

Also the following relation holds.

(v) $Z_G(A) = \overline{Z_G(-A)}$.

This follows from the relation

(v′) $I(f) = \overline{I(f_-)}$ for all $f \in L_G^2$, where $f_-(x) = \overline{f(-x)}$.

Relations (i)—(iv) simply follow from the properties of the operator I. The proof of (v), more precisely of its strengthened form (v′) demands some more work. This property is needed to decide when a random integral with respect to the complex valued random measure Z_G is a real valued random variable. (We shall soon define the random spectral measure Z_G and the (random) integral with respect to it.) I describe the proof of (v′) in the generalized field case. The proof in the discrete parameter case is similar, but simpler. Then I shall give the proof also of (iii).

Relation (v′) can be simply checked if f is a finite trigonometric polynomial in the discrete field case, or if $f = \tilde{\varphi}$, $\varphi \in \mathscr{S}^c$, in the generalized field case. (In the case $f = \tilde{\varphi}$, $\varphi \in \mathscr{S}^c$, the following argument works. Put $f(x) = \tilde{\varphi}_1(x) + i\tilde{\varphi}_2(x)$ with $\varphi_1, \varphi_2 \in \mathscr{S}$. Then $I(f) = X(\varphi_1) + iX(\varphi_2)$, and $f_-(x) = \bar{\tilde{\varphi}}_1(-x) - i\bar{\tilde{\varphi}}_2(-x) =$

$\tilde{\varphi}_1(x) + i(\widetilde{-\varphi_2}(x))$, hence $I(f_-) = X(\varphi_1) + iX(-\varphi_2) = X(\varphi_1) - iX(\varphi_2) = \overline{I(f)}$.)
Then a simple limiting procedure implies (v') in the general case.

Relation (iii) follows from the identity

$$EZ_G(A)\overline{Z_G(B)} = EI(\chi_A)\overline{I(\chi_B)} = \int \chi_A(x)\overline{\chi_B(x)}G(dx) = G(A \cap B).$$

We have constructed with the help of a stationary Gaussian random field with spectral measure G a set of complex valued random variables $Z_G(\cdot)$ which satisfy properties (i)—(v). In the next definition we shall call any class of sets of complex valued random variables with these properties (independently of how we have obtained them) a random spectral measure.

Definition of Random Spectral Measures. *Let G be a spectral measure. A set of random variables $Z_G(A)$, $G(A) < \infty$, satisfying (i)–(v) is called a (Gaussian) random spectral measure corresponding to the spectral measure G.*

Given a Gaussian random spectral measure Z_G corresponding to a spectral measure G we define the (one-fold) stochastic integral $\int f(x)Z_G(dx)$ for an appropriate class of functions f.

Let us first consider simple functions of the form $f(x) = \sum c_i \chi_{A_i}(x)$, where the sum is finite, the sets A_i are disjoint, and $G(A_i) < \infty$ for all indices i. In this case we define

$$\int f(x)Z_G(dx) = \sum c_i Z_G(A_i).$$

We have to justify that the above formula is meaningful. The problem is that the representation $f(x) = \sum c_i \chi_{A_i}(x)$ of a simple function is not unique. We can write a set A_i as the partition of finitely many disjoint sets $A_{i,j}$, and if we replace A_i with these sets $A_{i,j}$, and define the function to be c_i on all sets $A_{i,j}$, then we get a different representation of the same function f. Now the additivity property (iv) guarantees that the integral defined by the new representation has the same value. It is not difficult to check with the help of this observation that the value of the integral of a simple function with respect to Z_G does not depend on the representation of the simple function. Later we meet a generalized version of this problem in the definition of multiple integrals with respect to a spectral random measure. We shall explain there the above argument in more detail.

Then we have

$$E\left|\int f(x)Z_G(dx)\right|^2 = \sum c_i \bar{c}_j G(A_i \cap A_j) = \int |f(x)|^2 G(dx) \qquad (2.7)$$

for all elementary functions.

Since the simple functions are dense in L_G^2, relation (2.7) enables us to define $\int f(x)Z_G(dx)$ for all $f \in L_G^2$ via L_2-continuity. It can be seen that this integral satisfies the identity

$$E \int f(x)Z_G(dx) \overline{\int g(x)Z_G(dx)} = \int f(x)\overline{g(x)}G(dx) \qquad (2.8)$$

for all pairs of functions $f, g \in L_G^2$. Moreover, similar approximation with simple functions yields that

$$\int f(x)Z_G(dx) = \overline{\int \overline{f(-x)}Z_G(dx)} \qquad (2.9)$$

for a function $f \in L_G^2$. Here we exploit the identity $Z_G(A) = \overline{Z_G(-A)}$ formulated in property (v) of the random spectral measure Z_G.

Formula (2.9) implies in particular that $\int f(x)Z_G(dx)$ is real valued if $f(x) = \overline{f(-x)}$.

The last two identities together with the relations (2.1) and (2.3) imply that if we define the set of random variables X_n and $X(\varphi)$ by means of the formulas

$$X_n = \int e^{i(n,x)}Z_G(dx), \quad n \in \mathbb{Z}_\nu, \qquad (2.10)$$

and

$$X(\varphi) = \int \tilde{\varphi}(x)Z_G(dx), \quad \varphi \in \mathscr{S}, \qquad (2.11)$$

where we integrate with respect to the random spectral measure Z_G, then we get a Gaussian stationary random discrete and generalized field with spectral measure G, i.e. with correlation function given in formulas (2.1) and (2.3). To check this statement first we have to show that the random variables X_n and $X(\varphi)$ defined in (2.10) and (2.11) are real valued, or equivalently saying the identities $X_n = \overline{X_n}$ and $X(\varphi) = \overline{X(\varphi)}$ hold with probability 1. This follows from relation (2.9) and the identities $e^{i(n,x)} = \overline{e^{(i(n,-x)}}$ and $\tilde{\varphi}(x) = \overline{\tilde{\varphi}(-x)}$ for a (real valued) function $\varphi \in \mathscr{S}$. Then we can calculate the correlation functions $EX_nX_m = EX_n\overline{X}_m$ and $EX(\varphi)X(\psi) = EX(\varphi)\overline{X(\psi)}$ by means of formula (2.8), (2.10) and (2.11).

We also have

$$\int f(x)Z_G(dx) = I(f) \quad \text{for all } f \in L_G^2$$

if we consider the previously defined mapping $I(f)$ with the stationary random fields defined in (2.10) and (2.11). In particular, if we have a discrete or generalized stationary random field X_n, $n \in \mathbb{Z}_\nu$, or $X(\varphi)$, $\varphi \in \mathscr{S}$, and we construct the

random spectral measure Z_G with the help of the operator I in the way as we have written down at the beginning of this subsection, then we can write

$$X_n = I(e^{i(n,x)}) = \int e^{i(n,x)} Z_G(dx) \qquad \text{for all } n \in \mathbb{Z}_\nu,$$

and

$$X(\varphi) = I(\tilde{\varphi}) = \int \tilde{\varphi}(x) Z_G(dx) \qquad \text{for all } \varphi \in \mathscr{S}.$$

It is not difficult to prove the subsequent theorem with the help of the above results. I omit the details.

Theorem 2.1. *For a stationary Gaussian random field (a discrete or generalized one) with a spectral measure G there exists a unique Gaussian random spectral measure Z_G corresponding to the spectral measure G on the same probability space as the Gaussian random field such that relation (2.10) or (2.11) holds in the discrete or generalized field case respectively.*

Furthermore

$$\mathscr{B}(Z_G(A),\, G(A) < \infty) = \begin{cases} \mathscr{B}(X_n,\, n \in \mathbb{Z}_\nu) \text{ in the discrete field case,} \\ \mathscr{B}(X(\varphi),\, \varphi \in \mathscr{S}) \text{ in the generalized field case.} \end{cases}$$

$$(2.12)$$

Given a Gaussian stationary random field, a discrete field X_n, $n \in \mathbb{Z}_\nu$ or a generalized one $X(\varphi)$, $\varphi \in \mathscr{S}$ with some spectral measure G we call a random spectral measure Z_G adapted to it if it satisfies relation (2.10) or (2.11).

We have given a good representation of the random variables in the fields $\mathscr{H}_1$ by means of random integrals with respect to the random spectral measure. Later we shall show that one can also define multiple (Wiener–Itô) integrals with respect to the random spectral measure. In such a way we can get a good representation of all random variables with finite second moment which are measurable with respect to the σ-algebra generated by the elements of the original Gaussian stationary random field. Moreover, this representation is useful in the study of the limit theorem problems we are interested in.

To work out the theory of multiple Wiener–Itô integrals it is useful to have some additional knowledge about the properties of random spectral measures. I list below these properties, and I show how to prove them.

(vi) The random variables $\operatorname{Re} Z_G(A)$ are independent of the random variables $\operatorname{Im} Z_G(A)$.

(vii) The random variables of the form $Z_G(A \cup (-A))$ are real valued. If the sets $A_1 \cup (-A_1), \ldots, A_n \cup (-A_n)$ are disjoint, then the random variables $Z_G(A_1), \ldots, Z_G(A_n)$ are independent.

(viii) The relations $\operatorname{Re} Z_G(-A) = \operatorname{Re} Z_G(A)$ and $\operatorname{Im} Z_G(-A) = -\operatorname{Im} Z_G(A)$ hold, and if $A \cap (-A) = \emptyset$, then the (Gaussian) random variables $\operatorname{Re} Z_G(A)$ and $\operatorname{Im} Z_G(A)$ are independent with expectation zero and variance $G(A)/2$.

These properties easily follow from (i)–(v). Since $Z_G(\cdot)$ are complex valued Gaussian random variables, to prove the above formulated independence it is enough to show that the real and imaginary parts are uncorrelated. We show, as an example, the proof of (vi).

$$
\begin{aligned}
E \operatorname{Re} Z_G(A) \operatorname{Im} Z_G(B) &= \frac{1}{4i} E(Z_G(A) + \overline{Z_G(A)})(Z_G(B) - \overline{Z_G(B)}) \\
&= \frac{1}{4i} E(Z_G(A) + Z_G(-A))(\overline{Z_G(-B)} - \overline{Z_G(B)}) \\
&= \frac{1}{4i} G(A \cap (-B)) - \frac{1}{4i} G(A \cap B) \\
&\qquad + \frac{1}{4i} G((-A) \cap (-B)) - \frac{1}{4i} G((-A) \cap B) = 0
\end{aligned}
$$

for all pairs of sets A and B such that $G(A) < \infty$, $G(B) < \infty$, since $G(D) = G(-D)$ for all $D \in \mathscr{B}^\nu$. The fact that $Z_G(A \cup (-A))$ is real valued random variable, and the relations $\operatorname{Re} Z_G(-A) = \operatorname{Re} Z_G(A)$, $\operatorname{Im} Z_G(-A) = -\operatorname{Im} Z_G(A)$ under the conditions of (viii) follow directly from (v). The remaining statements of (vii) and (viii) can be proved similarly to (vi) only the calculations are simpler in this case.

The properties of the random spectral measure Z_G listed above imply in particular that the spectral measure G determines the joint distribution of the corresponding random variables $Z_G(B)$, $B \in \mathscr{B}^\nu$.

In the definition of random spectral measure we have imposed conditions (i)–(v) in the definition of random spectral measures, properties (vi)–(viii) were their consequences. Actually, we could have also omitted condition (iv) from the definition, because it can be deduced from the remaining conditions. This can be done by showing that the absolute value of the difference of the expressions at the two sides of the identity (iv) have zero second moment. (See the corresponding Remark at page 18 of my Lecture note.) Let me remark that if we want to define the distribution of a set of jointly Gaussian random variables, then it is enough to give the expected value and correlation function of these random variables. We followed a similar approach in the formulation of properties (i)–(iii) in the definition of random spectral measures. But since we work here with complex valued random variables, we had to add property (v) to these conditions to get a definition which determines the distribution of the random spectral measures.

2.3.   An application of the results on random spectral measures

We can define generalized stationary Gaussian field on the space $\mathscr{S}$ with such spectral measures G which satisfy (2.4), because this guarantees that the integral (2.3) is meaningful for all pairs of test-functions $\varphi, \psi \in \mathscr{S}$. Let us enlarge the space of test-functions $\mathscr{S}$ to a larger linear linear space $\mathscr{T} \supset \mathscr{S}$ which has the property that it is invariant under the shift transformations, i.e. if $\varphi(x) \in \mathscr{T}$, then $\varphi(x-t) \in \mathscr{T}$ for all arguments t. If the integral (2.3) is meaningful for all pairs of functions $\varphi, \psi \in \mathscr{T}$, then we can define a Gaussian stationary random field $X(\varphi)$, on a larger class of test functions $\varphi \in \mathscr{T}$, i.e. there is a Gaussian random field $X(\varphi)$ with test functions $\varphi \in \mathscr{T}$ which satisfies the identity $X(a_1\varphi_1 + a_2\varphi_2) = a_1X(\varphi_1) + a_2X(\varphi_2)$ with probability 1 for all constants a_1, a_2 and functions $\varphi_1, \varphi_2 \in \mathscr{T}$, and it has expectation $EX(\varphi) = 0$ and correlation function satisfying (2.3).

An especially interesting case appears when the linear space contains the indicator function of all rectangles of the form $\prod_{j=1}^{v}[a_j, b_j)$. In this case we get, restricting the stationary random field $X(\varphi)$, $\varphi \in \mathscr{T}$, to the set of cubes $\prod_{j=1}^{v}[a_j - \frac{1}{2}, a_j + \frac{1}{2})$, and identifying this cube with its center point $(a_1, \ldots, a_v)$ a discrete Gaussian stationary random field which can be considered the discretization of the original generalized field $X(\varphi)$, $\varphi \in \mathscr{S}$.

The above construction is especially interesting in the case when the generalized random field has the spectral measure G with density function $|X|^{-\alpha}$, with a parameter α is chosen so that the relation (2.4) holds with it. Such a spectral measure defines a self-similar stationary generalized Gaussian random field, and the above sketched method helps us to construct a discrete Gaussian stationary random field. I explain how we can construct the so-called fractional Brownian motions in such a way.

A fractional Brownian motion with Hurst parameter H, $0 < H < 1$, is defined as a Gaussian process $X(t)$, $t \geq 0$, with continuous trajectories and zero expectation, i.e. $EX(t) = 0$ for all $t \geq 0$, and with correlation function $R_H(s,t) = EX(s)X(t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$ for all $0 \leq s, t < \infty$. Naturally we must prove that such a process really exists.

I briefly explain that the correlation function of a fractional Brownian motion has a natural representation as the correlation function of the discretized version of an appropriately defined Gaussian stationary generalized self-similar field.

To understand this approach observe that a fractional Brownian motion with Hurst parameter H has the self-similarity property $EX(as)X(at) = a^{2H}EX(s)X(t)$ for all $a > 0$, and simple calculation shows that it also has the following stationary increments property: $EX(0)^2 = 0$, hence $X(0) = 0$ with probability 1, and $E[X(s+u) - X(u)][X(t+u) - X(u)] = EX(s)X(t)$ for all $0 \leq s, t, u < \infty$. To construct a fractional Brownian motion $X(t)$ first we define an appropriate stationary,

Gaussian generalized self-similar random field $\bar{X}(\varphi)$, $\varphi \in \mathscr{S}_1$ in the space of the real valued functions of the Schwartz space, and then we extend this field it to a larger parameter set (of functions), containing the indicator functions $\chi_{[0,t]}$ of the intervals $[0,t]$ for all $t \geq 0$. Finally we define the process $X(t)$ as $X(t) = \bar{X}(\chi_{[0,t]})$.

More explicitly, we can define for a parameter α a stationary generalized Gaussian field $\bar{X}(\varphi)$, $\varphi \in \mathscr{S}^1$, with zero expectation and spectral density $|u|^{-2\alpha}$, i.e. put $E\bar{X}(\varphi)\bar{X}(\psi) = \int \tilde{\varphi}(u)\bar{\tilde{\phi}}(u)|u|^{-2\alpha}\,du$. Then we introduce its natural extension to a function space containing the functions $\chi_{[0,t]}$ for all $t > 0$. Then we have

$$E\bar{X}(\chi_{[0,s]})\bar{X}(\chi_{[0,t]}) = \int \tilde{\chi}_{[0,s]}(u)\bar{\tilde{\chi}}_{[0,t]}(u)|u|^{-2\alpha}\,du = \int \frac{e^{isu}-1}{iu}\frac{e^{-itu}-1}{-iu}|u|^{-2\alpha}\,du,$$

provided that these integrals are convergent.

The above defined generalized fields exist if $2\alpha > -1$, and their extension to an appropriate space $\mathscr{T}$ containing the indicator functions $\chi_{[0,t]}$ exists if $-1 < 2\alpha < 1$. The first condition is needed to guarantee that the singularity of the integrand in the formula expressing the correlation function is not too large in the origin, and the second condition is needed to guarantee that the singularity of this integrand is not too large at the infinity even if we work with the Fourier transform of the indicator functions $\chi_{[0,t]}$ in the discretized case.

Simple calculation shows that the correlation function of the above defined random field satisfies the identity $E\bar{X}(\varphi_a)\bar{X}(\psi_a) = a^{-(1+2\alpha)}E\bar{X}(\varphi)\bar{X}(\psi)$, with the functions $\varphi_a(x) = \varphi(ax)$, $\psi_a(x) = \psi(ax)$, and similarly, we have $EX(as)X(at) = a^{(1+2\alpha)}EX(s)X(t)$ for all $a > 0$. Besides, the Gaussian stochastic process $X(t)$, $t > 0$, has stationary increments, i.e. $E[X(s+u) - X(u)][X(t+u) - X(u)] = EX(s)X(t)$ for all $0 \leq s,t,u < \infty$, and $EX(0)^2 = 0$. This follows from its construction with the help of a stationary Gaussian random field.

The above calculations imply that with the choice $\alpha = H - 1/2$ we get the correlation function of a fractional Brownian motion with Hurst parameter H for all $0 < H < 1$, more precisely the correlation function of this process multiplied by an appropriate constant. Indeed, it follows from the stationary increments property of the process that $E(X(t) - X(s))^2 = EX(t-s)^2$, if $t \geq s$, and the self-similarity property of this process implies that $EX(s)X(t) = \frac{1}{2}[EX(s)^2 + EX(t)^2 - E(X(t) - X(s))^2] = \frac{1}{2}EX(1)^2[s^{2H} + t^{2H} - |t-s|^{2H}]$.

We can get a representation of this process by means of a random integral with respect to a random spectral measure. This representation has the form

$$X(t) = \int \frac{e^{itu}-1}{iu}|u|^{-H+1/2}Z(du), \quad t > 0,$$

with the random spectral measure $Z(\cdot)$ corresponding to the Lebesgue measure on the real line. Here I omit the proof that such a stochastic process also has a version with continuous trajectories.

The representation of the fractional Brownian processes may be useful in the study of this process, but it seems to me that this was not fully exploited in the research about this subject. Finally I remark that there is a rather complete description of the stationary, self-similar Gaussian processes (both discrete and generalized ones) together with their Gaussian domain of attraction. (See P. Major: On renormalizing Gaussian fields. Z. *Wahrscheinlichkeitstheorie verw. Gebiete* **59** (1982), 515–533.)

Here I explain the content of this paper in a very informal way. The self-similar stationary Gaussian random fields (with random elements of zero expectation) are those, whose spectral measures are homogeneous functions. The stationary Gaussian random fields in their domain of attraction are those random fields, whose spectral measures are close (in a natural sense) to the spectral measure of the limiting field.

3.   Multiple Wiener–Itô integrals

Let us take a Gaussian stationary random field (either a discrete field X_n, $n \in \mathbb{Z}_\nu$, or a generalized field $X(\varphi)$, $\varphi \in \mathscr{S}$). We considered in both cases the (real) Hilbert space $\mathscr{H}$ consisting of the square integrable random variables, measurable with respect to the σ-algebra generated by the random variables of the stationary random field with the usual scalar product $(\xi, \eta) = E\xi\eta$, and we defined on it the group of shift transformations T_n, $n \in \mathbb{Z}_\nu$ and T_t, $t \in R^\nu$, for the discrete and generalized random fields respectively. We want to get a good representation of this Hilbert space together with the shift transformations on it.

First we decompose the Hilbert space $\mathscr{H}$ into the direct sum of orthogonal subspaces which are invariant with respect to all shift transformations.

We construct the invariant subspaces of the Hilbert space $\mathscr{H}$ with the help of its subspace $\mathscr{H}_1$ which is the closure of the finite linear combinations $\sum_{j=1}^k c_j X_{n_j}$ of the elements X_n, $n \in \mathbb{Z}_\nu$, in the case of discrete stationary random fields and of the random variables $X(\varphi)$, $\varphi \in \mathscr{S}$, in the case of generalized stationary random fields, where the closure is taken in the Hilbert space $\mathscr{H}$. First we define for all $n = 1, 2, \ldots$ the Hilbert subspace $\mathscr{H}_{\leq n} \subset \mathscr{H}$, $n = 1, 2, \ldots$, as the subspace which is the closure of the linear space consisting of the elements $P_n(X_{t_1}, \ldots, X_{t_m})$, where P_n runs through all polynomials of degree less than or equal to n, the integer m is arbitrary, and $X_{t_1}, \ldots, X_{t_m}$ are elements of $\mathscr{H}_1$. Let $\mathscr{H}_0 = \mathscr{H}_{\leq 0}$ consist of the constant functions, and put $\mathscr{H}_n = \mathscr{H}_{\leq n} \ominus \mathscr{H}_{\leq n-1}$, $n = 1, 2, \ldots$, where $\ominus$ denotes orthogonal completion. It is clear that the Hilbert space $\mathscr{H}_1$ given in this definition agrees with the previously defined Hilbert space $\mathscr{H}_1$.

The following theorem holds.

Theorem 3.1 on the decomposition of the Hilbert space $\mathscr{H}$ consisting of the

square integrable random variables measurable with respect to a stationary random field. *The Hilbert space $\mathscr{H}$ has the following decomposition with the help of the above defined Hilbert spaces $\mathscr{H}_n$, $n = 0, 1, 2, \ldots$.*

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1 + \mathscr{H}_2 + \cdots, \tag{3.1}$$

where $+$ denotes direct sum. Besides, all subspaces $\mathscr{H}_n$ of this decomposition are invariant subspaces of all shift transformations of the Hilbert space $\mathscr{H}$.

I explain the main ideas of the proof together with the formulation of some basic, classical results of the analysis that we need in the proof. First I recall the definition of Hermite polynomials, which play an important role both in this proof and in some further consideration. Then I also formulate some results about their properties.

Definition of Hermite polynomials. *The n-th Hermite polynomial $H_n(x)$ with leading coefficient 1 is the polynomial of order n defined by the formula $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2})$.*

The Hermite polynomials have the following property.

Theorem 3A. *The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \ldots$, form a complete orthogonal system in $L_2 \left(R, \mathscr{B}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \right)$. (Here $\mathscr{B}$ denotes the Borel σ-algebra on the real line.)*

We also need the following measure theoretical results in the proof of Theorem 3.1.

Let $(X_j, \mathscr{X}_j, \mu_j)$, $j = 1, 2, \ldots$, be countably many independent copies of a probability space $(X, \mathscr{X}, \mu)$. Let $(X^\infty, \mathscr{X}^\infty, \mu^\infty) = \prod_{j=1}^{\infty} (X_j, \mathscr{X}_j, \mu_j)$. With such a notation the following result holds.

Theorem 3B. *Let $\varphi_0, \varphi_1, \ldots, \varphi_0(x) \equiv 1$, be a complete orthonormal system in a Hilbert space $L_2(X, \mathscr{X}, \mu)$. Then the functions $\prod_{j=1}^{\infty} \varphi_{k_j}(x_j)$, where only finitely many indices k_j differ from 0, form a complete orthonormal basis in the product space $L_2(X^\infty, \mathscr{X}^\infty, \mu^\infty)$.*

Theorem 3C. *Let $Y_1, Y_2, \ldots$ be random variables on a probability space $(\Omega, \mathscr{A}, P)$ taking values in a measurable space $(X, \mathscr{X})$. Let ξ be a real valued random variable measurable with respect to the σ-algebra $\mathscr{B}(Y_1, Y_2, \ldots)$, and let $(X^\infty, \mathscr{X}^\infty)$ denote the infinite product $(X \times X \times \cdots, \mathscr{X} \times \mathscr{X} \times \cdots)$ of the space $(X, \mathscr{X})$ with itself. Then there exists a real valued, measurable function f on the space $(X^\infty, \mathscr{X}^\infty)$ such that $\xi = f(Y_1, Y_2, \ldots)$.*

Theorem 3.1 can be proved with the help of Theorems 3.A, 3B, and 3C in a natural way. One can show with the help of Theorems 3A and 3B that if we introduce the infinite product μ_∞ of the standard Gaussian probability distribution with itself on the infinite product space $(R^\infty, \mathscr{B}^\infty)$, then the set of all finite products of the form $H_{j_1}(x_{k_1}) \cdots H_{j_l}(x_{k_l})$ provide an orthogonal basis in the Hilbert space $L_2(R^\infty, \mathscr{B}^\infty, \mu_\infty)$. Then we can choose an orthonormal basis of standard Gaussian random variables $X_1, X_2, \ldots$ in $\mathscr{H}_1$, and we can construct with the help of this basis and Theorem 3C an appropriate embedding that implies Theorem 3.1. I omit the details of the proof.

I also formulate a result in the next Corollary 3.2 which we can get as a by-product of this proof. This result will be useful in our later considerations.

Corollary 3.2. *Let $\xi_1, \xi_2, \ldots$ be an orthonormal basis in $\mathscr{H}_1$, and let $H_j(x)$ denote the Hermite polynomial with order j and leading coefficient 1. Then the random variables $H_{j_1}(\xi_1) \cdots H_{j_k}(\xi_k)$, $k = 1, 2, \ldots$, $j_1 + \cdots + j_k = n$, and $j_k > 0$ form a complete orthogonal basis in $\mathscr{H}_n$.*

In a more detailed discussion we could have introduced the natural multivariate version of the Hermite polynomials, the so-called Wick polynomials. So I did in the lecture note which is the basis of this lecture. But here I omitted the discussion of Wick polynomials, because in the present text I do not work with them.

We constructed a good decomposition of the Hilbert space $\mathscr{H}$ into orthogonal invariant subspaces. Next we shall present the elements of these subspaces in the form of multiple Wiener–Itô integrals, because this is useful in the study of the limit problems we are interested in.

3.1.   The construction of multiple Wiener–Itô integrals

The multiple random integral I shall discuss here is actually different of the original Wiener–Itô integral. This is a version of it which was introduced by R. L. Dobrushin. Nevertheless, I shall apply the original name.

The original Wiener–Itô integral is taken with respect to a Gaussian orthogonal random measure Z_μ corresponding to a measure μ. This Gaussian orthogonal random measure Z_μ is defined on a measure space $(M, \mathscr{M}, \mu)$ with some σ-finite measure μ on $\mathscr{M}$, and it consists of (jointly) Gaussian random variables $Z_\mu(A)$ defined for all sets $A \in \mathscr{M}$ such that $\mu(A) < \infty$, and it has the properties that $EZ_\mu(A) = 0$, $EZ_\mu(A)^2 = \mu(A)$ for all $A \in \mathscr{M}$, the random variables $Z_\mu(A_j)$ are independent for disjoint sets A_j, and $Z_\mu(\cdot)$ is additive in the following sense. If $A_1, \ldots, A_k$, are disjoint sets, then $Z_\mu\left(\bigcup_{j=1}^{k} A_j\right) = \sum_{j=1}^{k} Z_\mu(A_j)$. Itô defined the k-fold random integral integral with respect to a Gaussian random measure Z_μ for

all multiplicities $k = 0, 1, 2, \ldots$. He defined the k-fold integral for such functions $f(x_1, \ldots, x_k)$ which are in the Hilbert space $L_2((M^k, \mathscr{M}^k, \mu^k))$. He could express all random variables which have finite second moment and are measurable with respect to the σ-algebra generated by the random variables $Z_\mu(\cdot)$ of the Gaussian orthogonal random measure as a sum of random integrals of different multiplicity. Moreover, this representation is unique. This result turned out to be useful in certain investigations.

Dobrushin worked out the theory of an analogous multiple random integral, where we integrate with respect to a random spectral measure instead of a Gaussian orthogonal random measure. The proof of the results in this case is somewhat more complicated. On the other hand, this integral is more appropriate in our investigations of limit theorems for non-linear functionals of Gaussian stationary random fields. First I briefly explain why such an integral is useful for us, and what kind of technical difficulties have to be overcome in their study which do not appear in the theory of the original Wiener–Itô integrals.

We want to study the Hilbert space $\mathscr{H}$ determined by our stationary random field together with the shift transformations on it. We can handle the shift transformation better with the help of Fourier transforms. To understand this let us take the following simple example. Let us consider a function $f(x)$ on the real line together with its shift $T_t f(x) = f(x - t)$. If we work with this shift transformation, then we can better calculate with the Fourier transform $\tilde{f}(u)$ since $\widetilde{T_t f}(u) = e^{itu} \tilde{f}(u)$. (Those who are familiar with the spectral theory of operators in Hilbert spaces can give the following interpretation to this example. If we take the Hilbert space of square integrable functions with respect to the Lebesgue measure, then the shift transformation $T_t \colon f(x) \to f(x - t)$ is a unitary operator on it. In the above formula we gave the spectral representation of this operator with the help of Fourier transforms.) Integration with respect to the random spectral measure plays a role similar to the Fourier transform on the real line, and as a consequence we shall have a representation of the shift transformation on $\mathscr{H}$ which is similar to the above example.

In the definition of the original Wiener–Itô integrals it was exploited that the Gaussian orthogonal random measure $Z_\mu(A_j)$ of disjoint sets A_j are independent. We want to apply a similar argument in the definition of random integrals with respect to random spectral measures. But here we have only a weaker independence property. We can state that $Z_G(A)$ and $Z_G(B)$ are independent only if $A \cup (-A)$ and $B \cup (-B)$ are disjoint. (See property (vii) of the random spectral measures.) Another point where we have to be careful is that we want to define the random integrals so that they are real valued, since $\mathscr{H}$ contains real valued random variables. Since the random spectral measure $Z_G(\cdot)$ is complex valued, we have to find the appropriate class of kernel functions to guarantee this property of the random

integrals.

Now I turn to the definition of the multiple Wiener–Itô integrals. First I introduce the(*real*) Hilbert space $\bar{\mathscr{H}}_G^n$ and its symmetrization $\mathscr{H}_G^n$ whose elements will be the kernel functions of the n-fold Wiener–Itô integrals with respect to a random spectral measure Z_G. As we shall later see, the Wiener–Itô integrals of a function and of its symmetrization agree. Hence it would be enough to work with Wiener–Itô integrals whose kernel functions are in the symmetrized space $\mathscr{H}_G^n$. But for some technical reasons it will be better to work with Wiener–Itô integrals with kernel functions from both spaces.

Let G be the spectral measure of a stationary Gaussian random field (discrete or generalized one). We define the following *real* Hilbert spaces $\bar{\mathscr{H}}_G^n$ and $\mathscr{H}_G^n$, $n = 1, 2, \ldots$.

We have $f_n \in \bar{\mathscr{H}}_G^n$ if and only if $f_n = f_n(x_1, \ldots, x_n)$, $x_j \in R^\nu$, $j = 1, 2, \ldots, n$, is a complex valued function of n variables, and

(a) $f_n(-x_1, \ldots, -x_n) = \overline{f_n(x_1, \ldots, x_n)}$,

(b) $\|f_n\|^2 = \int |f_n(x_1, \ldots, x_n)|^2 G(dx_1) \ldots G(dx_n) < \infty$.

Relation (b) also defines the norm in $\bar{\mathscr{H}}_G^n$. The subspace $\mathscr{H}_G^n \subset \bar{\mathscr{H}}_G^n$ contains those functions $f_n \in \bar{\mathscr{H}}_G^n$ which are invariant under permutations of their arguments, i.e.

(c) $f_n(x_{\pi(1)}, \ldots, x_{\pi(n)})) = f_n(x_1, \ldots, x_n)$ for all $\pi \in \Pi_n$, where Π_n denotes the group of all permutations of the set $\{1, 2, \ldots, n\}$.

The norm in $\mathscr{H}_G^n$ is defined in the same way as in $\bar{\mathscr{H}}_G^n$. Moreover, the scalar product is also similarly defined, namely if $f, g \in \bar{\mathscr{H}}_G^n$, then

$$
\begin{aligned}
(f, g) &= \int f(x_1, \ldots, x_n) \overline{g(x_1, \ldots, x_n)} G(dx_1) \ldots G(dx_n) \\
&= \int f(x_1, \ldots, x_n) g(-x_1, \ldots, -x_n) G(dx_1) \ldots G(dx_n).
\end{aligned}
$$

Because of the symmetry $G(A) = G(-A)$ of the spectral measure $(f, g) = \overline{(f, g)}$, i.e. the scalar product (f, g) is a real number for all $f, g \in \bar{\mathscr{H}}_G^n$. This means that $\bar{\mathscr{H}}_G^n$ is a real Hilbert space. We also define $\mathscr{H}_G^0 = \bar{\mathscr{H}}_G^0$ as the space of real constants with the norm $\|c\| = |c|$. I remark that $\bar{\mathscr{H}}_G^n$ is actually the n-fold direct product of $\mathscr{H}_G^1$, while $\mathscr{H}_G^n$ is the n-fold symmetrical direct product of $\mathscr{H}_G^1$. Condition (a) means heuristically that f_n is the Fourier transform of a real valued function.

We also define the so-called Fock space $\operatorname{Exp}\mathscr{H}_G$ whose elements are sequences of functions $f = (f_0, f_1, \dots)$, $f_n \in \mathscr{H}_G^n$ for all $n = 0, 1, 2, \dots$, such that

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \|f_n\|^2 < \infty,$$

where $\|f_n\|$ denotes the norm of the function f_n in the Hilbert space $\mathscr{H}_G^n$.

Given a function $f \in \bar{\mathscr{H}}_G^n$ we define $\operatorname{Sym} f$ as

$$\operatorname{Sym} f(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \Pi_n} f(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Clearly, $\operatorname{Sym} f \in \mathscr{H}_G^n$, and

$$\|\operatorname{Sym} f\| \leq \|f\|. \tag{3.2}$$

Let Z_G be a Gaussian random spectral measure corresponding to the spectral measure G on a probability space $(\Omega, \mathscr{A}, P)$. We shall define the n-fold Wiener–Itô integrals

$$I_G(f_n) = \frac{1}{n!} \int f_n(x_1, \dots, x_n) Z_G(dx_1) \dots Z_G(dx_n) \quad \text{for } f_n \in \bar{\mathscr{H}}_G^n$$

and

$$I_G(f) = \sum_{n=0}^{\infty} I_G(f_n) \quad \text{for } f = (f_0, f_1, \dots) \in \operatorname{Exp}\mathscr{H}_G.$$

We shall see that $I_G(f_n) = I_G(\operatorname{Sym} f_n)$ for all $f_n \in \bar{\mathscr{H}}_G^n$. Therefore, it would have been sufficient to define the Wiener–Itô integral only for functions in $\mathscr{H}_G^n$. Nevertheless, some arguments become simpler if we work in $\bar{\mathscr{H}}_G^n$. In the definition of Wiener–Itô integrals we restrict ourselves to the case when the spectral measure is non-atomic, i.e. $G(\{x\}) = 0$ for all $x \in R^{\nu}$. This condition is satisfied in all interesting cases. We could extend the definition of Wiener–itô integrals also to the case when G may be non-atomic, but we do not do that, because it seems so that we would not gain very much with such an extension.

In the definition of multiply Wiener–Itô integrals we follow a similar approach as in the definition of the one-fold integrals with respect to a random spectral measure. First we define them to a class of appropriately defined simple functions, and then we show that this integral can be extended because of an L_2-contraction property of this integral to the whole space $\bar{\mathscr{H}}_G^n$.

In the definition of the simple functions we have to take into account that they are elements of the space $\bar{\mathscr{H}}_G^n$. It will be natural to define them together with the notion of regular systems which are collections of disjoint subsets of R^{ν} with some additional properties. The simple functions of n-variables are those functions of

$\bar{\mathscr{H}}_G^n$ which are adapted in an appropriate way to a regular system. Here I give the definition of these notions.

Definition of Regular Systems and the Class of Simple Functions. *Let*

$$\mathscr{D} = \{\Delta_j, \ j = \pm 1, \pm 2, \ldots, \pm N\}$$

be a finite collection of bounded, measurable sets in R^ν indexed by the integers $\pm 1, \ldots, \pm N$. We say that $\mathscr{D}$ is a regular system if $\Delta_j = -\Delta_{-j}$, and $\Delta_j \cap \Delta_l = \emptyset$ if $j \neq l$ for all $j, l = \pm 1, \pm 2, \ldots, \pm N$. A function $f \in \bar{\mathscr{H}}_G^n$ is adapted to this system $\mathscr{D}$ if $f(x_1, \ldots, x_n)$ is constant on the sets $\Delta_{j_1} \times \Delta_{j_2} \times \cdots \times \Delta_{j_n}, \ j_l = \pm 1, \ldots, \pm N, \ l = 1, 2, \ldots, n$, it vanishes outside these sets and also on those sets of the form $\Delta_{j_1} \times \Delta_{j_2} \times \cdots \times \Delta_{j_n}$, for which $j_l = \pm j_{l'}$ for some $l \neq l'$.

A function $f \in \bar{\mathscr{H}}_G^n$ is in the class $\hat{\bar{\mathscr{H}}}_G^n$ of simple functions, and a (symmetric) function $f \in \mathscr{H}_G^n$ is in the class $\hat{\mathscr{H}}_G^n$ of simple symmetric functions if it is adapted to some regular system $\mathscr{D} = \{\Delta_j, \ j = \pm 1, \ldots, \pm N\}$.

Next we define the Wiener–Itô integrals of simple functions.

Definition of Wiener–Itô Integral of Simple Functions. *Let a simple function $f \in \hat{\bar{\mathscr{H}}}_G^n$ be adapted to some regular systems $\mathscr{D} = \{\Delta_j, \ j = \pm 1, \ldots, \pm N\}$. Its Wiener–Itô integral with respect to the random spectral measure Z_G is defined as*

$$\int f(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n) \tag{3.3}$$
$$= n! I_G(f) = \sum_{\substack{j_l = \pm 1, \ldots, \pm N \\ l = 1, 2, \ldots, n}} f(x_{j_1}, \ldots, x_{j_n}) Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}),$$

where $x_{j_l} \in \Delta_{j_l}, \ j_l = \pm 1, \ldots, \pm N, \ l = 1, \ldots, n$.

I remark that although the regular system $\mathscr{D}$ to which f is adapted is not uniquely determined (e.g. the elements of $\mathscr{D}$ can be divided to smaller sets in an appropriate way), the integral defined in (3.3) is meaningful, i.e. it does not depend on the choice of $\mathscr{D}$. This can be seen by observing that a refinement of a regular system $\mathscr{D}$ to which the function f is adapted yields the same value for the sum defining $n! I_G(f)$ in formula (3.3) as the original one. (We also exploit that if a function f is adapted to two different regular systems $\mathscr{D}_1$ and $\mathscr{D}_2$, then there is a regular system $\mathscr{D}_3$ which is a refinement of both of them, and the function f is adapted to it.) This follows from the additivity of the random spectral measure Z_G formulated in its property (iv), since this implies that each term $f(x_{j_1}, \ldots, x_{j_n}) Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n})$ in the sum at the right-hand side of formula (3.3) corresponding to the original regular system equals the sum of all such terms $f(x_{j_1}, \ldots, x_{j_n}) Z_G(\Delta'_{j_1'}) \cdots Z_G(\Delta'_{j_n'})$ in the sum corresponding to the refined partition for which $\Delta'_{j_1'} \times \cdots \times \Delta'_{j_n'} \subset \Delta_{j_1} \times \cdots \times \Delta_{j_n}$.

By property (vii) of the random spectral measures all products

$$Z_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})$$

with a non-zero coefficient in (3.3) are products of independent random variables. We had this property in mind when requiring the condition that the function f vanishes on a product $\Delta_{j_1}\times\cdots\times\Delta_{j_n}$ if $j_l=\pm j_{l'}$ for some $l\neq l'$. This condition is interpreted in the literature as discarding the hyperplanes $x_l=x_{l'}$ and $x_l=-x_{l'}$, $l,l'=1,2,\ldots,n$, $l\neq l'$, from the domain of integration. (Let me remark that here we also omitted the hyperplanes $x_l=-x_{l'}$ and not only the hyperplanes $x_l=x_{l'}$, $l\neq l'$, from the domain of integration. In particular, we omitted the points $x_l=-x_l$, i.e. $x_l=0$ for all $1\leq l\leq d$. This is a difference from the definition of the original Wiener–Itô integrals with respect to a Gaussian orthogonal random measure, where we omit only the hyperplanes $x_l=x_{l'}$, $l\neq l'$, from the domain of integration.) Property (a) of the functions in $\bar{\mathscr{H}}_G^n$ and property (v) of the random spectral measures imply that $I_G(f)=\overline{I_G(f)}$, i.e. $I_G(f)$ is a real valued random variable for all $f\in\hat{\bar{\mathscr{H}}}_G^n$. The relation

$$EI_G(f)=0,\quad\text{for }f\in\hat{\bar{\mathscr{H}}}_G^n,\quad n=1,2,\ldots\tag{3.4}$$

also holds. Let $\mathscr{H}_G^n=\mathscr{H}_G^n\cap\hat{\bar{\mathscr{H}}}_G^n$. If $f\in\hat{\bar{\mathscr{H}}}_G^n$, then $\mathrm{Sym}\,f\in\mathscr{H}_G^n$, and

$$I_G(f)=I_G(\mathrm{Sym}\,f).\tag{3.5}$$

Relation (3.5) holds, since $Z_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})=Z_G(\Delta_{\pi(j_1)})\cdots Z_G(\Delta_{\pi(j_n)})$ for all permutations $\pi\in\Pi_n$. I also claim that

$$EI_G(f)^2\leq\frac{1}{n!}\|f\|^2\quad\text{for }f\in\hat{\bar{\mathscr{H}}}_G^n,\tag{3.6}$$

and

$$EI_G(f)^2=\frac{1}{n!}\|f\|^2\quad\text{for }f\in\hat{\mathscr{H}}_G^n.\tag{3.7}$$

More generally, I claim that

$$EI_G(f)I_G(h)=\frac{1}{n!}(f,g)\;=\;\frac{1}{n!}\int f(x_1,\ldots,x_n)\overline{g(x_1,\ldots,x_n)}G(dx_1)\ldots G(dx_n)$$
$$\text{for }f,g\in\hat{\mathscr{H}}_G^n.\tag{3.8}$$

Because of (3.2) and (3.5) it is enough to check (3.8).

Let $\mathscr{D}$ be a regular system of sets in R^ν, and choose some sets $\Delta_{j_l}\in\mathscr{D}$ and $\Delta_{k_l}\in\mathscr{D}$ with indices $j_1,\ldots,j_n$ and $k_1,\ldots,k_n$ such that $j_l\neq\pm j_{l'}$, $k_l\neq\pm k_{l'}$ if $l\neq l'$. To prove (3.8) I show that

$$EZ_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1})\cdots Z_G(\Delta_{k_n})}$$
$$=\begin{cases}G(\Delta_{j_1})\cdots G(\Delta_{j_n})&\text{if }\{j_1,\ldots,j_n\}=\{k_1,\ldots,k_n\},\\0&\text{otherwise.}\end{cases}$$

To see the second relation in the last formula we will decompose the product whose expectation is investigated at the left-hand side of this formula to the product of two independent components in such a way that one of them has zero expectation.We shall find such a decomposition with the help of property (vii) of the random spectral measures.

The identity in the second relation of the last formula has to be proved under the condition $\{j_1, \ldots, j_n\} \neq \{k_1, \ldots, k_n\}$. Hence in this case there is an index l such that either $j_l \neq \pm k_{l'}$ for all $1 \leq l' \leq n$, or there exists an index l', $1 \leq l' \leq n$, such that $j_l = -k_{l'}$. In the first case $Z_G(\Delta_{j_l})$ is independent of the remaining coordinates of the vector $(Z_G(\Delta_{j_1}), \ldots, Z_G(\Delta_{j_n}), \overline{Z_G(\Delta_{k_1})}, \ldots, \overline{Z_G(\Delta_{k_n})})$, and $EZ_G(\Delta_{j_l}) = 0$. Hence the expectation of the investigated product equals zero, as we claimed. If $j_l = -k_{l'}$ with some index l', then a different argument is needed, since $Z_G(\Delta_{j_l})$ and $Z_G(-\Delta_{j_l})$ are not independent. In this case we can state that since $j_p \neq \pm j_l$ if $p \neq l$, and $k_q \neq \pm j_l$ if $q \neq l'$, the vector $(Z_G(\Delta_{j_l}), Z_G(-\Delta_{j_l}))$ is independent of the remaining coordinates of the above random vector. On the other hand, the product $Z_G(\Delta_{j_l})\overline{Z_G(-\Delta_{j_l})}$ has zero expectation, since $EZ_G(\Delta_{j_l})\overline{Z_G(-\Delta_{j_l})} = G(\Delta_{j_l} \cap (-\Delta_{j_l})) = 0$ by property (iii) of the random spectral measures and the relation $\Delta_{j_l} \cap (-\Delta_{j_l}) = \emptyset$ for the elements of a regular system. Hence the expectation of the considered product equals zero also in this case. If $\{j_1, \ldots, j_n\} = \{k_1, \ldots, k_n\}$, then

$$EZ_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1})\cdots Z_G(\Delta_{k_n})} = \prod_{l=1}^{n} EZ_G(\Delta_{j_l})\overline{Z_G(\Delta_{j_l})} = \prod_{l=1}^{n} G(\Delta_{j_l}).$$

If we have two functions $f, g \in \hat{\mathscr{H}}_G^n$, then we may assume that they are adapted to the same regular system $\mathscr{D} = \{\Delta_j, j = \pm 1, \ldots, \pm N\}$. Then, by exploiting that $I_G(g)$ is real valued, i.e. $I_G(g) = \overline{I_G(g)}$ we can calculate the expectation of $I_G(f)I_G(g)$ in the following way.

$$
\begin{aligned}
EI_G(f)I_G(g) &= EI_G(f)\overline{I_G(g)} = \left(\frac{1}{n}\right)^2 \sum\sum f(x_{j_1}, \ldots, x_{j_n})\overline{g(x_{k_1}, \ldots, x_{k_n})} \\
&\qquad EZ_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1})\cdots Z_G(\Delta_{k_n})} \\
&= \left(\frac{1}{n!}\right)^2 \sum f(x_{j_1}, \ldots, x_{j_n})\overline{g(x_{j_1}, \ldots, x_{j_n})} G(\Delta_{j_1})\cdots G(\Delta_{j_n}) \cdot n! \\
&= \frac{1}{n!} \int f(x_1, \ldots, x_n)\overline{g(x_1, \ldots, x_n)} G(dx_1)\cdots G(dx_n) = \frac{1}{n!}(f, g),
\end{aligned}
$$

where we took summation in the first sum for such pairs of indices $(j_1, \ldots, j_n)$ and $(k_1, \ldots, k_n)$ which are permutations of each other.

I claim that Wiener–Itô integrals of different order are uncorrelated. More explicitly, take two functions $f \in \hat{\bar{\mathscr{H}}}_G^n$ and $f' \in \hat{\bar{\mathscr{H}}}_G^{n'}$ such that $n \neq n'$. Then we

have

$$EI_G(f)I_G(f') = 0 \quad \text{if } f \in \hat{\bar{\mathscr{H}}}_G^n, \; f' \in \hat{\bar{\mathscr{H}}}_G^{n'}, \text{ and } n \neq n'. \tag{3.9}$$

To see this relation observe that a regular system $\mathscr{D}$ can be chosen is such a way that both f and f' are adapted to it. Then a similar, but simpler argument as the previous one shows that

$$EZ_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}) \overline{Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_{n'}})} = 0$$

for all sets of indices $\{j_1, \ldots, j_n\}$ and $\{k_1, \ldots, k_{n'}\}$ if $n \neq n'$, hence the sum expressing $EI_G(f)I_G(f')$ in this case equals zero.

We extend the definition of Wiener–Itô integrals to a more general class of kernel functions with the help of the following Lemma 3.3. This is a simple result whose proof is contained in Lemma 4.1 of my Lecture Note. Unfortunately this proof has a rather complicated notation, it contains several unpleasant technical details, hence it is rather unpleasant to read. I inserted an Appendix to this note, where I try to present a more accessible proof.

Lemma 3.3. *The class of simple functions $\hat{\bar{\mathscr{H}}}_G^n$ is dense in the (real) Hilbert space $\bar{\mathscr{H}}_G^n$, and the class of symmetric simple function $\hat{\mathscr{H}}_G^n$ is dense in the (real) Hilbert space $\mathscr{H}_G^n$.*

As the transformation $I_G(f)$ is a contraction from $\hat{\bar{\mathscr{H}}}_G^n$ into $L_2(\Omega, \mathscr{A}, P)$, it can uniquely be extended to the closure of $\hat{\bar{\mathscr{H}}}_G^n$, i.e. to $\bar{\mathscr{H}}_G^n$. (Here $(\Omega, \mathscr{A}, P)$ denotes the probability space where the random spectral measure $Z_G(\cdot)$ is defined.) At this point we exploit that if $f \in \hat{\bar{\mathscr{H}}}_G^n$, $N = 1, 2, \ldots$, is a convergent sequence in the space $\bar{\mathscr{H}}_G^n$, then the sequence of random variables $I_G(f_N)$ is convergent in the space $L_2(\Omega, \mathscr{A}, P)$, since it is a Cauchy sequence. With the help of this fact and Lemma 3.3 we can introduce the definition of Wiener–Itô integrals in the general case when the integral of a function $f \in \bar{\mathscr{H}}_G^n$ is taken.

Definition of Wiener–Itô Integrals. *Given a function $f \in \bar{\mathscr{H}}_G^n$ with a spectral measure G choose a sequence of simple functions $f_N \in \hat{\bar{\mathscr{H}}}_G^n$, $N = 1, 2, \ldots$, which converges to the function f in the space $\bar{\mathscr{H}}_G^n$. Such a sequence exists by Lemma 3.3. The random variables $I_G(f_N)$ converge to a random variable in the L_2-norm of the probability space where these random variables are defined, and the limit does not depend on the choice of the sequence f_N converging to f. This enables us to define the n-fold Wiener–Itô integral with kernel function f as*

$$\int f(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n) = n! I_G(f) = \lim_{N \to \infty} n! I_G(f_N),$$

where $f_N \in \hat{\bar{\mathscr{H}}}_G^n$, $N = 1, 2, \ldots$, is a sequence of simple functions converging to the function f in the space $\bar{\mathscr{H}}_G^n$.

We have defined the Wiener–Itô integral of a function $f(x_1, \ldots, x_n)$ with some nice properties. To simplify some later considerations I introduce the following convention. I shall sometimes consider the same function f with reindexed variables. We say that this reindexed function has the same Wiener–Itô integral. More explicitly, if $f(x_1, \ldots, x_n) = \bar{f}(x_{j_1}, \ldots, x_{j_n})$ with arbitrary (different) indices $j_1, \ldots, j_n$, then I shall say that

$$\int f(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n) = \int \bar{f}(x_{j_1}, \ldots, x_{j_n}) Z_G(dx_{j_1}) \ldots Z_G(dx_{j_n}n).$$

3.2. Further properties of Wiener–Itô integrals

I have claimed that the elements of the Hilbert spaces $\mathscr{H}_n$ defined before the formulation of Theorem 3.1 can be expressed by means of n-fold Wiener–Itô integrals, and the elements of Hilbert space $\mathscr{H}$ can be expressed by means of the sum of such integrals. I formulate this statement explicitly in the next Theorem 3.4. This theorem also states that this representation is unique if we allow only kernel functions from the space of symmetric functions, $\mathscr{H}_G^n$. Moreover, the mapping $I_G \colon \operatorname{Exp}\mathscr{H}_G \to \mathscr{H}$ defined after formula (3.2) is a unitary transformation between the Fock space $\operatorname{Exp}\mathscr{H}_G$ and $\mathscr{H}$. A similar statement holds also for the transformation $(n!)^{1/2} I_G \colon \mathscr{H}_G^n \to \mathscr{H}_n$.

I have also claimed that the shift transformations on $\mathscr{H}$ can be well expressed by means of Wiener–Itô integrals. This will be explained in Theorem 3.6.

One can see from the definition of the n-fold Wiener–Itô integrals that they are in $\mathscr{H}_{\leq n}$. Moreover, their orthogonality properties imply that they are in $\mathscr{H}_n$. To see that all elements of $\mathscr{H}_n$ can be expressed as an n-fold Wiener–Itô integral we need more information. This can be proved with the help of Corollary 3.2 and Itô's formula presented in Theorem 3.5 which has a central role in the theory of multiple Wiener–itô integrals. It is a very useful result, because it helps us to represent each elements of $\mathscr{H}_n$ in the form of an n-fold Wiener–Itô integral. This result also indicates the strong relation between multiple Wiener–Itô integrals and Hermite polynomials. I postpone its proof to the next section. We will get it as a consequence of the diagram formula, another important result in the theory of multiple Wiener–Itô integrals.

First I formulate Theorem 3.4.

Theorem 3.4. *Let a stationary Gaussian random field be given (discrete or generalized one), and let Z_G denote the random spectral measure adapted to it. If we integrate with respect to this Z_G, then the transformation $I_G \colon \operatorname{Exp}\mathscr{H}_G \to \mathscr{H}$, where $\mathscr{H}$ denotes the Hilbert space of those square integrable random variables which are measurable with respect to the σ-algebra generated by the random variables of the random spectral measure Z_G is unitary. More explicitly, formula*

(3.12) provides a unitary transformation between $\operatorname{Exp} \mathcal{H}_G$ *and* $\mathcal{H}$*. The transformation* $(n!)^{1/2} I_G \colon \mathcal{H}_G^n \to \mathcal{H}_n$ *is also unitary.*

Next I formulate Itô's formula which plays a central role in the proof of Theorem 3.4.

Theorem 3.5. (Itô's Formula.) *Let* $\varphi_1, \ldots, \varphi_m$, $\varphi_j \in \mathcal{H}_G^1$, $1 \le j \le m$, *be an orthonormal system in* L_G^2*. Let some positive integers* $j_1, \ldots, j_m$ *be given, and let* $j_1 + \cdots + j_m = N$*. Define for all* $i = 1, \ldots, N$ *the function* g_i *as* $g_i = \varphi_s$ *for* $j_1 + \cdots + j_{s-1} < i \le j_1 + \cdots + j_s$, $1 \le s \le m$*. (In particular,* $g_i = \varphi_1$ *for* $0 < i \le j_1$*.) Then*

$$H_{j_1} \left(\int \varphi_1(x) Z_G(dx) \right) \cdots H_{j_m} \left(\int \varphi_m(x) Z_G(dx) \right)$$

$$= \int g_1(x_1) \cdots g_N(x_N) Z_G(dx_1) \cdots Z_G(dx_N)$$

$$= \int \operatorname{Sym} \left[g_1(x_1) \cdots g_N(x_N) \right] Z_G(dx_1) \cdots Z_G(dx_N). \qquad (3.10)$$

($H_j(x)$ *denotes again the* j*-th Hermite polynomial with leading coefficient 1.)*
In particular, if $\varphi \in \mathcal{H}_G^1$, *and* $\int \varphi^2(x) G(dx) = 1$, *then*

$$H_n \left(\int \varphi(x) Z_G(dx) \right) = \int \varphi(x_1) \cdots \varphi(x_n) Z_G(dx_1) \cdots Z_G(dx_n). \qquad (3.11)$$

Proof of Theorem 3.4. We have already seen that I_G is an isometry. So it remains to show that it is a one to one map from $\operatorname{Exp} \mathcal{H}_G$ to $\mathcal{H}$ and from $\mathcal{H}_G^n$ to $\mathcal{H}_n$.

The one-fold integral $I_G(f)$, $f \in \mathcal{H}_G^1$, agrees with the stochastic integral $I(f)$ defined in Section 2. Hence $I_G(e^{i(n,x)}) = X(n)$ in the discrete field case, and $I_G(\tilde{\varphi}) = X(\varphi)$, $\varphi \in \mathscr{S}$, in the generalized field case. This implies that $I_G \colon \mathcal{H}_G^1 \to \mathcal{H}_1$ is a unitary transformation. Let $\varphi_1, \varphi_2, \ldots$ be a complete orthonormal basis in $\mathcal{H}_G^1$. Then $\xi_j = \int \varphi_j(x) Z_G(dx)$, $j = 1, 2, \ldots$, is a complete orthonormal basis in $\mathcal{H}_1$. Itô's formula implies that for all sets of positive integers $(j_1, \ldots, j_m)$ the random variable $H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m)$ can be written as a $j_1 + \cdots + j_m$-fold Wiener–Itô integral. Therefore Theorem 3.1 implies that the image of $\operatorname{Exp} \mathcal{H}_G$ is the whole space $\mathcal{H}$, and since $E H_{j_k}(\xi_k)^2 = j_k!$ the operator $I_G \colon \operatorname{Exp} \mathcal{H}_G \to \mathcal{H}$ is unitary.

The image of $\mathcal{H}_G^n$ contains $\mathcal{H}_n$ because of Corollary 3.2 and Itô's formula. Since these images are orthogonal for different n, formula (3.1) implies that the image of $\mathcal{H}_G^n$ coincides with $\mathcal{H}_n$. Hence $(n!)^{1/2} I_G \colon \mathcal{H}_G^n \to \mathcal{H}_n$ is a unitary transformation.

In Theorem 3.6 I shall describe the action of the shift transformations in $\mathcal{H}$. To do this let us first remark that by Theorem 3.4 all $\eta \in \mathcal{H}$ can be written in the

form

$$\eta = f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_n(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n) \tag{3.12}$$

with $f = (f_0, f_1, \ldots) \in \operatorname{Exp} \mathscr{H}_G$ in a unique way, where Z_G is the random measure adapted to the stationary Gaussian field. Now, we have

Theorem 3.6. *Let $\eta \in \mathscr{H}$ have the form (3.12). Then*

$$T_t \eta = f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int e^{i(t, x_1 + \cdots + x_n)} f_n(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n)$$

for all $t \in R^{\nu}$ in the generalized field and for all $t \in \mathbb{Z}_{\nu}$ in the discrete field case.

Proof of Theorem 3.6. Because of formulas (2.10) and (2.11) and the definition of the shift operator T_t we have

$$T_t \left(\int e^{i(n,x)} Z_G(dx) \right) = T_t X_n = X_{n+t} = \int e^{i(t,x)} e^{i(n,x)} Z_G(dx), \quad t \in \mathbb{Z}_{\nu},$$

and because of the identity $\widetilde{T_t \varphi}(x) = \int e^{(i(u,x)} \varphi(u-t)\, du = e^{i(t,x)} \tilde{\varphi}(x)$ for $\varphi \in \mathscr{S}$

$$\begin{aligned} T_t \left(\int \tilde{\varphi}(x) Z_G(dx) \right) &= T_t X(\varphi) = X(T_t \varphi) \\ &= \int e^{i(t,x)} \tilde{\varphi}(x) Z_G(dx), \quad \varphi \in \mathscr{S}, \quad t \in R^{\nu}, \end{aligned}$$

in the discrete and generalized field cases respectively. Observe, that the finite linear combinations of the functions $e^{i(n,x)}$, $n \in \mathbb{Z}_{\nu}$, is dense in the space $\mathscr{H}_1$ in the case of discrete stationary random fields, and the functions $\varphi \in \mathscr{S}_{\nu}$ are dense in $\mathscr{H}_1$ in the generalized stationary field case. Hence

$$T_t \left(\int f(x) Z_G(dx) \right) = \int e^{i(t,x)} f(x) Z_G(dx) \quad \text{if } f \in \mathscr{H}_G^1$$

for all $t \in \mathbb{Z}_{\nu}$ in the discrete field and for all $t \in R^{\nu}$ in the generalized field case. This means that Theorem 3.6 holds in the special case when η is a one-fold Wiener–Itô integral. Let $f_1(x), \ldots, f_m(x)$ be an orthogonal system in $\mathscr{H}_G^1$. The set of functions $e^{i(t,x)} f_1(x), \ldots, e^{i(t,x)} f_m(x)$ is also an orthogonal system in $\mathscr{H}_G^1$. (We choose $t \in \mathbb{Z}_{\nu}$ in the discrete and $t \in R^{\nu}$ in the generalized field case.) Hence Itô's formula implies that Theorem 3.6 also holds for random variables of the form

$$\eta = H_{j_1} \left(\int f_1(x) Z_G(dx) \right) \cdots H_{j_m} \left(\int f_m(x) Z_G(dx) \right)$$

and for their finite linear combinations. Since these linear combinations are dense in $\mathscr{H}$ Theorem 3.6 holds true.

I formulate at the end of this section a somewhat technical, but rather natural result. It is a formula for the change of variables in Wiener–Itô integrals. It can be interpreted so that we describe how to deal with the situation when we integrate with respect to the random spectral measure $Z_{G'}(dx) = g^{-1}(x)Z_G(dx)$ instead of $Z_G(dx)$ with some function $g(\cdot)$. (We have to assume that $g(x) = \overline{g(-x)}$ to get a random spectral measure again.) This new random measure corresponds to the spectral measure $G'(dx) = |g^{-2}(x)|G(dx)$, and to preserve the value of the (sum of) integrals we are working with we have to multiply the kernel function $f_n(x_1,\ldots,x_n)$ by $\prod\limits_{j=1}^{n} g(x_j)$ to compensate the multiplying factor $g^{-1}(x)$ in the definition of $Z_{G'}(dx)$.

Theorem 3.7. *Let G and G' be two non-atomic spectral measures such that G is absolutely continuous with respect to G', and let g(x) be a complex valued function such that*

$$
\begin{aligned}
g(x) &= \overline{g(-x)}, \\
|g^2(x)| &= \frac{dG(x)}{dG'(x)}.
\end{aligned}
$$

For every $f = (f_0, f_1, \ldots) \in \operatorname{Exp}\mathscr{H}_G$, we define

$$f'_n(x_1,\ldots,x_n) = f_n(x_1,\ldots,x_n)g(x_1)\cdots g(x_n), \quad n = 1,2,\ldots, \quad f'_0 = f_0.$$

Then $f' = (f'_0, f'_1, \ldots) \in \operatorname{Exp}\mathscr{H}^n_{G'}$, and

$$
\begin{aligned}
f_0 &+ \sum_{n=1}^{\infty} \int \frac{1}{n!} f_n(x_1,\ldots,x_n) Z_G(dx_1)\ldots Z_G(dx_n) \\
&\overset{\Delta}{=} f'_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int f'_n(x_1,\ldots,x_n) Z_{G'}(dx_1)\ldots Z_{G'}(dx_n),
\end{aligned}
$$

where Z_G and $Z_{G'}$ are Gaussian random spectral measures corresponding to G and G'.

Proof of Theorem 3.7. We have $\|f'_n\|_{G'} = \|f_n\|_G$, hence $f' \in \operatorname{Exp}\mathscr{H}_{G'}$. Let us choose a complete orthonormal system $\varphi_1, \varphi_2, \ldots$ in $\mathscr{H}^1_G$. Then $\varphi'_1, \varphi'_2, \ldots$, with $\varphi'_j(x) = \varphi_j(x)g(x)$ for all $j = 1, 2, \ldots$ is a complete orthonormal system in $\mathscr{H}^1_{G'}$. All functions $f_n \in \mathscr{H}^n_G$ can be written in the form

$$f(x_1,\ldots,x_n) = \sum c_{j_1,\ldots,j_n} \operatorname{Sym}\left(\varphi_{j_1}(x_1)\cdots \varphi_{j_n}(x_n)\right).$$

Then $f'(x_1, \ldots, x_n) = \sum c_{j_1, \ldots, j_n} \text{Sym}(\varphi'_{j_1}(x_1) \cdots \varphi'_{j_n}(x_n))$. Rewriting all terms

$$\int \text{Sym}(\varphi_{j_1}(x_1) \cdots \varphi_{j_n}(x_n)) Z_G(dx_1) \ldots Z_G(, dx_n)$$

and

$$\int \text{Sym}(\varphi'_{j_1}(x_1) \cdots \varphi'_{j_n}(x_n)) Z_{G'}(dx_1) \ldots Z_{G'}(, dx_n)$$

by means of Itô's formula we get that f and f' depend on a sequence of independent standard normal random variables in the same way. Theorem 3.7 is proved.

4.   The diagram formula and Itô's formula

The first main subject of this section is the diagram formula for the product of Wiener–Itô integrals. This formula enables us to rewrite the product of Wiener–Itô integrals in the form of a sum of Wiener–Itô integrals of different multiplicity. A Wiener–Itô integral is an element of the Hilbert space $\mathscr{H}$ consisting from square integrable random variables measurable with respect to the σ-algebra generated by the random variables of the underlying Gaussian stationary random field. Moreover, every element of the Hilbert space $\mathscr{H}$ can be expressed as the sum of Wiener–Itô integrals of different multiplicity. Thus, by the diagram formula the product of two Wiener–Itô integrals also belongs to the Hilbert space $\mathscr{H}$. The measurability property of the product is obvious, but the fact that its second moment is finite requires some explanation. Besides, the diagram formula is not a simple existence result, it also gives an explicit formula about how to rewrite the product of Wiener–Itô integrals as a sum of such integrals.

The other subject of this section which will be discussed in a special subsection is the proof of Itô's formula formulated in Theorem 3.5. The proof is made by induction which is based on the similarity of a recursion formula for a product of special form of Wiener–Itô integrals and a recursion formula about the relation of Hermite polynomials of different order. The recursion formula for Wiener–Itô integrals we apply is a simple consequence of the diagram formula.

To formulate the diagram formula first we have to introduce some notations.

Let us fix some functions $h_1 \in \mathscr{H}_G^{n_1}, \ldots, h_m \in \mathscr{H}_G^{n_m}$. In the diagram formula, we shall express the product $n_1! I_G(h_{n_1}) \cdots n_m! I_G(h_{n_m})$ as the sum of Wiener–Itô integrals. There is no unique terminology for this result in the literature. I shall follow the notation of Dobrushin's paper *Gaussian and their subordinated fields.* Annals of Probability 7, 1–28 (1979).

We introduce a class of diagrams γ denoted by $\Gamma(n_1, \ldots, n_m)$, and define with the help of each diagram γ in this class a function h_γ which will be the kernel function of one of the Wiener–Itô integrals taking part in the sum expressing the

product of the Wiener–Itô integrals we investigate. First we define the diagrams γ and the functions h_γ corresponding to them, and then we formulate the diagram formula with their help. After the formulation of this result I present an example together with some figures which may help in understanding better what the diagram formula is like.

We shall use the term diagram of order $(n_1, \ldots, n_m)$ for an undirected graph of $n_1 + \cdots + n_m$ vertices which are indexed by the pairs of integers (j, l), $l = 1, \ldots, m$, $j = 1, \ldots, n_l$ if the second term in the pair (j, l) equals l, and we shall call the set of vertices (j, l), $1 \leq j \leq n_l$ the l-th row of the diagram. The diagrams of order $(n_1, \ldots, n_m)$ are those undirected graphs with these vertices which have the properties that no more than one edge enters into each vertex, and edges can connect only pairs of vertices from different rows of a diagram, i.e. such vertices (j_1, l_1) and (j_2, l_2) for which $l_1 \neq l_2$. Let $\Gamma = \Gamma(n_1, \ldots, n_m)$ denote the set of all diagrams of order $(n_1, \ldots, n_m)$.

Given a diagram $\gamma \in \Gamma$ let $|\gamma|$ denote the number of edges in γ. Let there be given a set of functions $h_1 \in \bar{\mathscr{H}}_G^{n_1}, \ldots, h_m \in \bar{\mathscr{H}}_G^{n_m}$. Let us denote the variables of the function h_l by $x_{(j,l)}$ instead of x_j, i.e. let us write $h_l(x_{(1,l)}, \ldots, x_{(n_l,l)})$ instead of $h_l(x_1, \ldots, x_{n_l})$. Put $N = n_1 + \cdots + n_m$. We introduce the function of N variables corresponding to the vertices of the diagram by the formula

$$h(x_{(j,l)}, \; l = 1, \ldots, m, \; j = 1, \ldots, n_l) = \prod_{l=1}^{m} h_l(x_{(j,l)}, \; j = 1, \ldots, n_l). \qquad (4.1)$$

For each diagram $\gamma \in \Gamma = \Gamma(n_1, \ldots, n_m)$ we define the re-enumeration of the indices of the function in formula (4.1) in the following way. We enumerate the variables $x_{(j,l)}$ in such a way that the vertices into which no edge enters will have the indices $1, 2, \ldots, N - 2|\gamma|$, and the vertices connected by an edge will have the indices p and $p + |\gamma|$, where $p = N - 2|\gamma| + 1, \ldots, N - |\gamma|$. In such a way we have defined a function $h(x_1, \ldots, x_N)$ (with an enumeration of the indices of the variables depending on the diagram γ). After the definition of this function $h(x_1, \ldots, x_N)$ we take that function of $N - |\gamma|$ variables which we get by replacing the arguments $x_{N-|\gamma|+p}$ by the arguments $-x_{N-2|\gamma|+p}$, $1 \leq p \leq |\gamma|$, in the function $h(x_1, \ldots, x_N)$. Then we define the function h_γ appearing in the diagram formula by integrating this function by the product measure $\prod_{p=1}^{|\gamma|} G(dx_{N-2|\gamma|+p})$.

More explicitly, we write

$$h_\gamma(x_1, \ldots, x_{N-2|\gamma|}) \;\; = \;\; \int \cdots \int h(x_1, \ldots, x_{N-|\gamma|}, -x_{N-2|\gamma|+1}, \ldots, -x_{N-|\gamma|})$$
$$G(dx_{N-2|\gamma|+1}) \ldots G(dx_{N-|\gamma|}). \qquad (4.2)$$

The function h_γ depends only on the variables $x_1, \ldots, x_{N-2|\gamma|}$, i.e. it is independent of how the vertices connected by edges are indexed. Indeed, it follows from the

evenness of the spectral measure that by interchanging the indices s and $s + \gamma$ of two vertices connected by an edge we do not change the value of the integral h_γ. Let us now consider the Wiener–Itô integrals $(N - 2|\gamma|)!I_G(h_\gamma)$. In the diagram formula we shall show that the product of the Wiener–Itô integrals we considered can be expressed as the sum of these Wiener–Itô integrals. To see that the identity appearing in the diagram formula is meaningful observe that although the function h_γ may depend on the numbering of those vertices of γ from which no edge starts, the function $\operatorname{Sym} h_\gamma$, and therefore the Wiener–Itô integral $I_G(h_\gamma)$ does not depend on it.

Now I formulate the diagram formula. Then I make a remark about the definition of the function h_γ in it, and discuss an example to show how to calculate the terms appearing in this result.

Theorem 4.1. (Diagram Formula.) *For all functions $h_1 \in \bar{\mathscr{H}}_G^{n_1}, \ldots, h_m \in \bar{\mathscr{H}}_G^{n_m}$, $n_1, \ldots, n_m = 1, 2, \ldots$, the following relations hold with $N = n_1 + \cdots + n_m$:*

(A) $h_\gamma \in \bar{\mathscr{H}}_G^{N-2|\gamma|}$, and $\|h_\gamma\| \leq \prod\limits_{j=1}^{m} \|h_j\|$ for all $\gamma \in \Gamma$.

(B) $n_1!I_G(h_1) \cdots n_m!I_G(h_m) = \sum\limits_{\gamma \in \Gamma} (N - 2|\gamma|)!I_G(h_\gamma)$.

Here $\Gamma = \Gamma(n_1, \ldots, n_m)$, and the functions h_γ agree with the functions h_γ defined before the formulation of Theorem 4.1. In particular, h_γ was defined in (4.2).

Remark. Observe that at the end of the definition of the function h_γ we replaced the variable $x_{N-|\gamma|+p}$ by the variable $-x_{N-2|\gamma|+p}$ and not by $x_{N-2|\gamma|+p}$. This is related to the fact that in the Wiener–Itô integral we integrate with respect a complex valued random measure Z_G which has the property $EZ_G(\Delta)Z_G(-\Delta) = EZ_G(\Delta)\overline{Z_G(\Delta)} = G(\Delta)$, while $EZ_G(\Delta)Z_G(\Delta) = 0$ if $\Delta \cap (-\Delta) = \emptyset$. In the case of the original Wiener–Itô integral with respect to a Gaussian orthogonal random measure the situation is a bit different. In that case we integrate with respect to a real valued Gaussian orthonormal random measure Z_μ which has the property $EZ_\mu^2(\Delta) = \mu(\Delta)$. In that case a diagram formula also holds, but it has a slightly different form. The main difference is that in that case we define the function h_γ (because of the above mentioned property of the random measure Z_μ) by replacing the variable $x_{N-|\gamma|+p}$ by the variable $x_{N-2|\gamma|+p}$.

To make the notation in the diagram formula more understandable let us consider the following example.

Example. Let us take four functions $h_1 = h_1(x_1, x_2, x_3) \in \bar{\mathscr{H}}_G^3$, $h_2 = h_2(x_1, x_2) \in \bar{\mathscr{H}}_G^2$, $h_3 = h_3(x_1, x_2, x_3, x_4, x_5) \in \bar{\mathscr{H}}_G^5$ and $h_4 = h_4(x_1, x_2, x_3, x_4) \in \bar{\mathscr{H}}_G^4$, and consider the product of Wiener–Itô integrals $3!I_G(h_1)2!I_G(h_2)5!I_G(h_3)4!I_G(h_4)$. Let

us look how to calculate the kernel function h_γ of a Wiener–Itô integral $(14 - 2|\gamma|)!I_G(h_\gamma)$, corresponding to a diagram $\gamma \in \Gamma(3,2,5,4)$, in the diagram formula.

We have to consider the class of diagrams $\Gamma(3,2,5,4)$, i.e. the diagrams with vertices which are indexed in the first row as $(1,1)$, $(2,1)$, $(3,1)$, in the second row as $(1,2)$, $(2,2)$, in the third row in as $(1,3)$, $(2,3)$, $(3,3)$, $(4,3)$, $(5,3)$ and in the fourth row as $(1,4)$, $(2,4)$, $(3,4)$, $(4,4)$. (See Fig. 1.)

$$
\begin{array}{ccccc}
\textbf{(1.1)} & \textbf{(2.1)} & \textbf{(3.1)} & & \\
\bullet & \bullet & \bullet & & \\
\textbf{(1.2)} & \textbf{(2.2)} & & & \\
\bullet & \bullet & & & \\
\textbf{(1.3)} & \textbf{(2.3)} & \textbf{(3.3)} & \textbf{(4.3)} & \textbf{(5.3)} \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\textbf{(1.4)} & \textbf{(2.4)} & \textbf{(3.4)} & \textbf{(4.4)} & \\
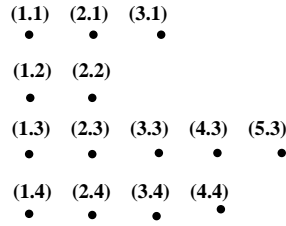\bullet & \bullet & \bullet & \bullet & \\
\end{array}
$$

Figure 1: The vertices of the diagrams $\gamma \in \Gamma(3,2,5,4)$

Let us take a diagram $\gamma \in \Gamma(3,2,5,4)$, and let us see how we can calculate the kernel function h_γ of the Wiener–Itô integral corresponding to it. We also draw some pictures which may help in following this calculation. Let us consider for instance the diagram $\gamma \in \Gamma(3,2,5,4)$ with edges $((2,1),(4,3))$, $((3,1),(1,3))$, $((1,2),(2,4))$, $((2,2),(5,3))$, $((3,3),(3,4))$. Let us draw the diagram γ with its edges and with such a re-enumeration of the vertices which helps in writing up the function $h(\cdot)$ (with $N = 14$ variables) corresponding to this diagram γ and introduced before the definition of the function h_γ.

The function defined in (4.1) equals in the present case

$$
\begin{aligned}
& h_1\big(x_{(1,1)},x_{(2,1)},x_{(3,1)}\big)h_2\big(x_{(1,2)},x_{(2,2)}\big)h_3\big(x_{(1,3)},x_{(2,3)},x_{(3,3)},x_{(4,3)},x_{(5,3)}\big) \\
& \quad h_4\big(x_{(1,4)},x_{(2,4)},x_{(3,4)},x_{(4,4)}\big).
\end{aligned}
$$

The variables of this function are indexed by the labels of the vertices of γ. We made a relabelling of the vertices of the diagram γ in such a way that by changing the indices of the above function with the help of this relabelling we get the function $h(\cdot)$ corresponding to the diagram γ. In the next step we shall make such a new relabelling of the vertices of γ which helps to write up the functions h_γ we are interested in. (See Fig. 2)

The function $h(\cdot)$ (with $N = 14$ variables) corresponding to the diagram γ can be written (with the help of the labels of the vertices in the second diagram) as

$$
\begin{aligned}
& h(x_1,x_2,\ldots,x_{14}) \\
& \quad = h_1(x_1,x_5,x_6)h_2(x_7,x_8)h_3(x_{11},x_2,x_9,x_{10},x_{13})h_4(x_3,x_{12},x_{14},x_4).
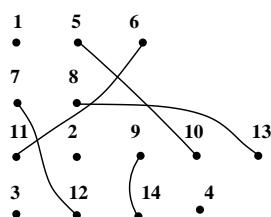\end{aligned}
$$

Figure 2: The diagram γ we are working with and the re-enumeration of its vertices.

Let us change the enumeration of the vertices of the diagram in a way that corresponds to the change of the arguments $x_{N-|\gamma|+p}$ by the arguments $-x_{N-2|\gamma|+p}$. This is done in the next picture. (In this notation the sign $(-)$ denotes that the variable corresponding to this vertex is $-x_{N-2|\gamma|+p}$ and not $x_{N-2|\gamma|+p}$. (See Fig 3.)
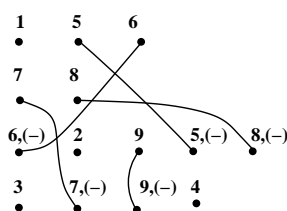

Figure 3: The diagram applied for the calculation of h_γ. The sign $-$ indicates that the corresponding argument is multiplied by -1.

With the help of the above diagram we can write up the function

$$h(x_1,\ldots,x_{N-|\gamma|},-x_{N-2|\gamma|+1},\ldots,-x_{N-|\gamma|})$$

corresponding to the diagram γ in a simple way. This yields that in the present case the function h_γ defined in (4.2) can be written in the form

$$\begin{aligned}
h_\gamma(x_1,x_2,x_3,x_4) \quad &= \int \cdots \int h_1(x_1,x_5,x_6)h_2(x_7,x_8)h_3(-x_6,x_2,x_9,-x_5,-x_8) \\
&\quad h_4(x_3,-x_7,-x_9,x_4)G(dx_5)G(dx_6)G(dx_7)G(dx_8)G(dx_9).
\end{aligned}$$

Here we integrate with respect to those variables x_j whose indices correspond to such a vertex of the last diagram from which an edge starts. Then the contribution of the diagram γ to the sum at the right-hand side of diagram formula equals $4!I_G(h_\gamma)$ with this function h_γ.

Let me remark that we had some freedom in choosing the enumeration of the vertices of the diagram γ. Thus e.g. we could have enumerated the four vertices

of the diagram from which no edge starts with the numbers 1, 2, 3 and 4 in an arbitrary order. A different indexation of these vertices would lead to a different function h_γ whose Wiener–itô integral is the same. I have chosen that enumeration of the vertices which seemed to be the most natural for me.

I shall omit the details of the proof of Theorem 4.1, because it contains several complicated, unpleasant details. I only briefly explain the main ideas.

The idea of the proof of Theorem 4.1. It suffices to prove Theorem 4.1 in the special case $m = 2$. Then the case $m > 2$ follows by induction.

We shall use the notation $n_1 = n$, $n_2 = m$, and we write $x_1, \ldots, x_{n+m}$ instead of $x_{(1,1)}, \ldots, x_{(n,1)}, x_{(1,2)} \ldots, x_{(m,2)}$. It is clear that the function h_γ satisfies Property (a) of the classes $\bar{\mathscr{H}}_G^{n+m-2|\gamma|}$ defined in subsection 3.1. We show that Part (A) of Theorem 4.1 is a consequence of the Schwartz inequality. I write down this part of the proof, because this is simple. The validity of this inequality means in particular that the functions h_γ satisfy also Property (b) of the class of functions $\bar{\mathscr{H}}_G^{n+m-2|\gamma|}$.

To prove this estimate on the norm of h_γ it is enough to restrict ourselves to such diagrams γ in which the vertices $(n, 1)$ and $(m, 2)$, $(n-1, 1)$ and $(m-1, 2), \ldots, (n-k, 1)$ and $(m-k, 2)$ are connected by edges with some number $0 \leq k \leq \min(n, m)$. In this case we can write

$$
\begin{aligned}
&\left| h_\gamma(x_1, \ldots, x_{n-k-1}, x_{n+1}, \ldots, x_{n+m-k-1}) \right|^2 \\
&= \left| \int h_1(x_1, \ldots, x_n) h_2(x_{n+1}, \ldots, x_{n+m-k-1}, -x_{n-k}, \ldots, -x_n) \right. \\
&\qquad \left. G(dx_{n-k}) \ldots G(dx_n) \right|^2 \\
&\leq \int |h_1(x_1, \ldots, x_n)|^2 G(dx_{n-k}) \ldots G(dx_n) \\
&\qquad \int |h_2(x_{n+1}, \ldots, x_{n+m})|^2 G(dx_{n+m-k}) \ldots G(dx_{n+m})
\end{aligned}
$$

by the Schwartz inequality and the symmetry $G(-A) = G(A)$ of the spectral measure G. Integrating this inequality with respect to the free variables we get Part (A) of Theorem 4.1.

In the proof of Part (B) first we restrict ourselves to the case when $h_1 \in \hat{\bar{\mathscr{H}}}_G^n$ and $h_2 \in \hat{\bar{\mathscr{H}}}_G^m$, i.e. to the case when they are simple functions. Moreover, we may assume that they are adapted to such a regular system of subsets $\Delta_j \in \mathscr{D}$ which satisfies the inequality $G(\Delta_j) < \varepsilon$ with a very small number $\varepsilon > 0$. At this reduction we exploit that the measure G is non-atomic. This enables us to split up the elements of a regular system to very small subsets. By making a good approximation of the function h_1 and h_2 with such elementary functions and then

taking a limiting procedure with $\varepsilon \to 0$ we get the proof of Part (B). During the limiting procedure we may exploit the already proven Part (A) of Theorem 4.1.

To prove Part (B) in this case let us consider a regular system $\mathscr{D} = \{\Delta_j, \; j = \pm 1, \ldots, \pm N\}$ of subsets of R^n such that the functions h_1 and h_2 are adapted to it, and its elements satisfy the inequality $G(\Delta_j) < \varepsilon$ with a very small $\varepsilon > 0$. Let us fix a point $u_j \in \Delta_j$ in all sets $\Delta_j \in \mathscr{D}$. We can express the product $n! I_G(h_1) m! I_G(h_2)$ as

$$I = n! I_G(h_1) m! I_G(h_2) \;\; = \;\; {\sum}' h_1(u_{j_1}, \ldots, u_{j_n}) h_2(u_{k_1}, \ldots, u_{k_m})$$
$$Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}) Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_m})$$

with the numbers $u_{j_p} \in \Delta_{j_p}$ and $u_{k_r} \in \Delta_{k_r}$ we have fixed. Here the summation in $\sum'$ goes through all pairs $((j_1, \ldots, j_n), (k_1, \ldots, k_m))$, $j_p, k_r \in \{\pm 1, \ldots, \pm N\}$, $p = 1, \ldots, n$, $r = 1, \ldots, m$, such that $j_p \neq \pm j_{\bar{p}}$ and $k_r \neq \pm k_{\bar{r}}$ if $p \neq \bar{p}$ or $r \neq \bar{r}$.

Write

$$I \;\; = \;\; \sum_{\gamma \in \Gamma} {\sum}^{\gamma} h_1(u_{j_1}, \ldots, u_{j_n}) h_2(u_{k_1}, \ldots, u_{k_m})$$
$$Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}) Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_m}), \qquad (4.3)$$

where $\sum^{\gamma}$ contains those terms of $\sum'$ for which $j_p = k_r$ or $j_p = -k_r$ if the vertices $(1, p)$ and $(2, r)$ are connected in γ, and $j_p \neq \pm k_r$ if $(1, p)$ and $(2, r)$ are not connected.

Let us introduce the notation

$$\Sigma^{\gamma} = {\sum}^{\gamma} h_1(u_{j_1}, \ldots, u_{j_n}) h_2(u_{k_1}, \ldots, u_{k_m}) Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n})$$

for all $\gamma \in \Gamma$. We prove Theorem 4.1 if we show that the inner sum Σ^{γ} in formula (4.3) is very close to $(n + m - 2|\gamma|)! I_G(h_\gamma)$ for all $\gamma \in \Gamma$ if $\varepsilon > 0$ is chosen very small. To explain why it is so we make a good approximation of Σ^{γ}. For this goal we introduce the following notations. Put

$$\begin{aligned} A_1 &\;=\; A_1(\gamma) = \{p \colon \; p \in \{1, \ldots, n\}, \text{ and no edge starts from } (p, 1) \text{ in } \gamma\}, \\ A_2 &\;=\; A_2(\gamma) = \{r \colon \; r \in \{1, \ldots, m\}, \text{ and no edge starts from } (r, 2) \text{ in } \gamma\} \end{aligned}$$

and

$$\begin{aligned} B = B(\gamma) \;\; = \;\; &\{(p, r) \colon \; p \in \{1, \ldots, n\}, \; r \in \{1, \ldots, m\}, \\ &\quad (p, 1) \text{ and } (r, 2) \text{ are connected in } \gamma\}. \end{aligned}$$

We define with the help of this notation the expression

$$\Sigma_1^{\gamma} \;\; = \;\; {\sum}^{\gamma} h_1(u_{j_1}, \ldots, u_{j_n}) h_2(u_{k_1}, \ldots, u_{k_m}) \prod_{p \in A_1} Z_G(\Delta_{j_p}) \prod_{r \in A_2} Z_G(\Delta_{k_r})$$
$$\cdot \prod_{(p,r) \in B} E\left(Z_G(\Delta_{j_p}) Z_G(\Delta_{k_r}) \right).$$

It can be shown that for one part Σ_1^γ is very close to Σ^γ, and on the other hand it approximates well $(n+m-2\gamma)!I_G(h_\gamma)$ if $\varepsilon > 0$ is small.

We can prove the first statement by showing that $E(\Sigma^\gamma - \Sigma_1^\gamma)^2$ is very small for small ε. To prove the second statement we show that the expression Σ_1^γ is very similar to the integral defining $(n+m-2\gamma)!I_G(h_\gamma)$. To see this observe that the terms $E\left(Z_G(\Delta_{j_p})Z_G(\Delta_{k_r})\right)$ in the expression Σ_1^γ can be simplified. Indeed, since the terms of these products have indices $(p,r) \in B$, we have $j_p = \pm k_r$, and the products with such indices satisfy either the identity $E\left(Z_G(\Delta_{j_p})Z_G(-\Delta_{j_p})\right) = G(\Delta_{j_p})$ or the identity $EZ_G(\Delta_{j_p})^2 = 0$. Writing these relations in the expression Σ_1^γ, and by exploiting the properties of the functions h_1 and h_2 we get that the sum Σ_1^γ can be written as an $(n+m-2|\gamma|)$-fold Wiener–Itô integral of an elementary function which almost agrees with the function h_γ. (There is a small difference because this elementary function disappears on a small set where the function h_γ may not disappear. This set contains such points x which have two different coordinates x_u, x_v with indices $u \neq v$ such that $x_u \in \Delta_j$ and $x_v \in \pm\Delta_j$ with the same element Δ_j of the regular system $\mathscr{D}$.)

A careful analysis shows that both properties mentioned before hold. Their proof is natural, but it requires the application of a rather complicated notation. Hence I omitted the explanation of the details. Next I turn to the proof of Itô's formula.

4.1.   The proof of Itô's formula

We shall prove Itô's formula with the help of two results. Here is the first one.

Proposition 4.2. *Let $f \in \bar{\mathscr{H}}_G^n$ and $h \in \bar{\mathscr{H}}_G^1$. Let us define the functions*

$$f \underset{k}{\times} h(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n) = \int f(x_1,\ldots,x_n)\overline{h(x_k)}G(dx_k), \quad k = 1,\ldots,n,$$

and

$$fh(x_1,\ldots,x_{n+1}) = f(x_1,\ldots,x_n)h(x_{n+1}).$$

Then $f \underset{k}{\times} h$, $k = 1,\ldots,n$, and fh are in $\bar{\mathscr{H}}_G^{n-1}$ and $\bar{\mathscr{H}}_G^{n+1}$ respectively, and their norms satisfy the inequality $\|f \underset{k}{\times} h\| \leq \|f\| \cdot \|h\|$ and $\|fh\| \leq \|f\| \cdot \|h\|$. The relation

$$n!I_G(f)I_G(h) = (n+1)!I_G(fh) + \sum_{k=1}^{n}(n-1)!I_G(f \underset{k}{\times} h)$$

holds true.

Proposition 4.2 is a simple consequence of the diagram formula if we apply it for the product of the Wiener–Itô integrals $n!I_G(f)$ and $I_G(h)$ with the functions

$f \in \bar{\mathscr{H}}_G^n$ and $h \in \bar{\mathscr{H}}_G^1$. We have to observe that in this case such diagrams appear which have two rows, the first row containing the vertices $(1,1)$, $(2,1)$,...,$(n,1)$, and the second row having one vertex $(1,2)$. There are two kind of diagrams in this model. The first kind of diagrams contains no edge, and it gives the kernel function fh. The other kind of diagrams contains one edge connecting the vertices $(k,1)$ an $(1,2)$, $1 \le k \le n$, giving the kernel function $f \underset{k}{\times} h$. In the last step we exploited that $h(-x) = \overline{h(x)}$, because $h \in \mathscr{H}_G^1$. These observations imply Proposition 4.2.

The other result we need is the following (well-known) recursion formula for Hermite polynomials.

Lemma 4.3. *The identity*

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x) \quad for \ n = 1, 2, \ldots,$$

holds with the notation $H_{-1}(x) \equiv 0$.

Proof of Lemma 4.3.

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}) \quad = \quad -e^{x^2/2} \frac{d}{dx}\left(H_{n-1}(x)e^{-x^2/2}\right)$$
$$= \quad xH_{n-1}(x) - \frac{d}{dx}H_{n-1}(x).$$

Since $\frac{d}{dx}H_{n-1}(x)$ is a polynomial of order $n-2$ with leading coefficient $n-1$ we can write

$$\frac{d}{dx}H_{n-1}(x) = (n-1)H_{n-2}(x) + \sum_{j=0}^{n-3} c_j H_j(x).$$

To complete the proof of Lemma 4.3 it remains to show that in the last expansion all coefficients c_j are zero. This follows from the fact that $e^{-x^2/2}H_{n-1}(x)$ is orthogonal to any polynomial whose order is not larger than $n - 2e^{-x^2/2}H_{n-1}(x)$ and the calculation

$$\int e^{-x^2/2}H_j(x)\frac{d}{dx}H_{n-1}(x)\,dx \quad = \quad -\int H_{n-1}(x)\frac{d}{dx}(e^{-x^2/2}H_j(x))\,dx$$
$$= \quad \int e^{-x^2/2}H_{n-1}(x)P_{j+1}(x)\,dx = 0$$

with the polynomial $P_{j+1}(x) = xH_j(x) - \frac{d}{dx}H_j(x)$ of order $j+1$ for $j \le n-3$.

The proof of Itô's formula. We prove Itô's formula by induction with respect to N. It holds for $N = 1$. Assume that it holds for $N - 1$. Let us define the functions

$$f(x_1,\ldots,x_{N-1}) \quad = \quad g_1(x_1)\cdots g_{N-1}(x_{N-1})$$
$$h(x) \quad = \quad g_N(x).$$

Then

$$J = \int g_1(x_1)\cdots g_N(x_N)Z_G(dx_1)\ldots Z_G(dx_N)$$

$$= N!I_G(fh) = (N-1)!I_G(f)I_G(h) - \sum_{k=1}^{N-1}(N-2)!I_G(f \underset{k}{\times} h)$$

by Proposition 4.1. We can write because of our induction hypothesis that

$$J = H_{j_1}\left(\int \varphi_1(x)Z_G(dx)\right)\cdots H_{j_{m-1}}\left(\int \varphi_{m-1}(x)Z_G(dx)\right)$$

$$H_{j_m-1}\left(\int \varphi_m(x)Z_G(dx)\right)\int \varphi_m(x)Z_G(dx)$$

$$-(j_m-1)H_{j_1}\left(\int \varphi_1(x)Z_G(dx)\right)\cdots H_{j_{m-1}}\left(\int \varphi_{m-1}(x)Z_G(dx)\right)$$

$$H_{j_m-2}\left(\int \varphi_m(x)Z_G(dx)\right),$$

where $H_{j_m-2}(x) = H_{-1}(x) \equiv 0$ if $j_m = 1$. This relation holds, since

$$f \underset{k}{\times} h(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{N-1}) = \int g_1(x_1)\cdots g_{N-1}(x_{N-1})\overline{\varphi_m(x_k)}G(dx_k)$$

$$= \begin{cases} 0 & \text{if } k \leq N - j_m \\ g_1(x_1)\cdots g_{k-1}(x_{k-1})g_{k+1}(x_{k+1})\cdots g_{N-1}(x_{N-1}) & \text{if } N - j_m < k \leq N - 1. \end{cases}$$

Hence Lemma 4.3 implies that

$$J = \prod_{s=1}^{m-1}H_{j_s}\left(\int \varphi_s(x)Z_G(dx)\right)\left[H_{j_m-1}\left(\int \varphi_m(x)Z_G(dx)\right)\int \varphi_m(x)Z_G(dx)\right.$$

$$\left.-(j_m-1)H_{j_m-2}\left(\int \varphi_m(x)Z_G(dx)\right)\right] = \prod_{s=1}^{m}H_{j_s}\left(\int \varphi_s(x)Z_G(dx)\right),$$

as claimed.

5.   Some applications of the diagram formula

In this section I discuss two kinds of results related to the diagram formula. The first of them is about the description of subordinated random fields and the construction of non-Gaussian self-similar random fields. I shall formulate the results both for discrete and generalized random fields. I shall omit some proofs, in particular the proof of the results about generalized random fields which are related to the theory of generalized function, a circle of problems not discussed in this note. The other problem I shall discuss is about the estimation of high moments and the tail distribution of Wiener–Itô integrals by means of the diagonal formula.

5.1. Description of subordinated random fields, construction of self-similar random fields

First I deal with the description of subordinated random fields.

Let X_n, $n \in \mathbb{Z}_\nu$, be a discrete stationary Gaussian random field with a non-atomic spectral measure, and let the random field ξ_n, $n \in \mathbb{Z}_\nu$, be subordinated to it. Let Z_G denote the random spectral measure adapted to the random field X_n. By Theorem 3.4 the random variable ξ_0 can be represented as

$$\xi_0 = f_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k)$$

with an appropriate function $f = (f_0, f_1, \ldots) \in \operatorname{Exp} \mathscr{H}_G$ in a unique way. This formula together with Theorem 4.4 yields the following

Theorem 5.1. *A random field ξ_n, $n \in \mathbb{Z}_\nu$, subordinated to the stationary Gaussian random field X_n, $n \in \mathbb{Z}_\nu$, with non-atomic spectral measure can be written in the form*

$$\xi_n = f_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int e^{i((n, x_1 + \cdots + x_k)} f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k), \quad n \in \mathbb{Z}_\nu,$$

(5.1)

with some $f = (f_0, f_1, \ldots) \in \operatorname{Exp} \mathscr{H}_G$, where Z_G is the random spectral measure adapted to the random field X_n. This representation is unique. On the other hand, formula (5.1) defines a subordinated field for all $f \in \operatorname{Exp} \mathscr{H}_G$.

We rewrite formula (5.1) in a slightly different form that shows the similarity between Theorem 5.1 and its analogue, Theorem 5.2 that gives a representation of subordinated generalized fields.

Let G denote the spectral measure of the underlying stationary Gaussian random field. If it has the property $G(\{x \colon x_p = u\}) = 0$ for all $u \in R^1$ and $1 \le p \le \nu$, where $x = (x_1, \ldots, x_\nu)$ (this is a strengthened form of the non-atomic property of G), then the functions

$$\bar{f}_k(x_1, \ldots, x_k) = f_k(x_1, \ldots, x_k) \tilde{\chi}_0^{-1}(x_1 + \cdots + x_k), \quad k = 1, 2, \ldots,$$

are meaningful, as functions in the measure space $(R^{k\nu}, \mathscr{B}^{k\nu}, G^k)$, where $\tilde{\chi}_n(x) = e^{i(n,x)} \prod_{p=0}^{\nu} \frac{e^{ix^{(p)}} - 1}{ix^{(p)}}$, $n \in \mathbb{Z}_\nu$, denotes the Fourier transform of the indicator function of the ν-dimensional unit cube $\prod_{p=1}^{\nu} [n^{(p)}, n^{(p)} + 1]$. Then the random variable ξ_n in formula (5.1) can be rewritten in the form

$$\xi_n = f_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\chi}_n(x_1 + \cdots + x_k) \bar{f}_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k), \quad n \in \mathbb{Z}_\nu.$$

(5.2)

Hence the following Theorem 5.2 can be considered as the continuous time version of Theorem 5.1.

Theorem 5.2. *Let a generalized random field $\xi(\varphi)$, $\varphi \in \mathscr{S}$, be subordinated to a stationary Gaussian generalized random field $X(\varphi)$, $\varphi \in \mathscr{S}$. Let G denote the spectral measure of the field $X(\varphi)$, and let Z_G be the random spectral measure adapted to it. Let the spectral measure G be non-atomic. Then $\xi(\varphi)$ can be written in the form*

$$\xi(\varphi) = f_0 \cdot \tilde{\varphi}(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k),$$

(5.3)

where the functions f_k are invariant under all permutations of their variables,

$$f_k(-x_1, \ldots, -x_k) = \overline{f_k(x_1, \ldots, x_k)}, \quad k = 1, 2, \ldots,$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int (1 + |x_1 + \cdots + x_k|^2)^{-p} |f_k(x_1 + \cdots + x_k)|^2 G(dx_1) \ldots G(dx_k) < \infty \quad (5.4)$$

with an appropriate number $p > 0$. This representation is unique.

On the other side, all random fields $\xi(\varphi)$, $\varphi \in \mathscr{S}$, defined by formulas (5.3) and (5.4) are subordinated to the stationary, Gaussian random field $X(\varphi)$, $\varphi \in \mathscr{S}$.

I shall omit the proof of Theorem 5.2. I only make some comments on it. The proof depends heavily on the theory of generalized functions. Even the proof of the statement that formula (5.3) defines a (generalized) stationary field is not simple. It is not enough to show that $T_t \xi(\varphi)) = \xi(T_t \varphi)$ in this case. We also have to prove that $E[\xi(\varphi_n) - \xi(\varphi)]^2 \to 0$ if $\varphi_n \to \varphi$ in the topology of $\mathscr{S}$, and this demands some special argument.

But the really hard part of Theorem 5.2 is to show that all subordinated fields can be represented in the form of (5.3). In particular we have to find the kernel functions f_k in this formula. To find them first we show that there is a function $\varphi_0 \in \mathscr{S}$, whose Fourier transform nowhere disappears, and the linear combinations made with the help of its shifts are everywhere dense in $\mathscr{S}$. Then writing the random variable $\xi(\varphi_0)$ in the form (3.12), and writing the functions $f_n(x_1, \ldots, x_n)$ in this representation as $f_n(x_1, \ldots, x_n) = \frac{f_n(x_1, \ldots, x_n)}{\tilde{\varphi}_0(x_1 + \cdots + x_n)} \tilde{\varphi}_0(x_1 + \cdots + x_n)$, we get that we have to choose the kernel functions $\frac{f_n(x_1, \ldots, x_n)}{\tilde{\varphi}_0(x_1 + \cdots + x_n)}$ in formula (5.3). A detailed proof of Theorem 5.2 would demand much work, and I omit it.

We shall call the representations given in Theorems 5.1 and 5.2 the canonical representation of a subordinated field. This notion will play an important role in

our investigation about limit problems. We shall rewrite the random fields Z_n^N defined in formula (1.1) in the form of their canonical representation with the help of Itô's formula, and this helps us to study their limit behaviour.

From now on we restrict ourselves to the case $E\xi_n = 0$ or $E\xi(\varphi) = 0$ respectively, i.e. to the case when $f_0 = 0$ in the canonical representation. Next I construct self-similar stationary random fields. To find such fields observe that if

$$\xi(\varphi) = \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k),$$

then

$$\xi(\varphi_t^A) = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{t^{\nu}}{A(t)} \int \tilde{\varphi}(t(x_1 + \cdots + x_k)) f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k)$$

with the function φ_t^A defined in (2.2), where we apply the function $A(t) > 0$ appearing in that formula. Define the spectral measures G_t by the formula $G_t(A) = G(tA)$ for all sets A. Then it is not difficult to see looking first at the definition of Wiener–Itô integrals when it is applied for elementary functions and then taking limit in the general case that

$$\xi(\varphi_t^A) \overset{\Delta}{=} \sum_{k=1}^{\infty} \frac{1}{k!} \frac{t^{\nu}}{A(t)} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k\left(\frac{x_1}{t}, \ldots, \frac{x_k}{t}\right) Z_{G_t}(dx_1) \ldots Z_{G_t}(dx_k).$$

If the spectral measure G and the kernel functions f_k in the formula expressing $\xi(\varphi)$ have the homogeneity properties that $G(tB) = t^{2\kappa} G(B)$ with some $\kappa > 0$ for all $t > 0$ and $B \in \mathscr{B}^{\nu}$, and the identity $f_k(\lambda x_1, \ldots, \lambda x_k) = \lambda^{\nu - \kappa k - \alpha} f_k(x_1, \ldots, x_k)$ holds, and $A(t)$ is chosen as $A(t) = t^{\alpha}$, then Theorem 3.7 (with the choice $G'(B) = G(tB) = t^{2\kappa} G(B)$) implies that $\xi(\varphi_t^A) \overset{\Delta}{=} \xi(\varphi)$. Hence we obtain the following

Theorem 5.3. *Let a generalized random field $\xi(\varphi)$ be given by the formula*

$$\xi(\varphi) = \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k). \qquad (5.5)$$

If $f_k(\lambda x_1, \ldots, \lambda x_k) = \lambda^{\nu - \kappa k - \alpha} f_k(x_1, \ldots, x_k)$ for all k, $(x_1, \ldots, x_k) \in R^{k\nu}$ and $\lambda > 0$, $G(\lambda A) = \lambda^{2\kappa} G(A)$ for all $\lambda > 0$ and $A \in \mathscr{B}^{\nu}$, then ξ is a self-similar random field with parameter α.

The discrete time version of this result can be proved in the same way. It states the following

Theorem 5.4. *If a discrete random field ξ_n, $n \in \mathbb{Z}_{\nu}$, has the form*

$$\xi_n = \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\chi}_n(x_1 + \cdots + x_k) f_k(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k), \quad n \in \mathbb{Z}_{\nu},$$

$$(5.6)$$

and $f_k(\lambda x_1,\ldots,\lambda x_k) = \lambda^{\nu-\kappa k-\alpha} f_k(x_1,\ldots,x_k)$ for all k, $G(\lambda A) = \lambda^{2\kappa} G(A)$, then ξ_n is a self-similar random field with parameter α.

Theorems 5.3 and 5.4 enable us to construct self-similar random fields. Nevertheless, we have to check whether formulas (5.5) and (5.6) are meaningful. The hard part of this problem is to check whether the inequality

$$\sum \frac{1}{k!} \int |\tilde{\chi}_n(x_1+\cdots+x_k)|^2 |f_k(x_1,\ldots,x_k)|^2 G(dx_1)\ldots G(dx_k) < \infty$$

holds in the discrete parameter case or whether the inequality

$$\sum \frac{1}{k!} \int |\tilde{\varphi}(x_1+\cdots+x_k)|^2 |f_k(x_1,\ldots,x_k)|^2 G(dx_1)\ldots G(dx_k) < \infty \quad \text{for all } \varphi \in \mathscr{S}$$

holds in the generalized field case.

It is a rather hard problem to decide when these expressions are finite. This is a hard question even if we consider a single integral and not an infinite sum of integrals. One may consider the question whether an integral is convergent or divergent a technical problem, but one should not underestimate it. The question whether some integrals are convergent or divergent is closely related to the problem whether in a certain model we have a new type of limit theorem with a non-standard normalization and a new non-Gaussian limit or the classical central limit theorem holds in that model. In the next result I prove such a result about the finiteness of a certain integral which is needed to guarantee the existence of an important self-similar field. This self-similar field will appear in the limit theorems we shall prove.

Let us define the measure G

$$G(A) = \int_A |x|^{2\kappa-\nu} a\left(\frac{x}{|x|}\right) dx, \quad A \in \mathscr{B}^\nu, \tag{5.7}$$

where $a(\cdot)$ is a non-negative, measurable and even function on the ν-dimensional unit sphere $S_{\nu-1}$, and $\kappa > 0$. (The condition $\kappa > 0$ is imposed to guarantee the relation $G(A) < \infty$ for all bounded sets $A \in \mathscr{B}^\nu$.) We prove the following

Proposition 5.5. *Let the measure G be defined in formula (5.7).*

If the function $a(\cdot)$ is bounded on the unit sphere $S_{\nu-1}$, and $\frac{\nu}{k} > 2\kappa > 0$, then

$$D(n) = \int |\tilde{\chi}_n(x_1+\cdots+x_k)|^2 G(dx_1)\ldots G(dx_k) < \infty \quad \text{for all } n \in \mathbb{Z}_\nu,$$

and

$$\begin{aligned} D(\varphi) &= \int |\tilde{\varphi}(x_1+\cdots+x_k)|^2 G(dx_1)\ldots G(dx_k) \\ &\leq C \int (1+|x_1+\cdots+x_k|^2)^{-p} G(dx_1)\ldots G(dx_k) < \infty \end{aligned}$$

for all $\varphi \in \mathcal{S}$ and $p > \frac{v}{2}$ with some $C = C(\varphi, p) < \infty$.

Remark. In the lecture note which is the basis of these lectures I also proved that this result is sharp. Namely, if the function $a(\cdot)$ in the definition of the spectral measure G is always larger than some number $\varepsilon > 0$, and $2\kappa \leq 0$ or $2\kappa \geq \frac{v}{k}$, then the integrals defining $D(n)$ and $D(\varphi)$ are divergent. This means that the conditions imposed on κ in Proposition 5.5 cannot be improved. Actually, the condition about the property I imposed on $a(\cdot)$ can be weakened in this statement.

Proof of Proposition 5.5. We may assume that $a(x) = 1$ for all $x \in S_{v-1}$. Define

$$J_{\kappa,k}(x) = \int_{x_1+\cdots+x_k=x} |x_1|^{2\kappa-v} \cdots |x_k|^{2\kappa-v} dx_1 \ldots dx_k, \quad x \in R^v,$$

for $k \geq 2$, where $dx_1 \ldots dx_k$ denotes the Lebesgue measure on the hyperplane $x_1 + \cdots + x_k = x$, and let $J_{\kappa,1}(x) = |x|^{2\kappa-v}$. The identity

$$J_{\kappa,k}(\lambda x) = |\lambda|^{k(2\kappa-v)+(k-1)v} J_{\kappa,k}(x) = |\lambda|^{2k\kappa-v} J_{\kappa,k}(x), \quad x \in R^v, \ \lambda > 0,$$

holds because of the homogeneity of the integral (provided that the integral $J_{\kappa,k}(x)$ is finite). We can write, because of (5.7) with $a(x) \equiv 1$

$$D(n) = \int_{R^v} |\tilde{\chi}_n(x)|^2 J_{\kappa,k}(x) \, dx, \tag{5.8}$$

and

$$D(\varphi) = \int_{R^v} |\tilde{\varphi}(x)|^2 J_{\kappa,k}(x) \, dx.$$

We prove by induction on k that

$$J_{\kappa,k}(x) \leq C(\kappa,k)|x|^{2\kappa k-v} \tag{5.9}$$

with an appropriate constant $C(\kappa,k) < \infty$ if $\frac{v}{k} > 2\kappa > 0$.

Inequality (5.9) holds for $k = 1$, and we have

$$J_{\kappa,k}(x) = \int J_{\kappa,k-1}(y)|x-y|^{2\kappa-v} dy$$

for $k \geq 2$. Hence

$$J_{\kappa,k}(x) \ \leq \ C(\kappa,k-1) \int |y|^{(2\kappa(k-1)-v)}|x-y|^{2\kappa-v} dy$$

$$= \ C(\kappa,k-1)|x|^{2\kappa k-v} \int |y|^{(2\kappa(k-1)-v)}\left| \frac{x}{|x|} - y \right|^{2\kappa-v} dy = C(\kappa,k)|x|^{2\kappa k-v},$$

since $\int |y|^{(2\kappa(k-1)-v)}\left| \frac{x}{|x|} - y \right|^{2\kappa-v} dy < \infty$.

The last integral is finite, since its integrand behaves at zero asymptotically as $C|y|^{2\kappa(k-1)-\nu}$, at the point $e = \frac{x}{|x|} \in S_{\nu-1}$ as $C_2|y-e|^{2\kappa-\nu}$ and at infinity as $C_3|y|^{2\kappa k - 2\nu}$. Relations (5.8) and (5.9) imply that

$$
\begin{aligned}
D(n) &\leq C' \int |\tilde{\chi}_0(x)|^2 |x|^{2\kappa k - \nu} \, dx \leq C'' \int |x|^{2\kappa k - \nu} \prod_{l=1}^{\nu} \frac{1}{1+|x^{(l)}|^2} \, dx \\
&\leq C''' \int_{|x^{(1)}| = \max_{1 \leq l \leq \nu} |x^{(l)}|} |x^{(1)}|^{2\kappa k - \nu} \prod_{l=1}^{\nu} \frac{1}{1+|x^{(l)}|^2} \, dx \\
&= \sum_{p=0}^{\infty} C''' \int_{|x^{(1)}| = \max_{1 \leq l \leq \nu} |x^{(l)}|, \, 2^p \leq |x^{(1)}| < 2^{p+1}} + C''' \int_{|x^{(1)}| = \max_{1 \leq l \leq \nu} |x^{(l)}|, |x^{(1)}| < 1}.
\end{aligned}
$$

The second term in the last sum can be simply bounded by a constant, since $B = \left\{ x\colon \ |x^{(1)}| = \max_{1 \leq l \leq \nu} |x^{(l)}|, \ |x^{(1)}| < 1 \right\} \subset \{ x\colon \ |x| \leq \sqrt{\nu} \}$, and we have $|x^{(1)}|^{2\kappa k - \nu} \prod_{l=1}^{\nu} \frac{1}{1+|x^{(l)}|^2} \leq \text{const.} |x|^{2\kappa k - \nu}$ on the set B. Hence

$$
D(n) \leq C_1 \sum_{p=0}^{\infty} 2^{p(2\kappa k - \nu)} \left[\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx \right]^{\nu} + C_2 < \infty.
$$

We have $|\varphi(x)| \leq C(1+|x^2|)^{-p}$ with some $C > 0$ and $D > 0$ if $\varphi \in \mathscr{S}$. The proof of the estimate $D(\varphi) < \infty$ for $\varphi \in \mathscr{S}$ is similar but simpler.

We can prove some similar theorems, but they have smaller importance, so I omit them. I discuss instead another useful application of the diagram formula, the estimation of high moments of Wiener–itô integrals.

5.2. Moment estimates on Wiener–Itô integrals

Next I show that the diagram formula, Theorem 4.1, enables us to estimate the expectation of a product of Wiener–itô integrals, in particular the moments of a Wiener–Itô integral.

By applying the diagram formula we can rewrite the product of Wiener–Itô integrals as a sum of Wiener integrals of different multiplicity. The expected value of this sum equals the sum of the expected value of the individual terms. On the other hand, each Wiener–Itô integral of multiplicity $n \geq 1$ has zero expectation. Only the constant terms, i.e. Wiener–Itô integrals of zero multiplicity can have non-zero expectation. The constant terms in the diagram formula correspond to those diagrams in which there starts an edge from each vertex. This makes natural to introduce the notion of complete diagrams, defined in the following way. Let

$\bar{\Gamma} \subset \Gamma$ denote the set of complete diagrams, i.e. let a diagram $\gamma \in \bar{\Gamma}$ if an edge enters in each vertex of γ.

Clearly, we have $EI(h_\gamma) = 0$ for all $\gamma \in \Gamma \setminus \bar{\Gamma}$, since (3.4) holds for all $f \in \bar{\mathscr{H}}_G^n$, $n \geq 1$, and if $\gamma \in \bar{\Gamma}$, then $I(h_\gamma) \in \bar{\mathscr{H}}_G^0$. Let h_γ denote the value of $I(h_\gamma)$ in this case. Let us also observe that in part (A) of Theorem 4.1 we gave an upper bound for $|I(h_\gamma)| = \|h_\gamma\|$ if γ is a closed diagram. These facts imply the following

Proposition 5.6. *For all $h_1 \in \bar{\mathscr{H}}_G^{n_1}, \ldots, h_n \in \bar{\mathscr{H}}_G^{n_m}$*

$$E n_1! I_G(h_1) \cdots n_m! I_G(h_m) = \sum_{\gamma \in \bar{\Gamma}} h_\gamma.$$

(The sum on the right-hand side equals zero if $\bar{\Gamma}$ is empty.) Besides, we have

$$|h_\gamma| \leq \prod_{j=1}^{m} \|h_j\| \quad \text{for all } \gamma \in \bar{\Gamma}.$$

Proposition 5.6 enables us to give a good estimate on the high moments of a Wiener–Itô integral. We may assume that the kernel function of this integral is a symmetric function. In the next Corollary I formulate an estimate on the $2N$-th moment of an m-fold Wiener–Itô integral. The interesting case is when N is large.

Corollary 5.7. *Let $h \in \mathscr{H}_G^m$. Then*

$$\begin{aligned} E\left[(m! I_G(h))^{2N}\right] &\leq C(m,N)\|h\|^{2N} = C(m,N)(E(m! I_G(h))^2)^N \\ &\leq (2mN - 1)(2mN - 3) \cdots 3 \cdot 1 (E(m! I_G(h))^2)^N, \end{aligned}$$

where $C(m,N)$ denotes the number of complete diagrams consisting of $2N$ rows with m elements in each row.

Proof of Corollary 5.7. The first inequality in Corollary 5.7 follows immediately from Proposition 5.6 if we apply it to the $2N$-fold product of the Wiener–Itô integral $I_G(h)$ with itself. To prove the next identity it is enough to observe that

$$E(m! I_G(h))^2 = m! \|h\|^2 \quad \text{if } h \in \mathscr{H}_G^m.$$

Finally, we have to give an upper bound on the number of complete diagrams $C(m,N)$. Let us calculate the number of those 'generalized' closed diagrams with the same number of rows $2N$ and m vertices in each row, where also one edge starts from each vertex, but an edge also may connect vertices from the same row. Then it is not difficult to see that he number of such 'generalized' closed diagrams is $(2mN - 1)(2nN - 3) \cdots 3 \cdot 1$, and this is an upper bound for $C(m,N)$.

Next I formulate some results which can be considered as a consequence of the above statements. I shall not work out the details of the proofs. Finally I make some comments about the content of these results.

First I formulate the following

Theorem 5.8. *Let* $(\xi_1,\dots,\xi_k)$ *be a normal random vector, and* $P(x_1,\dots,x_k)$ *a polynomial of degree m. Then*

$$E\left[P(\xi_1,\dots,\xi_k)^{2N}\right] \leq Cm,N)(m+1)^N \left(EP(\xi_1,\dots,\xi_k)^2\right)^N$$

with the constant $C(m,N)$ *introduced in Corollary 5.7.*

I omit the proof of Theorem 5.8, I only explain its main idea. The random variable $P(\xi_1,\dots,\xi_k)$ can be expressed as the sum of j-fold Wiener–Itô integrals with $0 \leq j \leq m$. The moments of each integral can be bounded by means of Corollary 5.7. For $j < m$ we have a better estimate than for $j = m$. A careful analysis provides the proof of Theorem 5.8.

The next result gives an interesting estimate on the tail-distribution of Wiener–Itô integrals.

Theorem 5.9. *Let G be a non-atomic spectral measure and* Z_G *a random spectral measure corresponding to G. For all* $h \in \mathscr{H}_G^m$ *there exist some constants* $K_1 > K_2 > 0$ *and* $x_0 > 0$ *depending on the function h such that*

$$e^{-K_1 x^{2/m}} \leq P(|I_G(h)| > x) \leq e^{-K_2 x^{2/m}}$$

for all $x > x_0$.

Remark 1. As the proof of Theorem 5.9 shows the constant K_2 in the upper bound of the above estimate can be chosen as $K_m = C_m(EI_G(h)^2)^{-1/m}$ with a constant C_m depending only on the order m of the Wiener–Itô integral of $I_G(h)$. This means that for a fixed number m the constant K_2 in the above estimate can be chosen as a constant depending only on the variance of the random variable $I_G(h)$. On the other hand, no simple characterization of the constant $K_1 > 0$ appearing in the lower bound of this estimate is possible.

Remark 2. Theorem 5.9 has some interesting consequences. For instance, we know that all bounded random variables in the space $\mathscr{H}$ can be written as a sum of Wiener–Itô integrals. Theorem 5.9 implies that this representation of a bounded random variables must be an infinite sum, since the tail distribution of a finite sum tends too slowly to zero at infinity.

The proof of the lower bound in Theorem 5.9 requires a special argument that I omit. On the other hand, the upper bound follows from Corollary 5.7 and the Markov inequality.

Proof of the upper estimate in Theorem 5.9. By the Markov inequality

$$P(|I_G(h)| > x) \leq x^{2N} E(I_G(h)|^{2N}).$$

On the other hand, by Corollary 5.7

$$E(I_G(h)|^{2N}) \leq \frac{(2mN-1)(2mN-3)\cdots 3 \cdot 1}{(m!)^N}(EI_G(h)^2)^N.$$

We get, by multiplying the inequalities

$$(2Nm-2j-1)(2Nm-2j-1-2N)\cdots(2Nm-2j-1-2N(m-1)) \leq (2N)^m m!,$$

for all $j = 1, \ldots, N$ that

$$\frac{(2mN-1)(2mN-3)\cdots 3 \cdot 1}{(m!)^N} \leq (2N)^{mN}.$$

(This inequality could be sharpened, but it is sufficient for our purpose.) Choose a sufficiently small number $\alpha > 0$, and define $N = [\alpha x^{2/m}]$, where $[\cdot]$ denotes integer part. With this choice we have

$$P(|I_G(h)| > x) \leq (x^{-2}(2\alpha)^m x^2)^N (EI_G(h)^2)^N = [(2\alpha)^m EI_G(h)^2]^N \leq e^{-K_2 x^{2/m}}$$

if α is chosen in such a way that $(2\alpha)^m E(I_G(h)^2 \leq \frac{1}{e}$, $K_2 = \frac{\alpha}{2}$, and $x > x_0$ with an appropriate $x_0 > 0$.

Observe that if ξ is a standard normal variable, then $P(|\xi|^m > x) = P(|\xi| > x^{1/m}) < e^{-x^{2/m}}$ for $x > 1$, and this estimate is sharp. Thus Theorem 5.9 means that an m-fold Wiener–Itô integral, i.e. a random variable $\eta \in \mathcal{H}_m$ has a similar tail distribution behaviour as the m-th power of a normal random variable with expectation zero. This shows a new property of the decomposition of the Hilbert space $\mathcal{H}$ (consisting of the square integrable random variables measurable with respect to the σ-algebra generated by the underlying Gaussian random field.)

In such a way we got a different characterization of the space of random variables which can be written down as an m-fold Wiener–Itô integral with a fixed number m. Let me also remark that Theorem 5.9 is closely related to the previous results of this subsection, since the growth behaviour of the high moments of a random variable and the behaviour of its tail distribution are closely related.

If we want to study the high moments (or the behaviour of the tail distribution) of the random variables in $\mathcal{H}$, then we can do this also by means of the theory of the original Wiener–itô integrals and the diagram formula about the calculation of their products. In the theory of the original Wiener–Itô integrals we integrate with respect to Gaussian orthogonal measures. This has some advantages. It is simpler, and moreover it has some useful modifications. We can work out the theory of multiple integrals with respect to such random measures which are not Gaussian, but they preserve that property that in these random measures the measure of

disjoint sets are independent random variables. Also a version of the diagram formula can be proved for such random integrals, and this result has some useful applications.

Thus for instance I could prove good estimates on the moments and on the behaviour of the tail distribution of U-statistics by applying the diagram formula for a version of the diagram formula for an appropriately defined random measure. These results played a very important role in my Springer Lecture Note *On the estimation of multiple random integrals and degenerate U-statistics.*

In this subsection I explained how to get good estimates on non-linear functionals of Gaussian random fields with the help of Wiener–Itô integrals. I would remark that there is another powerful method to deal with such problems. This is the theory of logarithmic Soboliev inequalities worked out by E. Nelson and L. Gross. I would refer to the paper of L. Gross *Logarithmic Soboliev inequalities* Am. J. Math. **97** (1061–1083) (1975) about this subject. The theory of logarithmic Soboliev inequality is based on a theory completely different from the subject of these lectures (on the theory of Markov processes), so I will not discuss it here.

6. Some limit theorems for non-linear functionals of Gaussian random fields

In this section I give the proof of some non-trivial limit theorems about the limit behaviour of the renormalizations Z_n^N, $N = 1, 2, \ldots, n \in \mathbb{Z}_v$, of a stationary random field ξ_n, $n \in \mathbb{Z}_v$, defined in formula (1.1) if the underlying random field ξ_n, $n \in \mathbb{Z}_v$, has some nice properties. In the first subsection I formulate the main results, and introduce the notions needed to formulate them. Here I also explain the main ideas of the proofs which also indicate what kind of results we can expect. In the next subsection I present the details of the proof. In that part I copy the original proofs from my lecture note with almost no change, only with some rearrangement of the text. Finally I discuss the content of our results and their relation to some other problems.

6.1. Formulation of the main results

To formulate our results first we have to introduce some notions. First I recall the definition of slowly varying functions, an important notion also in the theory of limit theorems for sums of independent variables, and also formulate the Karamata theorem that gives a useful characterization of them.

Definition 6A. (Definition of Slowly Varying Functions.) A function $L(t)$, $t \in$

$[t_0, \infty)$, $t_0 > 0$, is said to be a slowly varying function (at infinity) if

$$\lim_{t \to \infty} \frac{L(st)}{L(t)} = 1 \quad \text{for all } s > 0.$$

We shall apply the following description of slowly varying functions.

Theorem 6A. (Karamata's Theorem.) *If a slowly varying function $L(t)$, $t \geq t_0$, with some $t_0 > 0$ is bounded on every finite interval, then it can be represented in the form*

$$L(t) = a(t) \exp \left\{ \int_{t_0}^{t} \frac{\varepsilon(s)}{s} \, ds \right\},$$

where $a(t) \to a_0 \neq 0$, $\varepsilon(t)$ is integrable on any finite interval $[t_0, t]$, and $\varepsilon(t) \to 0$ as $t \to \infty$.

We shall consider a stationary Gaussian random field X_n, $n \in \mathbb{Z}_\nu$, with expectation zero and correlation function

$$r(n) = EX_0 X_n = |n|^{-\alpha} a \left(\frac{n}{|n|} \right) L(|n|), \quad n \in \mathbb{Z}_\nu, \quad \text{if } n \neq (0, \ldots, 0)), \quad (6.1)$$

where $0 < \alpha < \nu$, $L(t)$, $t \geq 1$, is a slowly varying function, bounded in all finite intervals, and $a(t)$ is a continuous function on the unit sphere $\mathscr{S}_{\nu-1}$, satisfying the symmetry property $a(x) = a(-x)$ for all $x \in \mathscr{S}_{\nu-1}$. We shall prove limit theorems for such random fields which are subordinated to this Gaussian random field X_n, $n \in \mathbb{Z}_\nu$,

Let G denote the spectral measure of the field X_n, and let us define the measures G_N, $N = 1, 2, \ldots$, by the formula

$$G_N(A) = \frac{N^\alpha}{L(N)} G \left(\frac{A}{N} \right), \quad A \in \mathscr{B}^\nu, \quad N = 1, 2, \ldots. \quad (6.2)$$

To get our results we shall need a result about the asymptotic behaviour of the rescaled versions G_N of the spectral measure G. To formulate this result whose proof I postpone to the next subsection we have to introduce the notion of vague convergence of not necessarily finite measures on a Euclidean space. This notion is a natural counterpart of the notion of weak convergence, an important notion in probability theory if we work with not necessarily finite measures.

Definition of Vague Convergence of Measures. *Let G_n, $n = 1, 2, \ldots$, be a sequence of locally finite measures over R^ν, i.e. let $G_n(A) < \infty$ for all measurable bounded sets A. We say that the sequence G_n vaguely converges to a locally finite measure G_0 on R^ν (in notation $G_n \xrightarrow{v} G_0$) if*

$$\lim_{n \to \infty} \int f(x) G_n(dx) = \int f(x) G_0(dx)$$

for all continuous functions f with a bounded support.

I formulate the following

Lemma 6.1. *Let G be the spectral measure of a stationary random field with a correlation function $r(n)$ of the form (6.1). Then the sequence of measures G_N defined in (6.2) tends vaguely to a locally finite measure G_0. The measure G_0 has the homogeneity property*

$$G_0(A) = t^{-\alpha} G_0(tA) \quad \text{for all } A \in \mathscr{B}^\nu \quad \text{and } t > 0, \tag{6.3}$$

and it satisfies the identity

$$2^\nu \int e^{i(t,x)} \prod_{j=1}^\nu \frac{1-\cos x^{(j)}}{(x^{(j)})^2} G_0(dx) \tag{6.4}$$

$$= \int_{[-1,1]^\nu} (1-|x^{(1)}|)\cdots(1-|x^{(\nu)}|) \frac{a\left(\frac{x+t}{|x+t|}\right)}{|x+t|^\alpha} dx, \quad \text{for all } t \in R^\nu.$$

Formula (6.3) together with the vague convergence of G_n to G_0 can be heuristically so interpreted that the measure G is asymptotically homogeneous in the neighbourhood of zero. On the other hand, it can be proved that we get a correlation function of the form (6.1) by defining it as the Fourier transform of a (positive) measure with a density of the form $g(u) = |u|^{\alpha-\nu} b(\frac{u}{|u|}) h(|u|)$, $u \in R^\nu$, where $b(\cdot)$ is a non-negative smooth function on the unit sphere $\{u: \ u \in R^\nu, |u| = 1\}$, and $h(u)$ is a non-negative smooth function on R^1 which does not disappear at the origin, and it tends to zero at infinity sufficiently fast. The regularizing function $h(|u|)$ is needed in this formula to make the function $g(\cdot)$ integrable. Results of this type are studied in the theory of generalized functions.

The above mentioned result is interesting for us, because it shows that there are correlation functions of the form (6.1) with appropriate functions $a(\cdot)$ and $L(\cdot)$. The problem with the definition of a correlation function $r(n)$, $n \in \mathbb{Z}_\nu$, satisfying (6.1) is that this function must be positive definite. We can guarantee this property by defining it as the Fourier transform of a measure.

I remark that formulae (6.3) and (6.4) imply that the function $a(t)$ and number α in the definition (6.1) of a correlation function $r(n)$ uniquely determine the measure G_0. Indeed, by formula (6.4) they determine the (finite) measure $\prod_{j=1}^\nu \frac{1-\cos x^{(j)}}{(x^{(j)})^2} G_0(dx)$, since they determine its Fourier transform. Hence they also determine the measure G_0. (Formula (6.3) shows that G_0 is a locally finite measure). Let us also remark that since $G_N(A) = G_N(-A)$ for all $N = 1, 2, \ldots$, and $A \in \mathscr{B}^\nu$, the relation $G_0(A) = G_0(-A)$, $A \in \mathscr{B}^\nu$ also holds. These properties of

the measure G_0 imply that it can be considered as the spectral measure of a generalized random field.

Now I formulate the basic limit theorem of this section.

Theorem 6.2. *Let X_n, $n \in \mathbb{Z}_v$, be a stationary Gaussian field with a correlation function $r(n)$ satisfying relation (6.1) and such that $r(0) = EX_n^2 = 1$, $n \in \mathbb{Z}_v$. Let us define the stationary random field $\xi_j = H_k(X_j)$, $j \in \mathbb{Z}_v$, with some positive integer k, where $H_k(x)$ denotes the k-th Hermite polynomial with leading coefficient 1, and assume that the parameter α appearing in (6.1) satisfies the relation $0 < \alpha < \frac{v}{k}$ with this number k. If the random fields Z_n^N, $N = 1, 2, \ldots$, $n \in \mathbb{Z}_v$, are defined by formula (1.1) with $A_N = N^{v-k\alpha/2}L(N)^{k/2}$ and the above defined random variables $\xi_j = H_k(X_j)$, then their multi-dimensional distributions tend to those of the random field Z_n^*,*

$$Z_n^* = \int \tilde{\chi}_n(x_1 + \cdots + x_k) Z_{G_0}(dx_1) \ldots Z_{G_0}(dx_k), \quad n \in \mathbb{Z}_v.$$

Here Z_{G_0} is a random spectral measure corresponding to the spectral measure G_0 which appeared in Lemma 6.1. The function $\tilde{\chi}_n(\cdot)$, $n = (n^{(1)}, \ldots, n^{(v)})$, is (similarly to formula (5.2) in Section 5) the Fourier transform of the indicator function of the v-dimensional unit cube $\prod\limits_{p=1}^{v} [n^{(p)}, n^{(p)} + 1]$.

I give a heuristic explanation for this result. First I explain why the choice of the normalizing constant A_N in Theorem 6.2 was natural, then I explain the main ideas of the proof. I shall work out the details in the next subsection.

There is a fairly well-known result by which if (ξ, η) is a Gaussian random vector with $E\xi = E\eta = 0$ and $E\xi^2 = E\eta^2 = 1$, then they satisfy the identity $EH_k(\xi)H_k(\eta) = k!(E\xi\eta)^k$. I give a short sketch of a possible proof.

Put $r = E\xi\eta$. Then we can write $\eta = r\xi + (1-r^2)^{1/2}Z$, where $Z = (1-r^2)^{-1/2}(\eta - r\xi)$ is a standard Gaussian random variable, uncorrelated with, hence also independent of the random variable ξ. We can express the random variable $H_k(\eta) = H_k(r\xi + (1-r^2)^{1/2}Z)$, as a linear combination of the random variables $H_j(\xi)H_l(Z)$ with indices j, l, $0 \le j+l \le k$. In this linear combination the term $H_k(\xi)$ has coefficient $r^k = (E\xi\eta)^k$. This is the only term in this linear combination which is not orthogonal to the random variable $H_k(\xi)$. Hence $EH_k(\xi)H_k(\eta) = (E\xi\eta)^k EH_k^2(\xi)$. On the other hand, $EH_k^2(\xi) = k!$ that can be seen with the help of some calculation. It follows for instance from Itô's formula and the diagram formula, but it can be proved directly from the formula by which we defined the Hermite polynomials by applying partial integration.

The above identity implies that

$$E(Z_n^N)^2 = \frac{k!}{A_N^2} \sum_{j,l \in B_0^N} r(j-l)^k \sim \frac{k!}{A_N^2} \sum_{j,l \in B_0^N} |j-l|^{-k\alpha} a^k \left(\frac{j-l}{|j-l|} \right) L(|j-l|)^k,$$

with the set B_0^N introduced after formula (1.1). Some calculation with the help of the above formula shows that with our choice of A_N the expectation $E(Z_n^N)^2$ is separated both from zero and infinity, therefore this is the natural norming factor. In this calculation we have to exploit the condition $k\alpha < v$, which implies that in the sum expressing $E(Z_n^N)^2$ those terms are dominant for which $j - l$ is relatively large, more explicitly which are of order N. There are const. N^{2v} such terms.

The field ξ_n, $n \in \mathbb{Z}_v$, is subordinated to the Gaussian field X_n. It is natural to express its terms via multiple Wiener–Itô integrals, and to write up the canonical representation of the fields Z_n^N defined in Section 5.

Itô's formula yields the identity

$$\xi_j = H_k\left(\int e^{i(j,x)}Z_G(dx)\right) = \int e^{i(j,x_1+\cdots+x_k)}Z_G(dx_1)\ldots Z_G(dx_k),$$

where Z_G is the random spectral measure adapted to the random field X_n. Then

$$
\begin{aligned}
Z_n^N &= \frac{1}{A_N}\sum_{j\in B_n^N}\int e^{i(j,x_1+\cdots+x_k)}Z_G(dx_1)\ldots Z_G(dx_k) \\
&= \frac{1}{A_N}\int e^{i(Nn,x_1+\cdots+x_k)}\prod_{j=1}^{v}\frac{e^{iN(x_1^{(j)}+\cdots+x_k^{(j)})}-1}{e^{i(x_1^{(j)}+\cdots+x_k^{(j)})}-1}Z_G(dx_1)\ldots Z_G(dx_k).
\end{aligned}
$$

Let us make the substitution $y_j = Nx_j$, $j = 1,\ldots,k$, in the last formula, and let us rewrite it in a form resembling formula (5.6). To this end, let us introduce the measures G_N defined in (6.2). It is not difficult to see that

$$Z_n^N \stackrel{\Delta}{=} \int f_N(y_1,\ldots,y_k)\tilde{\chi}_n(y_1+\cdots+y_k)Z_{G_N}(dy_1)\ldots Z_{G_N}(dy_k)$$

with the measure G_N defined in (6.2) and

$$f_N(y_1,\ldots,y_k) = \prod_{j=1}^{v}\frac{i(y_1^{(j)}+\cdots+y_k^{(j)})}{\left(\exp\left\{i\frac{1}{N}(y_1^{(j)}+\cdots+y_k^{(j)})\right\}-1\right)N}, \tag{6.5}$$

where $\tilde{\chi}_n(\cdot)$ is the Fourier transform of the indicator function of the unit cube $\prod_{j=1}^{v}[n^{(j)},n^{(j)}+1)$.

(In the above calculations we made a small inaccuracy. We calculated freely with Wiener–Itô integrals, although we defined them only in the case when the spectral measure G is non-atomic. We shall prove a result in Lemma 6B below which states that if the correlation function satisfies relation (6.1), then this property holds. Moreover, we shall prove a stronger statement, namely that all

hyperplanes $x^{(j)} = t$, $1 \leq j \leq \nu$, $t \in R^1$, have zero G measure. This fact together Fubini's theorem imply that the set where the denominator of the functions f_N defined in formula (6.5) disappears, i.e. the set, where $y_1^{(j)} + \cdots + y_k^{(j)} = 2lN\pi$ with some integer $l \neq 0$ and $1 \leq j \leq \nu$ has zero $G_N \times \cdots \times G_N$ measure. This means that the functions f_N are well defined.)

The functions f_N tend to 1 uniformly in all bounded regions, and the measures G_N tend vaguely to G_0 as $N \to \infty$ by Lemma 6.1. These relations suggest the following limiting procedure. The limit of Z_n^N can be obtained by substituting f_N with 1 and the random spectral measure Z_{G_N} with Z_{G_0} in the Wiener–Itô integral expressing Z_n^N. This would provide that the large-scale limit of the fields Z_n^N equals the random field Z_n^* defined in the formulation of Theorem 6.2.

We have to justify the above formal limiting procedure. We shall do this by showing first that the Wiener–Itô integral expressing Z_n^N is essentially concentrated in a large bounded region independent of N. The L_2-isomorphism of Wiener–Itô integrals can help us to show this. We will justify this argument in Lemma 6.3.

I shall discuss Lemma 6.3 in the next subsection, where the technical details of the proofs is explained. In this section I shall still present the proof of Lemma 6B which is, as I mentioned before, is needed for the justification of some formal steps we made before. Then I finish this subsection with a result formulated in Theorem 6.2′ which is a natural continuation of Theorem 6.2.

Lemma 6B. *Let the correlation function of a stationary random field X_n, $n \in \mathbb{Z}_\nu$, satisfy the relation $r(n) \leq A|n|^{-\alpha}$ with some $A > 0$ and $\alpha > 0$ for all $n \in \mathbb{Z}_\nu$, $n \neq 0$. Then its spectral measure G is non-atomic. Moreover, the hyperplanes $x^{(j)} = t$ have zero G measure for all $1 \leq j \leq \nu$ and $t \in R^1$.*

Proof of Lemma 6B. Lemma 6B clearly holds if $\alpha > \nu$, because in this case the spectral measure G has even a density function $g(x) = \sum\limits_{n \in \mathbb{Z}_\nu} e^{-i(n,x)} r(n)$. On the other hand, the p-fold convolution of the spectral measure G with itself (on the torus $R^\nu / 2\pi\mathbb{Z}_\nu$) has Fourier transform, $r(n)^p$, $n \in \mathbb{Z}^\nu$, and as a consequence in the case $p > \frac{\nu}{\alpha}$ this measure is non-atomic. Hence it is enough to show that if the convolution $G * G$ is a non-atomic measure, then the measure G is also non-atomic. But this is obvious, because if there were a point $x \in R^\nu / 2\pi\mathbb{Z}_\nu$ such that $G(\{x\}) > 0$, then the relation $G * G(\{x + x\}) > 0$ would hold, and this is a contradiction. (Here addition is taken on the torus.) The second statement of the lemma can be proved with some small modifications of the previous proof, by reducing it to the one-dimensional case.

I finish this subsection with the following

Theorem 6.2′. *Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian field with a correlation function $r(n)$ defined in (6.1) and such that $r(0) = EX_n^2 = 1$, $n \in \mathbb{Z}_\nu$. Let $H(x)$ be*

a real valued function with the properties $EH(X_n) = 0$ and $EH(X_n)^2 < \infty$. Let us consider the orthogonal expansion

$$H(x) = \sum_{j=1}^{\infty} c_j H_j(x), \quad \sum c_j^2 j! < \infty, \tag{6.6}$$

of the function $H(\cdot)$ by the Hermite polynomials H_j (with leading coefficients 1). Let k be the smallest index in this expansion such that $c_k \neq 0$. If $0 < k\alpha < \nu$ for the parameter α in (6.1), and the field Z_n^N is defined by the field $\xi_n = H(X_n)$, $n \in \mathbb{Z}_\nu$, and formula (1.1), then the multi-dimensional distributions of the fields Z_n^N with $A_N = N^{\nu - k\alpha/2} L(N)^{k/2}$ tend to those of the fields $c_k Z_n^*$, $n \in \mathbb{Z}_\nu$, where the field Z_n^* is the same as in Theorem 6.2.

Proof of Theorem 6.2' *with the help of Theorem 6.2.* Define $H'(x) = \sum_{j=k+1}^{\infty} c_j H_j(x)$ and $Y_n^N = \frac{1}{A_N} \sum_{l \in B_n^N} H'(X_l)$. Because of Theorem 6.2 in order to prove Theorem 6.2' it is enough to show that

$$E(Y_n^N)^2 \to 0 \quad \text{as } N \to \infty.$$

It can be proved similarly to the identity $EH_k(\xi)H_k(\eta) = k!(E\xi\eta)^k EH_k^2(\xi)$ for a Gaussian vector (ξ, η) such that $E\xi = E\eta = 0$, $E\xi_k^2 = E\eta_k^2 = 1$, that also the identity $EH_j(\xi)H_l(\eta) = j!\delta_{j,l}(E\xi\eta)^j$ holds, where $\delta_{j,l} = 0$ if $j \neq l$, and $\delta_{j,l} = 1$ if $j = l$.

This means in our case that

$$EH_j(X_n)H_l(X_m) = \delta_{j,l} j!(EX_n X_m)^j = \delta_{j,l} j! r(n-m)^j.$$

Hence

$$E(Y_n^N)^2 = \frac{1}{A_N^2} \sum_{j=k+1}^{\infty} c_j^2 j! \sum_{s,t \in B_n^N} [r(s-t)]^j.$$

Some calculation yields with the help of this identity and formula (6.1) that

$$E(Y_n^N)^2 = \frac{1}{A_N^2} \left[O(N^{2\nu - (k+1)\alpha} L(N)^{k+1}) + O(N^\nu) \right] \to 0.$$

(Observe that we imposed the condition $\sum c_j^2 j! < \infty$ which is equivalent to the condition $EH(X_n)^2 < \infty$.) Theorem 6.2' is proved.

The main difference between Theorem 6.2 and Theorem 6.2' is that in Theorem 6.2 we considered random variables $\xi_n = H_k(X_n)$, while in Theorem 6.2' $\xi_n = H(\xi_n)$ with such a function H in whose expansion with respect to Hermite

polynomials the Hermite polynomial $H_k(x)$ was the term with the smallest index with a non-zero coefficient. We saw that in these two cases a very similar result holds. In both theorems we imposed the condition $0 < k\alpha < \nu$ for the parameter α in (6.1). One may ask what happens if this condition is violated. I mentioned before the proof of Proposition 5.5 that in this case the multiple Wiener–Itô integral defining the limiting field Z_n^*, $n \in \mathbb{Z}_\nu$, in these theorems does not exist. I shall briefly discuss this case by pointing out that in this case the central limit theorem holds. But because of lack of time I cannot discuss the details of the proof.

6.2.   The details of the proofs

First I prove a lemma that enables us to prove the convergence of Wiener–Itô integrals under some conditions.

Lemma 6.3. *Let G_N, $N = 1, 2, \ldots$, be a sequence of non-atomic spectral measures on R^ν tending vaguely to a non-atomic spectral measure G_0. Let a sequence of measurable functions $K_N = K_N(x_1, \ldots, x_k)$, $N = 0, 1, 2, \ldots$, be given such that $K_N \in \bar{\mathscr{H}}_{G_N}^k$ for $N = 1, 2, \ldots$. Assume further that these functions satisfy the following properties: For all $\varepsilon > 0$ there exist some constants $A = A(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon) > 0$ such that the conditions (a) and (b) formulated below are satisfied.*

(a) *The function K_0 is continuous on the set $B = [-A, A]^{k\nu}$, and $K_N \to K_0$ uniformly on the set B as $N \to \infty$. Besides, the hyperplanes $x_p = \pm A$ have zero G_0 measure for all $1 \leq p \leq \nu$.*

(b) *$\int_{R^{k\nu} \setminus B} |K_N(x_1, \ldots, x_k)|^2 G_N(dx_1) \ldots G_N(dx_k) < \frac{\varepsilon^2}{k!}$ if $N = 0$ or $N \geq N_0$, and $K_0(-x_1, \ldots, -x_k) = \overline{K_0(x_1, \ldots, x_k)}$ for all $(x_1, \ldots, x_k) \in R^{k\nu}$.*

Then $K_0 \in \bar{\mathscr{H}}_{G_0}^k$, and

$$\int K_N(x_1, \ldots, x_k) Z_{G_N}(dx_1) \ldots Z_{G_N}(dx_k) \xrightarrow{\mathscr{D}} \int K_0(x_1, \ldots, x_k) Z_{G_0}(dx_1) \ldots Z_{G_0}(dx_k)$$

as $N \to \infty$, where $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution.

Remark. In my Lecture Note a somewhat more general result is proved that allows to handle also such cases where the function K_0 may have some discontinuities. There are results whose proof demands that more general result, but Theorem 6.2 can be proved with the help of this simpler result. I shall present a proof simpler than in the Lecture Note. The main difference is that in the Lecture Note I exploited that the weak convergence of probability measures is metrizable for

instance by means of the so-called Prokhorov metric, and I also applied some of its properties. Here I use instead that well-known result that a sequence of random variables U_n converge weakly to some random variable U_0 in the Euclidean space R^p if and only if their characteristic functions satisfy the relation $\lim\limits_{n\to\infty} E^{i(t,U_n)} = E^{i(t,U_0)}$ for all $t \in R^p$. I shall give a complete proof of Lemma 6.3 that applies Lemma 3.3 whose proof is given in the Appendix. Actually I apply a slightly stronger version of this result which also formulates an additional property of the approximation constructed in the proof of Lemma 3.3 which is mentioned at the end of the proof.

Proof of Lemma 6.3. First I show that $K_0 \in \bar{\mathscr{H}}_{G_0}^k$. Indeed, Conditions (a) and (b) obviously imply that

$$\int |K_0(x_1,\ldots,x_k)|^2 G_0(dx_1)\ldots G_0(dx_k) < \infty,$$

hence $K_0 \in \bar{\mathscr{H}}_{G_0}^k$.

Let us fix an $\varepsilon > 0$, and let us choose some $A = A(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon) > 0$ for which conditions (a) and (b) hold with this ε. Then

$$E\left[\int [1 - \chi_B(x_1,\ldots,x_k)]K_N(x_1,\ldots,x_k)Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k)\right]^2$$
$$\leq k! \int_{R^{kv}\setminus B} |K_N(x_1,\ldots,x_k)|^2 G_N(dx_1)\ldots G_N(dx_k) < \varepsilon^2 \qquad (6.7)$$

for $N = 0$ or $N > N_0$, where χ_B denotes the indicator functions of the set B introduced in the formulation of condition (a).

Since $B = [-A,A]^{kv}$, and $G_N \xrightarrow{v} G_0$, hence $G_N \times \cdots \times G_N(B) < C(A)$ with an appropriate constant $C(A) < \infty$ for all $N = 0, 1, \ldots$. Because of this estimate and the uniform convergence $K_N \to K_0$ on the set B we have

$$E\left[\int (K_N(x_1,\ldots,x_k) - K_0(x_1,\ldots,x_k))\chi_B(x_1,\ldots,x_k)Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k)\right]^2$$
$$\leq k! \int_B |K_N(x_1,\ldots,x_k) - K_0(x_1,\ldots,x_k)|^2 G_N(dx_1)\ldots G_N(dx_k) < \varepsilon^2 \quad (6.8)$$

for $N > N_1$ with some $N_1 = N_1(A,\varepsilon)$.

With the help of relations (6.7) and (6.8) I reduce the proof of Lemma 6.3 to the proof of the relation

$$\int K_0(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k)Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k)$$
$$\xrightarrow{\mathscr{D}} \int K_0(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k)Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k). \quad (6.9)$$

For this goal I introduce the random variables

$$T_N = \int K_N(x_1,\ldots,x_k) Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k),$$

$$U_N = \int K_N(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k) Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k),$$

$$V_N = \int K_0(x_1,\ldots,x_k))\chi_B(x_1,\ldots,x_k) Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k),$$

$$W = \int K_0(x_1,\ldots,x_k))\chi_B(x_1,\ldots,x_k) Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k).$$

By inequality (6.7) we have for all $t \in R^1$ and $N > N_0$

$$
\begin{aligned}
|E(e^{itT_N} - e^{itU_N})| &\leq E|(1 - e^{it(U_n - V_N)})| \leq E|(t(T_N - U_N)| \\
&\leq |t|(E(T_N - U_N)^2)^{1/2} \leq |t|\varepsilon.
\end{aligned}
$$

Similarly, $|E(e^{itU_N} - e^{itV_N})| \leq |t|(E(U_N - V_N)^2)^{1/2} \leq |t|\varepsilon$ for all $t \in R^1$ and $N > N_0$ by inequality (6.8). Finally, $Ee^{itV_N} \to Ee^{itW}$ for all $t \in R^1$ by (6.9). These relations together imply that $E|e^{itT_N} - Ee^{itW}| \leq C(t)|\varepsilon|$ if $N > N_0(t,\varepsilon)$ with some numbers $C(t)$ and $N_0(t,\varepsilon)$. Since this inequality holds for all $\varepsilon > 0$, it implies that $T_N \overset{\mathscr{D}}{\to} W$ which agrees with the statement of Lemma 6.3,

We shall prove formula (6.9) by showing that $K_0(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k)$ can be well approximated by simple functions from $\hat{\bar{\mathscr{H}}}_{G_0}^k$.

More explicitly, I claim that for all $\varepsilon > 0$ there exists a simple function $f_\varepsilon \in \hat{\bar{\mathscr{H}}}_{G_0}^k$ such that

$$E \int (K_0(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k) - f_\varepsilon(x_1,\ldots,x_k))^2 G_0(dx_1)\ldots G_0(dx_k) \leq \frac{\varepsilon^2}{k!} \tag{6.10}$$

and also

$$E \int (K_0(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k) - f_\varepsilon(x_1,\ldots,x_k))^2 G_N(dx_1)\ldots G_N(dx_k) \leq \frac{\varepsilon^2}{k!} \tag{6.11}$$

if $N \geq N_0$ with some threshold index $N_0 = N_0(\varepsilon, K_0(\cdot)\chi_B(\cdot))$. Moreover, this simple function f_ε can be chosen in such a way that it is adapted to such a regular system $\mathscr{D} = \{\Delta_j, \, j = \pm 1,\ldots,\pm M\}$ whose elements have boundaries of zero G_0 measure, i.e. $G_0(\partial \Delta_j) = 0$ for all $1 \leq |j| \leq M$.

Relation (6.9) can be proved with the help of the estimates (6.10), (6.11) and the relation (6.12) formulated below similarly to the reduction of the proof of

Lemma 6.3 to (6.9). Relation (6.12) states that the function f_ε appearing in formulas (6.10) and (6.11) satisfies also the relation

$$\int f_\varepsilon(x_1,\ldots,x_k) Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k) \overset{\mathscr{D}}{\to} \int f_\varepsilon(x_1,\ldots,x_k) Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k)$$
(6.12)

as $N \to \infty$. Formula (6.12) is a simplified version of the relation (6.9) where the kernel function K_0 in the integral is replaced by a simple function f_ε.

We can get the proof of (6.9) with the help of the estimates (6.10), (6.11) and (6.12) similarly to the argument leading to the proof of the limit relation of Lemma 6.3.

In the proof of (6.12) we exploit that the function $f_\varepsilon \in \hat{\bar{\mathscr{H}}}_{G_0}^k$ is adapted to such a regular system $\mathscr{D} = \{\Delta_j, \ j = \pm 1,\ldots,\pm M\}$ for which all Δ_j has the property $G_0(\partial\Delta_j) = 0$. In the proof of Lemma 3.3 in the Appendix I showed that the function f_ε and the regular system $\mathscr{D}$ to which it is adapted can be chosen in such a way. Besides, the spectral measures G_N have the property $G_N \overset{v}{\to} G_0$. Hence the (Gaussian) random vectors $(Z_{G_N}(\Delta_j), \ j = \pm 1,\ldots,\pm M)$ converge in distribution to the (Gaussian) random vector $(Z_{G_0}(\Delta_j), \ j = \pm 1,\ldots,\pm M)$ as $N \to \infty$. This implies that if we put the random variables $(Z_{G_N}(\Delta_j), \ j = \pm 1,\ldots,\pm M)$ to the arguments of a continuous function of $2M$ variables, then these random variables converge to the random variable we obtain if we put the random variables $(Z_{G_0}(\Delta_j), \ j = \pm 1,\ldots,\pm M)$ to the arguments of this function. Formula (6.12) follows from this statement because the random vectors we consider in it can be written as an appropriate polynomial of these random vectors.

Relation (6.10) follows directly from Lemma 3.3 if we apply it to the function $K_0(\cdot)\chi_B(\cdot)$. But we need a stronger version of this result, because we want to find such a function f_ε which also satisfies relations (6.11) and (6.12). We have seen that relation (6.12) holds if the approximating function f_ε has the additional property that it is adapted to a regular system $\mathscr{D}$ consisting of sets with zero G_0 measure.

A more careful analysis shows that a function f_ε with this extra property satisfies not only (6.10) but also (6.11) for $N \geq N_0$ with a sufficiently large threshold index N_0. We can get another explanation of the estimate (6.11) by exploiting that the function $h_0(x_1,\ldots,x_k)$ defined as

$$h_0(x_1,\ldots,x_k) = K_0(x_1,\ldots,x_k)\chi_B(x_1,\ldots,x_k) - f_\varepsilon(x_1,\ldots,x_k)$$

is almost everywhere continuous with respect to the measure $G_0 \times \cdots \times G_0$, and it disappears outside the compact set B. It can be shown that the vague convergence has similar properties as the weak convergence. In particular, the above mentioned almost everywhere continuity implies that

$$\lim_{N\to\infty} \int h_0(x_1,\ldots,x_k) G_N(dx_1)\ldots G_N(dx_k) = \int h_0(x_1,\ldots,x_k) G_0(dx_1)\ldots G_0(dx_k).$$

In such a way we can reduce the proof of (6.10) to the proof of (6.11). The proof of Lemma 6.3 is finished.

Now I show the proof of Theorem 6.2 with the help of Lemma 6.3, the still unproved Lemma 6.1 and still another result which will be formulated in Lemma 6.4.

Proof of Theorem 6.2. We want to prove that for all positive integers p, real numbers $c_1, \ldots, c_p$ and $n_l \in \mathbb{Z}_\nu$, $l = 1, \ldots, p$,

$$\sum_{l=1}^{p} c_l Z_{n_l}^N \overset{\mathcal{D}}{\to} \sum_{l=1}^{p} c_l Z_{n_l}^*,$$

since this relation also implies the convergence of the multi-dimensional distributions. Applying the same calculation as in the heuristic justification of Theorem 6.2 we get that

$$\sum_{l=1}^{p} c_l Z_{n_l}^N = \frac{1}{A_N} \sum_{l=1}^{p} c_l \int \sum_{j \in B_{n_l}^N} e^{i(j, x_1 + \cdots + x_k)} Z_G(dx_1) \ldots Z_G(dx_k),$$

and by applying the scaling $y_j = Nx_j$ we can show that

$$\sum_{l=1}^{p} c_l Z_{n_l}^N \overset{\Delta}{=} \int K_N(x_1, \ldots, x_k) Z_{G_N}(dx_1) \ldots Z_{G_N}(dx_k)$$

with

$$
\begin{aligned}
K_N(x_1, \ldots, x_k) &= \frac{1}{N^\nu} \sum_{l=1}^{p} c_l \sum_{j \in B_{n_l}^N} \exp \left\{ i \left(\frac{j}{N}, x_1 + \cdots + x_k \right) \right\} \\
&= f_N(x_1, \ldots, x_k) \sum_{l=1}^{p} c_l \tilde{\chi}_{n_l}(x_1 + \cdots + x_k).
\end{aligned}
\tag{6.13}
$$

with the function f_N defined in (6.5) and the measure G_N defined in (6.2), The function $\tilde{\chi}_n(\cdot)$ denotes again the Fourier transform of the indicator function of the unit cube $\prod_{j=1}^{\nu} [n^{(j)}, n^{(j)} + 1)$, $n = (n^{(1)}, \ldots n^{(\nu)})$.

Let us define the function

$$K_0(x_1, \ldots, x_k) = \sum_{l=1}^{p} c_l \tilde{\chi}_{n_l}(x_1 + \cdots + x_k)$$

and the measures μ_N on $R^{k\nu}$ by the formula

$$\mu_N(A) = \int_A |K_N(x_1, \ldots, x_k)|^2 G_N(dx_1) \ldots G_N(dx_k),$$
$$A \in \mathscr{B}^{k\nu} \text{ and } N = 0, 1, \ldots. \tag{6.14}$$

The measure G_0 defined with parameter $N = 0$ is the vague limit of the measures G_N.

We prove Theorem 6.2 by showing that Lemma 6.3 can be applied with these spectral measures G_N and functions K_N. Since $G_N \xrightarrow{v} G_0$, and $K_N \to K_0$ uniformly in all bounded regions in R^{kv}, it is enough to show, beside the proof of Lemma 6.1, that the measures μ_N, $N = 1, 2, \ldots$, tend weakly to the (necessarily finite) measure μ_0 which is also defined in (6.14), (in notation $\mu_N \xrightarrow{w} \mu_0$), i.e. $\int f(x)\mu_N(dx) \to \int f(x)\mu_0(dx)$ for all continuous and bounded functions f on R^{kv}. Then this convergence implies condition (b) in Lemma 6.3. Moreover, it is enough to show the slightly weaker statement by which there exists some finite measure $\bar{\mu}_0$ such that $\mu_N \xrightarrow{w} \bar{\mu}_0$, since then $\bar{\mu}_0$ must coincide with μ_0 because of the relations $G_N \xrightarrow{v} G_0$ and $K_N \to K_0$ uniformly in all bounded regions of R^{kv}, and K_0 is a continuous function. This implies that $\mu_N \xrightarrow{v} \mu_0$, and $\mu_0 = \bar{\mu}_0$.

There is a well-known theorem in probability theory about the equivalence between weak convergence of finite measures and the convergence of their Fourier transforms. It would be natural to apply this theorem for proving $\mu_N \xrightarrow{w} \bar{\mu}_0$. But actually we shall need a version of this result. In this version we exploit that we have the additional information that the measures μ_N, $N = 1, 2, \ldots$, are concentrated in the cubes $[-N\pi, N\pi)^{kv}$, since the spectral measure G is concentrated in $[-\pi, \pi)^v$. On the other hand, formula (6.1) provides only restricted information about the Fourier transform of μ_N. We have an asymptotic relation on the function $r(n)$, i.e. on the Fourier transform of G only in the points $n \in \mathbb{Z}_v$. As we shall see, this implies that we have control on the Fourier transform μ_N only in the points $\frac{n}{N}$, $n \in \mathbb{Z}_v$.

Hence it will be more appropriate for us to work with such a version of the result about the equivalence of weak convergence of probability measures which takes into account that we have only restricted information about the Fourier transform, but on the other hand we have the additional information that the probability measures we are working with are concentrated in some well-defined cubes. We will formulate such a result in the next Lemma 6.4. Its proof will be postponed after the proof of Theorem 6.2.

Lemma 6.4. *Let $\mu_1, \mu_2, \ldots$ be a sequence of finite measures on R^l such that $\mu_N(R^l \setminus [-C_N\pi, C_N\pi)^l) = 0$ for all $N = 1, 2, \ldots$, with some sequence $C_N \to \infty$ as $N \to \infty$. Define the modified Fourier transform*

$$\varphi_N(t) = \int_{R^l} \exp\left\{i\left(\frac{[tC_N]}{C_N}, x\right)\right\} \mu_N(dx), \quad t \in R^l,$$

where $[tC_N]$ is the integer part of the vector $tC_N \in R^l$. (For an $x \in R^l$ its integer part $[x]$ is the vector $n \in \mathbb{Z}_l$ for which $x^{(p)} - 1 < n^{(p)} \leq x^{(p)}$ if $x^{(p)} \geq 0$, and $x^{(p)} \leq n^{(p)} < x^{(p)} + 1$ if $x^{(p)} < 0$ for all $p = 1, 2, \ldots, l$.) If for all $t \in R^l$ the sequence

$\varphi_N(t)$ tends to a function $\varphi(t)$ continuous at the origin, then the measures μ_N weakly tend to a finite measure μ_0, and $\varphi(t)$ is the Fourier transform of μ_0.

In the proof of Theorem 6.2 we apply Lemma 6.4 with $C_N = N$ and $l = k\nu$ for the measures μ_N defined in (6.14). Because of the middle term in (6.13) we can write the modified Fourier transform φ_N of the measure μ_N as

$$\varphi_N(t_1,\ldots,t_k) = \sum_{r=1}^{p}\sum_{s=1}^{p} c_r c_s \psi_N(t_1 + n_r - n_s,\ldots,t_k + n_r - n_s) \tag{6.15}$$

with

$$\psi_N(t_1,\ldots,t_r) = \frac{1}{N^{2\nu}}\int \exp\left\{i\frac{1}{N}((j_1,x_1)+\cdots+(j_k,x_k))\right\}$$
$$\sum_{u\in B_0^N}\sum_{v\in B_0^N}\exp\left\{i\left(\frac{u-v}{N},x_1+\cdots+x_k\right)\right\}G_N(dx_1)\ldots G_N(dx_k)$$
$$= \frac{1}{N^{2\nu-k\alpha}L(N)^k}\sum_{u\in B_0^N}\sum_{v\in B_0^N} r(u-v+j_1)\cdots r(u-v+j_k), \tag{6.16}$$

where $j_p = [t_p N]$, $t_p \in R^\nu$, $p = 1,\ldots,k$.

The asymptotic behaviour of $\psi_N(t_1,\ldots,t_k)$ for $N \to \infty$ can be investigated by the help of the last relation and formula (6.1). Rewriting the last double sum in the form of a single sum by fixing first the variable $l = u - v \in [-N,N]^\nu \cap \mathbb{Z}_\nu$, and then summing up for l one gets

$$\psi_N(t_1,\ldots,t_k) = \int_{[-1,1]^\nu} f_N(t_1,\ldots,t_k,x)\,dx$$

with

$$f_N(t_1,\ldots,t_k,x)$$
$$= \left(1 - \frac{[|x^{(1)}N|]}{N}\right)\cdots\left(1 - \frac{[|x^{(\nu)}N|]}{N}\right)\frac{r([xN]+j_1)}{N^{-\alpha}L(N)}\cdots\frac{r([xN]+j_k)}{N^{-\alpha}L(N)}.$$

(In the above calculation we exploited that in the last sum of formula (6.16) the number of pairs (u,v) for which $u - v = l = (l_1,\ldots,l_\nu)$ equals $(N - |l_1|)\cdots(N - |l_\nu|)$.)

Let us fix some vector $(t_1,\ldots,t_k) \in R^{k\nu}$. It can be seen with the help of formula (6.1) that for all $\varepsilon > 0$ the convergence

$$f_N(t_1,\ldots,t_k,x) \to f_0(t_1,\ldots,t_k,x) \tag{6.17}$$

holds uniformly with the limit function

$$f_0(t_1,\ldots,t_k,x) = (1 - |x^{(1)}|)\ldots(1 - |x^{(\nu)}|)\frac{a\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha} \cdots \frac{a\left(\frac{x+t_k}{|x+t_k|}\right)}{|x+t_k|^\alpha} \qquad (6.18)$$

on the set $x \in [-1,1]^\nu \setminus \bigcup_{p=1}^{k} \{x \colon |x+t_p| < \varepsilon\}$.

I claim that

$$\psi_N(t_1,\ldots,t_k) \to \psi_0(t_1,\ldots,t_k) = \int_{[-1,1]^\nu} f_0(t_1,\ldots,t_k,x)\,dx,$$

and ψ_0 is a continuous function.

This relation implies that $\mu_N \xrightarrow{w} \mu_0$. To prove it, it is enough to show beside formula (6.17) that

$$\left|\int_{|x+t_p|<\varepsilon} f_0(t_1,\ldots,t_k,x)\,dx\right| < C(\varepsilon), \quad p = 1,\ldots,k, \qquad (6.19)$$

and

$$\int_{|x+t_p|<\varepsilon} |f_N(t_1,\ldots,t_k,x)|\,dx < C(\varepsilon), \quad p = 1,\ldots,k, \quad \text{and } N = 1,2,\ldots \quad (6.20)$$

with a constant $C(\varepsilon)$ such that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By formula (6.18) and Hölder's inequality

$$\left|\int_{|x+t_p|<\varepsilon} f_0(t_1,\ldots,t_k,x)\,dx\right| \leq C \prod_{1\leq l\leq k, l\neq p} \left[\int_{x\in[-1,1]^\nu} |x+t_l|^{-k\alpha}\,dx\right]^{1/k}$$

$$\left[\int_{|x+t_p|\leq\varepsilon} |x+t_p|^{-k\alpha}\,dx\right]^{1/k} \leq C'\varepsilon^{\nu/k-\alpha}$$

with some appropriate $C > 0$ and $C' > 0$, since $\nu - k\alpha > 0$, and $a(\cdot)$ is a bounded function. Similarly,

$$\int_{|x+t_p|<\varepsilon} |f_N(t_1,\ldots,t_k,x)|\,dx \leq \prod_{1\leq l\leq k, l\neq p} \left[\int_{x\in[-1,1]^\nu} \frac{|r([xN]+j_l)|^k}{N^{-k\alpha}L(N)^k}\,dx\right]^{1/k},$$

$$\left[\int_{|x+t_p|\leq\varepsilon} \frac{|r([xN]+j_p)|^k}{N^{-k\alpha}L(N)^k}\,dx\right]^{1/k}. \qquad (6.21)$$

It is not difficult to see with the help of Karamata's theorem that if $L(t)$, $t \geq 1$, is a slowly varying function which is bounded in all finite intervals, then for all

numbers $\eta > 0$ and $K > 0$ there are some constants $K_1 = K_1(\eta, K) > 0$, and $C = C(\eta, K) > 0$ together with a threshold index $N_0 = N_0(\eta, K)$ such that

$$\frac{L(uN)}{L(N)} \leq Cu^{-\eta} \quad \text{if } uN > K_1, \ u \leq K, \ \text{and } N \geq N_0.$$

Hence formula (6.1) implies that

$$|r([xN] + [t_lN])| = |r([xN] + j_l)| \leq CN^{-\alpha}L(N)|x + t_l|^{-\alpha - \eta}$$
$$\text{if } |x + t_l| \leq K \text{ and } N \geq N_0. \qquad (6.22)$$

Relation (6.22) follows from the previous relation and (6.1) if $|[xN] + [t_lN]| \geq K_1$. It also holds if $|[xN] + [t_lN]| \leq K_1$, since in this case the left-hand side can be bounded by the inequality $|r([xN] + [t_lN]| \leq 1$, while the right-hand side of (6.22) is greater than 1 with the choice of a sufficiently large constant C (depending on η and K_1). This follows from the relation $|x + t|^{-\alpha - \eta} = N^{\alpha + \eta}|N(x + t)|^{-\alpha - \eta} \geq C_1N^{\alpha + \eta}$ if $|[xN] + [t_lN]| \leq K_1$, and $L(N) \geq N^{-\eta}$.

We get with the help of (6.22) that

$$\int_{|x + t_p| < \varepsilon} \frac{|r([xN] + j_p)|^k}{N^{-k\alpha}L(N)^k} \, dx \leq B \int_{|x + t_p| < \varepsilon} |x + t_p|^{-k(\alpha + \eta)} \, dx \leq B'\varepsilon^{\nu - k(\alpha + \eta)}$$

$$\int_{x \in [-1,1]^\nu} \frac{|r([xN] + j_l)|^k}{N^{-k\alpha}L(N)^k} \, dx \leq B''.$$

for a sufficiently small constant $\eta > 0$ with some constants $B, B', B'' < \infty$ depending on η and t_p, $1 \leq p \leq k$.

Therefore we get from (6.21), by choosing an $\eta > 0$ such that $k(\alpha + \eta) < \nu$, that the inequality

$$\int_{|x + t_p| < \varepsilon} |f_N(t_1, \ldots, t_k, x)| \, dx \leq C\varepsilon^{\nu/k - (\alpha + \eta)}$$

holds with some $C < \infty$. The right-hand side of this inequality tends to zero as $\varepsilon \to 0$. Hence we proved beside (6.17) formulae (6.19) and (6.20), and they have the consequence that $\psi_N(t_1, \ldots, t_k) \to \psi_0(t_1, \ldots, t_k)$. Since $\psi_0(t_1, \ldots, t_k)$ is a continuous function relation (6.15) with Lemma 6.4 imply that the measures μ_N introduced in (6.25) converge weakly to a probability measure as $N \to \infty$, and as we saw at the beginning of the proof of Theorem 6.2 this limit measure must be μ_0.

Hence we can apply Lemma 6.3 for the spectral measures G_N and functions $K_N(\cdot)$, $N = 0, 1, 2, \ldots$, defined in Theorem 6.2. The convergence $G_N \xrightarrow{v} G_0$ follows from Lemma 6.1. Conditions (a) and (b) also hold with the choice of a sufficiently large number $A = A(\varepsilon)$. The hard point of the proof was to check condition (b).

This follows from the relation $\mu_N \xrightarrow{w} \mu_0$. Thus we have proved Theorem 6.2 with the help of Lemmas 6.1, 6.3 and 6.4.

Now I turn to the proof of Lemma 6.4. Before writing it down I make some comments on its conditions. Let us observe that if the measures μ_N or a part of them are shifted with a vector $2\pi C_N u$ with some $u \in \mathbb{Z}_l$, then their modified Fourier transforms $\varphi_N(t)$ do not change because of the periodicity of the trigonometric functions $e^{i(j/C_N,x)}$, $j \in \mathbb{Z}_l$. On the other hand, these new measures which are not concentrated in $[-C_N\pi, C_N\pi)^l$, have no limit. Lemma 6.4 states that if the measures μ_N are concentrated in the cubes $[-C_N\pi, C_N\pi)^l$, then the convergence of their modified Fourier transforms defined in Lemma 6.4, which is a weaker condition, than the convergence of their Fourier transforms, also implies their convergence to a limit measure.

Proof of Lemma 6.4. The proof is a natural modification of the proof about the equivalence of weak convergence of measures and the convergence of their Fourier transforms. First we show that for all $\varepsilon > 0$ there exists some $K = K(\varepsilon)$ such that

$$\mu_N(x \colon x \in R^l, \; |x^{(1)}| > K) < \varepsilon \quad \text{for all } N \geq 1. \tag{6.23}$$

As $\varphi(t)$ is continuous in the origin there is some $\delta > 0$ such that

$$|\varphi(0,\ldots,0) - \varphi(t,0,\ldots,0)| < \frac{\varepsilon}{2} \quad \text{if } |t| < \delta. \tag{6.24}$$

We have

$$0 \leq \operatorname{Re}\left[\varphi_N(0,\ldots,0) - \varphi_N(t,0,\ldots,0)\right] \leq 2\varphi_N(0,\ldots,0) \tag{6.25}$$

for all $N = 1, 2, \ldots$. The sequence in the middle term of (6.25) tends to

$$\operatorname{Re}\left[\varphi(0,\ldots,0) - \varphi(t,0,\ldots,0)\right]$$

as $N \to \infty$. The right-hand side of (6.25) is a bounded sequence, since it is convergent. Hence the dominated convergence theorem can be applied for the functions $\operatorname{Re}\left[\varphi_N(0,\ldots,0) - \varphi_N(t,0,\ldots,0)\right]$. We get because of the condition $C_N \to \infty$ and relation (6.24) that

$$\lim_{N \to \infty} \int_0^{[\delta C_N]/C_N} \frac{1}{\delta} \operatorname{Re}\left[\varphi_N(0,\ldots,0) - \varphi_N(t,0,\ldots,0)\right] dt$$
$$= \int_0^{\delta} \frac{1}{\delta} \operatorname{Re}\left[\varphi(0,\ldots,0) - \varphi(t,0,\ldots,0)\right] dt < \frac{\varepsilon}{2}$$

with the number $\delta > 0$ appearing in (6.24). Hence

$$
\begin{aligned}
\frac{\varepsilon}{2} &> \lim_{N\to\infty} \int_0^{[\delta C_N]/C_N} \frac{1}{\delta} \mathrm{Re}\left[\varphi_N(0,\ldots,0) - \varphi_N(t,0,\ldots,0)\right] dt \\
&= \lim_{N\to\infty} \int \left(\frac{1}{\delta}\int_0^{[\delta C_N]/C_N} \mathrm{Re}\left[1 - e^{i[tC_N]x^{(1)}/C_N}\right] dt\right) \mu_N(dx) \\
&= \lim_{N\to\infty} \int \frac{1}{\delta C_N} \sum_{j=0}^{[\delta C_N]-1} \mathrm{Re}\left[1 - e^{ijx^{(1)}/C_N}\right] \mu_N(dx) \\
&\geq \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \frac{1}{\delta C_N} \sum_{j=0}^{[\delta C_N]-1} \mathrm{Re}\left[1 - e^{ijx^{(1)}/C_N}\right] \mu_N(dx) \\
&= \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \left(1 - \frac{1}{\delta C_N}\mathrm{Re}\frac{1 - e^{i[\delta C_N]x^{(1)}/C_N}}{1 - e^{ix^{(1)}/C_N}}\right) \mu_N(dx)
\end{aligned}
$$

with an arbitrary $K > 0$. (In the last but one step of this calculation we have exploited that $\frac{1}{\delta C_N}\sum_{j=0}^{[\delta C_N]-1} \mathrm{Re}\left[1 - e^{ijx^{(1)}/C_N}\right] \geq 0$ for all $x^{(1)} \in R^1$.)

Since the measure μ_N is concentrated in $\{x\colon x \in R^l, |x^{(1)}| \leq C_N\pi\}$, and

$$
\begin{aligned}
\mathrm{Re}\frac{1 - e^{i[\delta C_N]x^{(1)}/C_N}}{1 - e^{ix^{(1)}/C_N}} &= \frac{\mathrm{Re}\left(ie^{-ix^{(1)}/2C_N}\left(1 - e^{i[\delta C_N]x^{(1)}/C_N}\right)\right)}{i(e^{-ix^{(1)}/2CN} - e^{ix^{(1)}/2C_N})} \\
&\leq \frac{1}{\left|\sin\left(\dfrac{x^{(1)}}{2C_N}\right)\right|} \leq \frac{C_N\pi}{|x^{(1)}|}
\end{aligned}
$$

if $|x^{(1)}| \leq C_N\pi$, (here we exploit that $|\sin u| \geq \frac{2}{\pi}|u|$ if $|u| \leq \frac{\pi}{2}$), hence we have with the choice $K = \frac{2\pi}{\delta}$

$$
\frac{\varepsilon}{2} > \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \left(1 - \left|\frac{\pi}{\delta x^{(1)}}\right|\right) \mu_N(dx) \geq \limsup_{N\to\infty} \frac{1}{2}\mu_N(|x^{(1)}| > K).
$$

As the measures μ_N are finite the inequality $\mu_N(|x^{(1)}| > K) < \varepsilon$ holds for each index N with a constant $K = K(N)$ that may depend on N. Hence the above inequality implies that formula (6.23) holds for all $N \geq 1$ with a possibly larger index K that does not depend on N.

Applying the same argument to the other coordinates we find that for all $\varepsilon > 0$ there exists some $C(\varepsilon) < \infty$ such that

$$
\mu_N\left(R^l \setminus [-C(\varepsilon), C(\varepsilon)]^l\right) < \varepsilon \quad \text{for all } N = 1, 2, \ldots.
$$

Consider the usual Fourier transforms

$$\tilde{\varphi}_N(t) = \int_{R^l} e^{i(t,x)} \mu_N(dx), \quad t \in R^l.$$

Then

$$|\varphi_N(t) - \tilde{\varphi}_N(t)| \leq 2\varepsilon + \int_{[-C(\varepsilon),C(\varepsilon)]} \left| e^{i(t,x)} - e^{i([tC_N]/C_N,x)} \right| \mu_N(dx)$$

$$\leq 2\varepsilon + \frac{lC(\varepsilon)}{C_N} \mu_N(R^l)$$

for all $\varepsilon > 0$. Hence $\tilde{\varphi}_N(t) - \varphi_N(t) \to 0$ as $N \to \infty$, and $\tilde{\varphi}_N(t) \to \varphi(t)$. (Observe that $\mu_N(R^l) = \varphi_N(0) \to \varphi(0) < \infty$ as $N \to \infty$, hence the measures $\mu_N(R^l)$ are uniformly bounded, and $C_N \to \infty$ by the conditions of Lemma 6.4.) Then Lemma 6.4 follows from standard theorems on Fourier transforms.

It remained to prove Lemma 6.1.

Proof of Lemma 6.1. Introduce the notation

$$K_N(x) = \prod_{j=1}^{v} \frac{e^{ix^{(j)}} - 1}{N(e^{ix^{(j)}/N} - 1)}, \quad N = 1, 2, \dots,$$

and

$$K_0(x) = \prod_{j=1}^{v} \frac{e^{ix^{(j)}} - 1}{ix^{(j)}}.$$

Let us consider the measures μ_N defined in formula (6.14) in the special case $k = 1$ with $p = 1$, $c_1 = 1$ in the definition of the function $K_N(\cdot)$, i.e. put

$$\mu_N(A) = \int_A |K_N(x)|^2 G_N(dx), \quad A \in \mathscr{B}^v, \quad N = 1, 2, \dots.$$

We have already seen in the proof of Theorem 6.2 that $\mu_N \overset{w}{\to} \mu_0$ with some finite measure μ_0, and the Fourier transform of μ_0 is

$$\varphi_0(t) = \int_{[-1,1]^v} (1 - |x^{(1)}|) \cdots (1 - |x^{(v)}|) \frac{a\left(\frac{x+t}{|x+t|}\right)}{|x+t|^\alpha} dx.$$

Moreover, since $|K_N(x)|^2 \to |K_0(x)|^2$ uniformly in any bounded domain, it is natural to expect that $G_N \overset{v}{\to} G_0$ with $G_0(dx) = \frac{1}{|K_0(x)|^2} \mu_0(dx)$. But $K_0(x) = 0$ in some points (if $x^{(j)} = 2k\pi$ with some integer $k \neq 0$ for a coordinate of x), and the function $K_0(\cdot)^{-2}$ is not continuous here. As a consequence, we cannot give direct

proof of the above statement. Hence we apply instead a modified version of this method. First we prove the following result about the behaviour of the restrictions of the measures G_N to appropriate cubes:

For all $T \geq 1$ there is a finite measure G_0^T concentrated on $(-T\pi, T\pi)^\nu$ such that

$$\lim_{N \to \infty} \int f(x) G_N(dx) = \int f(x) G_0^T(dx) \tag{6.26}$$

for all continuous functions f which vanish outside the cube $(-T\pi, T\pi)^\nu$.

Indeed, let a continuous function f vanish outside the cube $(-T\pi, T\pi)^\nu$ with some $T \geq 1$. Put $M = [\frac{N}{2T}]$. Then

$$
\begin{aligned}
\int f(x) G_N(dx) &= \frac{N^\alpha}{L(N)} \cdot \frac{L(M)}{M^\alpha} \int f\left(\frac{N}{M}x\right) G_M(dx) \\
&= \frac{N^\alpha L(M)}{M^\alpha L(N)} \int f\left(\frac{N}{M}x\right) |K_M(x)|^{-2} \mu_M(dx) \\
&\to (2T)^\alpha \int f(2Tx) |K_0(x)|^{-2} \mu_0(dx) \\
&= \int f(x) \frac{(2T)^\alpha}{|K_0(\frac{x}{2T})|^2} \mu_0\left(\frac{dx}{2T}\right) \quad \text{as } N \to \infty,
\end{aligned}
$$

because $f(\frac{N}{M}x)|K_M(x)|^{-2}$ vanishes outside the cube $[-\pi, \pi]^\nu$, the limit relation

$$f(\frac{N}{M}x)|K_M(x)|^{-2} \to f(2Tx)|K_0(x)|^{-2}$$

holds uniformly, (the function $K_0(\cdot)^{-2}$ is continuous in the cube $[-\pi, \pi]^\nu$), and $\mu_M \xrightarrow{w} \mu_0$ as $N \to \infty$. Hence relation (6.26) holds if we define G_0^T as the restriction of the measure $\frac{(2T)^\alpha}{|K_0(\frac{x}{2T})|^2} \mu_0\left(\frac{dx}{2T}\right)$ to the cube $(-T\pi, T\pi)^\nu$. The measures G_0^T appearing in (6.26) are consistent for different parameters T, i.e. G_0^T is the restriction of the measure $G_0^{T'}$ to the cube $(-T\pi, T\pi)^\nu$ if $T' > T$. This follows from the fact that $\int f(x) G_0^T(dx) = \int f(x) G_0^{T'}(dx)$ for all continuous functions with support in $(-T, T)^\nu$. We claim that by defining the measure G_0 by the relation $G_0(A) = G_0^T(A)$ for a bounded set A and such number $T > 1$ for which $A \subset (-T\pi, T\pi)^\nu$ we get such a locally finite measure G_0 for which $G_N \xrightarrow{v} G_0$. The above mentioned vague convergence is a direct consequence of (6.26) and the definition of G_0, but to give a complete proof we have to show that G_0 is really a (σ-additive) measure.

Actually it is enough to prove that the restriction of G_0 to the bounded, measurable sets is σ-additive, because it follows then from standard results in measure theory that it has a unique σ-additive extension to $\mathscr{B}^\nu$. But this is an almost direct

consequence of the definition of G_0. The desired σ-additivity clearly holds, since if $A = \bigcup\limits_{n=1}^{\infty} A_n$, the set A is bounded, and the sets A_n, $n = 1, 2, \ldots$, are disjoint, then there is a number $T > 1$ such that $A \subset (-T\pi, T\pi)^\nu$, the same relation holds for the sets A_n, and the σ-additivity of G_0^T implies that $G_0(A) = \sum\limits_{n=1}^{\infty} G_0(A_n)$.

As $G_N \xrightarrow{v} G_0$, and $|K_N(x)|^2 \to |K_0(x)|^2$ uniformly in all bounded regions, the relation $\mu_N \xrightarrow{v} \bar\mu_0$ holds with the measure $\bar\mu_0$ defined by the formula $\bar\mu_0(A) = \int_A |K_0(x)|^2 G_0(dx)$, $A \in \mathscr{B}^\nu$. Since $\mu_N \xrightarrow{w} \mu_0$ the measures μ_0 and $\bar\mu_0$ must coincide, i.e.

$$\mu_0(A) = \int_A |K_0(x)|^2 G_0(dx), \quad A \in \mathscr{B}^\nu.$$

Relation (6.4) expresses the fact that φ_0 is the Fourier transform of μ_0. It remained to prove the homogeneity property (6.3) of the measure G_0. For this goal let us extend the definition of the measures G_N given in (6.2) to all non-negative real numbers u. It is easy to see that the relation $G_u \xrightarrow{v} G_0$ as $u \to \infty$ remains valid. Hence we get for all fixed $s > 0$ and continuous functions f with compact support that

$$\begin{aligned}
\int f(x) G_0(dx) &= \lim_{u \to \infty} \int f(x) G_u(dx) = \lim_{u \to \infty} \frac{s^\alpha L(\frac{u}{s})}{L(u)} \int f(sx) G_{\frac{u}{s}}(dx) \\
&= s^\alpha \int f(sx) G_0(dx) = \int f(x) s^\alpha G_0\left(\frac{dx}{s}\right).
\end{aligned}$$

This identity implies the homogeneity property (6.3) of G_0. Lemma 6.1 is proved.

6.3.   A discussion about our results

Lemma 6.1 is a result about the limit behaviour of the spectral distribution of a stationary random fields if its correlation function satisfies formula (6.1). It states that under this condition the appropriately rescaled spectral measure has a limit in the vague convergence sense, and Lemma 6.1 also describes this limit. There is a closer relation between the behaviour of the correlation function and spectral measure which may be worthwhile for a more detailed discussion. Moreover, this comparison may help us to understand the relation between limit theorems about the large scale limit of stationary Gaussian random fields and such limit theorems about non-linear functionals of stationary Gaussian fields which are similar to Theorem 6.2 of this work.

In the subsequent slightly informal discussion I disregard the appearance of the slowly varying function $L(t)$ in our results, I assume simply that $L(t) \sim 1$ as $t \to \infty$. In this case we can interpret Lemma 6.1 so that if the correlation function $r(n)$ behaves like $L(n) \sim |n|^{-\alpha}$ in the neighbourhood of the infinity, then the

spectral measure behaves like $G(t) \sim \text{const.} t^\alpha$ as $t \to 0$. (Here $G(t)$ denotes the spectral measure of the ball with radius t and center point at the origin.) One may ask, what can be said in the opposite direction. What can we say about the asymptotic behaviour of the correlation function if we have some information about the behaviour of the spectral measure? For the sake of simplicity let us restrict our attention to the correlation function of one-dimensional random fields, i.e. to the case when $\nu = 1$. In the following consideration I shall apply some heuristic, not completely precise argument.

By some results about Fourier analysis we can say that the smoother is a function the faster tends its Fourier transform to zero at infinity. On the other hand, if a function has a singularity, but otherwise it is smooth enough, then the asymptotic behaviour of its Fourier transform at infinity is determined by this singularity. This means in particular that if the spectral measure behaves like $G(t) \sim C \cdot t^\alpha$, $0 < \alpha < 1$, (or it has a spectral density has the form $g(t) \sim |t|^{\alpha-1}$) in the neighbourhood of the origin, and this is the strongest irregularity of the spectral measure, then the correlation function of the random field satisfies condition (6.1) of Theorem 6.2.

In the paper P. Major: On renormalizing Gaussian fields. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **59** (1982), 515–533. the condition for a limit theorem for Gaussian fields was given by means of the spectral measure while in the result of Theorem 6.2 the condition of a limit theorem was given by means of the correlation function of the underlying Gaussian field. It may be worth comparing the conditions of these two results.

The limit theorem about the existence of the large scale limit of a stationary Gaussian field can be interpreted in a slightly informal way so that the limit exists if the spectral measure has the singularity of the form $G(t) \sim C|t|^{-\alpha}$ in a small neighbourhood of the origin. On the other hand, the condition of Theorem 6.2 was that $r(n) \sim C|n|^{-\alpha}$. As I mentioned before this is a stronger condition which implies that the spectral measure behaves in a small neighbourhood of the origin similarly to the previous case, but it also implies some additional restriction. The spectral measure cannot have a stronger singularity outside zero which would influence too strongly the behaviour of its Fourier transform at the infinity. I present an example taken from the fourth section of my paper with R. L. Dobrushin that shows such a picture which the above considerations suggest.

Take a stationary Gaussian sequence X_n, $EX_n = 0$, $EX_n^2 = 1$, $n = 0, \pm1, \pm2, \ldots$, with spectral density

$$g(x) = C_1|x|^{-\alpha} + C_2(|x-a|^{-\beta} + |x+a|^{-\beta}), \quad -\pi \le x < \pi,$$

where $0 < \alpha < \beta < 1$, $\beta > \frac{1}{2}$, $0 < x < \pi$, $C_1, C_2 > 0$. We are interested in what kind of limit theorems hold for the sums $S_n = \frac{1}{A_n} \sum_{j=1}^n X_j$ and $T_n = \frac{1}{B_n} \sum_{j=1}^n H_2(X_j) =$

$\frac{1}{B_n}\sum_{j=1}^{n}(X_j^2-1)$. In particular, how do we have to choose the norming constants A_n and B_n to get a limit. (In the definition of T_n when we are working with the Hermite polynomial $H_2(x) = x^2 - 1$.) It can be proved that in the example with this spectral density the correlation function has the following form.

$$
\begin{aligned}
EX_kX_{p+k} &= \int_{-\pi}^{\pi} e^{ipx}g(x)\,dx \\
&= K_1p^{\alpha-1}\left(1+O\left(\frac{1}{p}\right)\right) + K_2p^{\beta-1}\cos pa\left(1+O\left(\frac{1}{p}\right)\right)
\end{aligned}
$$

with some positive constants K_1 and K_2.

In the first problem, where we study the limit behaviour of S_n we have a Gaussian limit, and some calculation shows that the variance of the sum without the normalization is of order $n^{1+\alpha}$, which means that we get a limit with the norming constant $A_n = n^{(1+\alpha)/2}$. This means that in the limit behaviour the singularity $|t|^{-\alpha}$ of the spectral density at the origin is important.

In the case of the second limit problem the situation is different. In this case we can calculate the correlation function of the terms $H_2(X_j)$ e.g. with the help of the diagram formula, and some calculation yields that

$$
EH_2(X_k)H_2(X_{p+k}) = K_2^2p^{2\beta-1}(1+\cos 2pa+o(1)).
$$

Further calculation shows that in this case the right norming for T_n for which the variance of T_n is separated both from zero and infinity is $B_n = n^{\beta}$. Some further calculation shows that all moments of T_n has a limit, moreover these limits determine the limit distribution, hence there exists limit theorem in this case. Finally the third moment of the limit is positive, and this means that the limit is non-Gaussian.

This means that in the second problem the singularity $|x \pm a|^{\beta}$ gives the dominating factor that determines the limit distribution. A more complete description of the situation would demand further investigation.

In the results of this section we discussed the limit behaviour of the large scale limit of a random field $H(X_n)$, $n \in \mathbb{Z}_\nu$, defined with the help of a function $H(x)$ (square integrable with respect to the standard Gaussian measure) and a stationary Gaussian random sequence X_n, $n \in \mathbb{Z}_\nu$, whose correlation function satisfies relation (6.1) with some parameter $\alpha > 0$. It was proved that if this parameter α is not too large (this condition was formulated in a more explicit form), then we have a non-Gaussian limit theorem. To get a more complete picture one would like to know what can be said if this parameter α is relatively large, which means some sort of weak dependence. Next I formulate a result in this case, but because of lack of time I omit its proof. First I formulate the above problem in more detail.

Let us consider a slightly more general version of the problem investigated in Theorem 6.2'. Take a stationary Gaussian random field X_n, $EX_n = 0$, $EX_n^2 = 1$, $n \in \mathbb{Z}_\nu$, with a correlation function satisfying relation (6.1), and the field $\xi_n = H(X_n)$, $n \in \mathbb{Z}_\nu$, subordinated to it with a general function $H(x)$ such that $EH(X_n) = 0$ and $EH(X_n)^2 < \infty$. We are interested in the large-scale limit of such random fields. Take the Hermite expansion (6.6) of the function $H(x)$, and let k be the smallest such index for which $c_k \neq 0$ in the expansion (6.6). In Theorem 6.2' we solved this problem if $0 < k\alpha < \nu$. We are interested in the question what happens in the opposite case when $k\alpha > \nu$. Let me remark that in the case $k\alpha \geq \nu$ the field Z_n^*, $n \in \mathbb{Z}_\nu$, which appeared in the limit in Theorem 6.2' does not exist. The Wiener-Itô integral defining Z_n^* is meaningless, because the integral which should be finite to guarantee the existence of the Wiener–Itô integral is divergent in this case. I formulate a general result which contains the answer to the above question as a special case.

Theorem 6.5. *Let us consider a stationary Gaussian random field X_n, $EX_n = 0$, $EX_n^2 = 1$, $n \in \mathbb{Z}_n$, with correlation function $r(n) = EX_m X_{m+n}$, $m, n \in \mathbb{Z}_\nu$. Take a function $H(x)$ on the real line such that $EH(X_n) = 0$ and $EH(X_n)^2 < \infty$. Take the Hermite expansion (6.6) of the function $H(x)$, and let k be the smallest index in this expansion such that $c_k \neq 0$. If*

$$\sum_{n \in \mathbb{Z}_\nu} |r(n)|^k < \infty, \tag{6.27}$$

then the limit

$$\lim_{N \to \infty} EZ_n^N(H_l)^2 = \lim_{N \to \infty} N^{-\nu} \sum_{i \in B_n^N} \sum_{j \in B_n^N} r^l(i-j) = \sigma_l^2 l!$$

exists for all indices $l \geq k$, where $Z_n^N(H_l)$ is defined in (1.1) with $A_N = N^{\nu/2}$, and $\xi_n = H_l(X_n)$ with the l-th Hermite polynomial $H_l(x)$ with leading coefficient 1. Moreover, also the inequality

$$\sigma^2 = \sum_{l=k}^\infty c_l^2 l! \sigma_l^2 < \infty$$

holds.

The finite dimensional distributions of the random field $Z_n^N(H)$ defined in (1.1) with $A_N = N^{\nu/2}$ and $\xi_n = H(X_n)$ tend to the finite dimensional distributions of a random field σZ_n^* with the number σ defined in the previous relation, where Z_n^*, $n \in \mathbb{Z}_\nu$, are independent, standard normal random variables.

Theorem 6.5 can be applied if the conditions of Theorem 6.2' hold with the only modification that the condition $k\alpha < \nu$ is replaced by the relation $k\alpha > \nu$. In

this case the relation (6.27) holds, and the large-scale limit of the random field Z_n^N, $n \in \mathbb{Z}_v$ with normalization $A_N = N^{v/2}$ is a random field consisting of independent standard normal random variables multiplied with the number σ. There is a slight generalization of Theorem 6.5 which also covers the case $k\alpha = v$. In this result we assume instead of the condition (6.27) that $\sum\limits_{n \in \bar{B}_N} r(n)^k = L(N)$ with a slowly varying function $L(\cdot)$, where $\bar{B}_N = \{(n_1, \ldots, n_v) \in \mathbb{Z}_v\colon\ -N \leq n_j \leq N,\ 1 \leq j \leq v\}$, and some additional condition is imposed which states that an appropriately defined finite number $\sigma^2 = \lim\limits_{N \to \infty} \sigma_N^2$, which plays the role of the variance of the random variables in the limiting field, exists. There is a similar large scale limit in this case as in Theorem 6.5, the only difference is that the norming constant in this case is $A_N = N^{v/2}L(N)^{1/2}$. This result has the consequence that if the conditions of Theorem 6.2$'$ hold with the only difference that $k\alpha = v$ instead of $k\alpha < v$, then the large scale limit exists with norming constants $A_N = N^{v/2}L(N)$ with an appropriate slowly varying function $L(\cdot)$, and it consists of independent Gaussian random variables with expectation zero.

The proof of Theorem 6.5 and its generalization that I did not formulate here explicitly appeared in my paper with P. Breuer *Central limit theorems for non-linear functionals of Gaussian fields* Journal of Multivariate Analysis **13** (1983), 425–441. I omit its proof, I only make some short explanation about it.

In the proof we show that all moments of the random variables Z_n^N converge to the corresponding moments of the Gaussian random variables Z_n^* with expectation zero and the right variance as $N \to \infty$. The moments of the random variables Z_n^N can be calculated by means of the diagram formula if we either rewrite them in the form of a Wiener–Itô integral or apply a version of it for the moments of Hermite (or of their generalization, the Wick polynomials) instead of Wiener–Itô integrals. In both cases the moments can be expressed explicitly by means of the correlation function of the underlying Gaussian random field. The most important step of the proof is to show that we can select a special subclass of (closed) diagrams, called regular diagrams in my paper with P. Breuer which yield the main contribution to the moments $E(Z_n^N)^M$, and their contribution can be simply calculated. The contribution of all remaining diagrams is $o(1)$ (after norming), hence it is negligible. For the sake of simplicity let us restrict our attention to the case $H(x) = H_k(x)$, and let us explain the definition of the regular diagrams in this special case.

If the number of the rows M is an even number, then we call a closed diagram regular if there is a pairing of the rows, i.e. a partition $\{k_1, k_2\}$, $\{k_3, k_4\}, \ldots$, $\{k_{M-1}, k_M\}$ of the set $\{1, \ldots, M\}$ into subsets of two elements in such a way that an edge can connect only vertices in paired rows. If M is an odd number, then there is no regular diagram. The main step of the proof is to show that the contribution of all remaining closed diagrams is negligibly small.

7.   Appendix: The proof of Lemma 3.3

In the Appendix I present a new proof of Lemma 3.3 which is simpler than its original version that appeared as the proof of Lemma 4.1 in my Lecture Note *Multiple Wiener–Itô integrals*.

Our goal is to find for all functions $f \in \bar{\mathscr{H}}_G^n$ and $\varepsilon > 0$ a function $f' \in \hat{\bar{\mathscr{H}}}_G^n$ such that the distance of f and f' is smaller than ε in the Hilbert space $\bar{\mathscr{H}}_G^n$. Then the corresponding statement about functions in the Hilbert space $\mathscr{H}_G^n$ follows from a standard symmetrization procedure.

Let us first observe that if two functions $f_1 \in \bar{\mathscr{H}}_G^n$ and $f_2 \in \bar{\mathscr{H}}_G^n$ can be arbitrarily well approximated by functions from $\hat{\bar{\mathscr{H}}}_G^n$ in the norm of this space, then the same relation holds for any linear combination $c_1 f_1 + c_2 f_2$ with real coefficients c_1 and c_2. Indeed, if the functions f_i, $i = 1, 2$, are approximated by some functions $g_i \in \hat{\bar{\mathscr{H}}}_G^n$, then we may assume, by applying some refinement of the partitions if it is necessary, that the approximating functions g_1 and g_2 are adapted to the same regular partition. Hence also $c_1 g_1 + c_2 g_2 \in \hat{\bar{\mathscr{H}}}_G^n$, and it provides a good approximation of $c_1 f_1 + c_2 f_2$.

The above observation enables us to reduce the proof of Lemma 3.3 to the proof of a simpler statement formulated in the following *Statement A*. Here we have to approximate simpler functions $f \in \bar{\mathscr{H}}_G^n$. We have to consider two different cases. In the first case the function f is the indicator function of some set $A \in R^{n\nu}$. In the second case f is a simple function taking imaginary values. It takes the value $i = \sqrt{-1}$ in a set A, the value $-i$ in the set $-A$, and otherwise it equals zero. Here is the formulation of *Statement A*.

Statement A. Let $A \in \mathscr{B}^{n\nu}$ be a bounded, symmetric set, i.e. let $A = -A$. Then for any $\varepsilon > 0$ there is a function $g \in \hat{\bar{\mathscr{H}}}_G^n$ such that $g = \chi_B$ with some set $B \in \mathscr{B}^{n\nu}$, i.e. g is the indicator function of a set B such that the inequality $\|g - \chi_A\| < \varepsilon$ holds with the norm of the space $\bar{\mathscr{H}}_G^n$. (Here χ_A denotes the indicator function of the set A, and we have $\chi_A \in \bar{\mathscr{H}}_G^n$.)

If $\chi_A \in \bar{\mathscr{H}}_G^n$ is a bounded set, and there is such a set A_1 for which the set A can be written in the form $A = A_1 \cup (-A_1)$, and the sets A_1 and $-A_1$ have a positive distance from each other, i.e. $\rho(A_1, -A_1) = \inf\limits_{x \in A_1, y \in -A_1} \rho(x, y) > \delta$, with some $\delta > 0$, where ρ denotes the Euclidean distance in $R^{n\nu}$, then a good approximation of χ_A can be given with such a function $g = \chi_{B \cup (-B)} \in \hat{\bar{\mathscr{H}}}_G^n$ for which the sets B and $-B$ are disjoint, and the set B is close to A_1. More explicitly, for all $\varepsilon > 0$ there is a set $B \in \mathscr{B}^{n\nu}$ such that $B \subset A_1^{\delta/2} = \{x\colon \rho(x, A_1) \leq \frac{\delta}{2}\}$, $g = \chi_{B \cup (-B)} \in \hat{\bar{\mathscr{H}}}_G^n$, where $\delta > 0$ may depend on $\varepsilon > 0$, and $G^n(A_1 \Delta B) < \frac{\varepsilon}{2}$. Here $A \Delta B$ denotes the symmetric difference of the sets A and B, and G^n is the n-fold direct product of the spectral measure G on the space $R^{n\nu}$. (The above properties of the set B imply

that the function $g = \chi_{B \cup (-B)} \in \hat{\tilde{\mathscr{H}}}_G^n$ satisfies the relation $\|g - \chi_A\| < \varepsilon$.)

The reduction of Lemma 3.3 to *Statement A* is relatively simple. Given a function $f \in \bar{\mathscr{H}}_G^n$ we can write $f = f_1 + i f_2$ with $f_1 = \operatorname{Re} f$, $i f_2 = \operatorname{Im} f$, and both $f_1 \in \bar{\mathscr{H}}_G^n$ and $i f_2 \in \bar{\mathscr{H}}_G^n$. Hence it is enough to prove the arbitrarily good approximability of $\operatorname{Re} f \in \bar{\mathscr{H}}_G^n$ and $\operatorname{Im} f \in \bar{\mathscr{H}}_G^n$ by a function in $\hat{\tilde{\mathscr{H}}}_G^n$.

Moreover, the real part and imaginary part of the function f can be arbitrarily well approximated by such real or imaginary valued functions from the space $\bar{\mathscr{H}}_G^n$ which take only finitely many values, and which take a non-zero value only on a bounded set. Since we know that the if some functions $f_1, \ldots, f_m$ from $\bar{\mathscr{H}}_G^n$ can be approximated arbitrary well by a function from $\hat{\tilde{\mathscr{H}}}_G^n$, then the same relation holds for their linear combination $\sum_{j=1}^m c_j f_j$ with real coefficients c_j, the good approximability of $\operatorname{Re} f$ follows from the first part of *Statement A*.

The proof of the good approximability of $\operatorname{Im} f$ is similar, but it demands an additional argument. We can reduce the statement we want to prove to the good approximability of such a function f for which $f(x) = i$ on a bounded set A_0, $f(x) = -i$ on the set $-A_0$, and $f(x) = 0$ otherwise. Naturally the sets A_0 and $-A_0$ are disjoint, but their distance may be zero. Let us observe that for any $\varepsilon > 0$ there is such a compact set $A_1 \subset A_0$ for which $G^n(A_1 \setminus A_0) < \varepsilon$. Then $\rho(A_1, -A_1) > \delta$ with some $\delta > 0$, and we can reduce the statement about the good approximability of the function $\operatorname{Im} f$ to the good approximability of the function g which is defined as $g(x) = i$ on the set A_1, $g(x) = -i$ on the set $-A_1$, and it equals zero otherwise. But the latter statement follows from the second part of *Statement A* if it is applied for $A = A_1 \cup (-A_1)$.

To prove *Statement A* first I make the following observation.

For all numbers $M > 0$ and $\varepsilon > 0$ there is a number $\delta = \delta(\varepsilon, M) > 0$ such that the set

$$
\begin{aligned}
K(\delta) \;=\; & \left\{ x = (x^{(1)}, \ldots, x^{(n)}) \colon \; |x_j \pm x_k| < \delta \text{ for a pair } (j,k), \; 1 \le j < k \le n \right\} \\
& \cap \{ x \colon \; x \in R^{n\nu}, \; |x| \le M \}
\end{aligned}
$$

satisfies the inequality $G^n(K(\delta)) < \varepsilon$.

Similarly, for all $\varepsilon > 0$ and $M > 0$ there is a number $\eta = \eta(\varepsilon, M) > 0$ such that

$$
G^n(L(\eta)) < \varepsilon \quad \text{with } L(\eta) = \left(\bigcup_{j=1}^n L_j(\eta) \right) \cap \{ x \colon \; x \in R^{n\nu}, \; |x| \le M \},
$$

where $L_j(\eta) = \{ (x_1, \ldots, x_n) \colon \; x_l \in R^\nu, \; l = 1, \ldots, n, \; \rho(x_j, 0) \le \eta \}$.

Indeed, because of condition that $G(\{x\}) = 0$ for all $x \in R^\nu$ we get by means of the Fubini theorem that for all $j \ne k$, $1 \le j, k \le n$, $G^n(\{x_j \ne \pm x_k\}) = 0$. The

first statement follows from this relation, since it implies that the intersection of the sets $K(\delta)$ with $\delta \to 0$ is contained in a set with zero G^n measure.

The second statement follows similarly from the relation $G(\{0\}) = 0$, since it implies that for all $1 \le j \le n$ $G^n(x = (x_1, \ldots, x_n): x_j = 0) = 0$.

Since the set A considered in *Statement A* is bounded, the above relations enable us to replace the set A by the set $A' = A \setminus (K(\delta) \cup L(\eta))$ with a sufficiently small $\delta > 0$ and $\eta > 0$ in the formulation of *Statement A*. It is enough to consider the case when A' is a non-empty set. First we consider the first part of *Statement A*. Observe that the property $A' = -A'$ is preserved.

We can choose some open rectangles

$$
\begin{aligned}
D(j) \;=\; & (a_{(1,1)}(j), b_{(1,1)}(j)) \times \cdots \times (a_{(1,v)}(j), b_{(1,v)}(j)) \times \cdots \\
& \cdots \times (a_{(n,1)}(j), b_{(n,1)}(j)) \times \cdots \times (a_{(n,v)}(j), b_{(n,v)}(j)),
\end{aligned}
$$

$j = 1, \ldots, M$ with some number $M > 0$ which satisfy the following relations: $G^n(x_{(p-1)v+s} = \pm a_{(p,s)}(j)) = 0$ and $G^n(x_{(p-1)v+s} = \pm b_{(p,s)}(j)) = 0$ for all $1 \le p \le n$ and $1 \le s \le v$, and also the inequality $G^n\left(\left(\bigcup_{j=1}^{M} D(j)\right) \Delta A'\right) < \frac{\varepsilon}{2}$ holds. Let us define the rectangles $D(-j) = -D(j)$ for all $1 \le j \le n$. Since $A' = -A'$ the last inequality also implies that $G^n\left(\left(\bigcup_{j=-M}^{M} D(j)\right) \Delta A'\right) < \varepsilon$.

We split up the set $\bigcup_{j=-M}^{M} D(j)$ (by omitting some hyperplanes with zero G_N measure) to disjoint open rectangles in the following way. First we choose some disjoint intervals $S_l = (a'_l, b'_l)$, $-P' \le l \le P$ with some $P > 0$ in such a way that it satisfies the following properties. $S_l = -S_{-l}$, for all $-P \le l \le P$, (in particular, $S_0 = -S_0$). The relations $G(x_k = a'_l) = G(x_k = b'_l) = 0$ hold for all $-P \le l \le P$ and $1 \le k \le v$. Besides, $b'_l - a'_l \le \min(\frac{\delta}{2nv}, \frac{\eta}{2nv})$ for all $-P \le l \le P$ with the parameters δ and η of those sets $K(\delta)$ and $L(\eta)$ which we chose in the definition of the set A', and all edges $(a_{(p,s)}(j), b_{(p,s)}(j))$ of the rectangles $D(j)$, $1 \le s \le n$, $1 \le p \le v$, $-M \le j \le M$, (except finitely many points of the form a'_l or b'_l) can be presented as the union of some elements from the set of intervals (a'_l, b'_l), $-P \le l \le P$.

Then we take all those rectangles $D'(k)$ of the form

$$
\begin{aligned}
D'(k) \;=\; & (a'_{u(1,1,k)}, b'_{u(1,1,k)}) \times \cdots \times (a'_{u(1,v,k)}, b'_{u(1,v,k)}) \times \cdots \\
& \cdots \times (a'_{u(n,1,k)}, b'_{u(n,1,k)}) \times \cdots \times (a'_{u(n,v,k)}, b'_{u(n,v,k)})
\end{aligned}
$$

for which $D'(k) \subset D(j)$ with some $-M \le j \le M$. The union of these rectangles equals $\bigcup_{j=-M}^{M} D(j)$ minus finitely many hyperplanes of dimension $nv - 1$ with zero G^n measure.

In the next step of our construction we preserve those elements from the set of these rectangles whose intersection with the set A' is non-empty. Let us reindex the set of these preserved rectangles $D'(k)$ by the numbers $1 \le k \le M'$ with some number M'. Clearly we have $G^n\left(\left(\bigcup_{k=1}^{M'} D'(k)\right) \Delta A'\right) \le G^n\left(\left(\bigcup_{j=-M}^{M} D(j)\right) \Delta A'\right) < \varepsilon$.

Let us define set of rectangles $\mathscr{D}$ that consists of those rectangles $\Delta_s \in \mathscr{B}^v$, $1 \le s \le P$, which are a side of one of the above defined rectangles $D'(k)$, $1 \le k \le M'$. More precisely $\mathscr{D}$ consists of those rectangles in R^v which can be written in the form $\Delta_p = (a'_{u(p,1,k)}, b'_{u(p,1,k)}) \times \cdots \times (a'_{u(p,v,k)}, b'_{u(p,v,k)})$, $1 \le p \le n$, $1 \le k \le M'$, where the intervals $(a'_{u(p,l,k)}, b'_{u(p,l,k)})$ appear in the representation of one of the rectangles $D'(k)$, $1 \le k \le M'$.

I claim that the class of sets $\mathscr{D}$ (with an appropriate indexation) is a regular system, and if we define the function $g(x)$ as $g(x) = 1$ if $x \in D'(k)$ with some $1 \le k \le M'$, and $g(x) = 0$ otherwise, then $g(x)$ is a simple function adapted to the regular system $\mathscr{D}$. This fact together with the above mentioned inequality imply the first part of *Statement A*.

It is clear that $\mathscr{D}$ consists of disjoint sets, and if $\Delta_l \in \mathscr{D}$, then also $-\Delta_l \in \mathscr{D}$. We still have to show that $-\Delta_l \neq \Delta_l$ for all sets $\Delta_l \in \mathscr{D}$, since Δ_l and $-\Delta_l = \Delta_{-l}$ must be two different elements of $\mathscr{D}$. To prove this statement let us first observe that $D'(k) \cap L(\frac{\eta}{2}) = \emptyset$ for all $1 \le k \le M'$. Indeed, there is some point $x \in D'(k) \setminus L(\eta)$, because $D'(k) \cap A'$ is non-empty. As the diameter of $D'(k)$ is less than $\frac{\eta}{2}$ this implies that $D'(k) \cap L(\frac{\eta}{2}) = \emptyset$. Since this relation holds for all sets $D'(k)$, $1 \le k \le M'$, the definition of the set $L(\frac{\eta}{2})$ and of the class of sets $\mathscr{D}$ imply that $-\Delta_l \neq \Delta_l$ for all sets $\Delta_l \in \mathscr{D}$.

To prove that $g(x)$ is a simple function adapted to $\mathscr{D}$ we still have to show that for all rectangles $D'(k) = \Delta_{k_1} \times \cdots \times \Delta_{k_n}$, $1 \le k \le M'$, the relation $k_l \neq \pm k_{l'}$ holds if $l \neq l'$, $1 \le l, l' \le n$. To prove this statement observe that $D'(k) \cap K(\frac{\delta}{2}) = \emptyset$ for all $0 \le k \le M'$. Indeed, there is some point $x \in D'(k) \setminus K(\delta)$, since $D'(k) \cap A' \neq \emptyset$. Since the length of all edges of $D'(k)$ is less than $\frac{\delta}{2nv}$, this implies this statement. This statement and the definition of the set $K(\frac{\delta}{2})$ imply the desired property.

The proof of the second part of *Statement A* can be proved with some small modifications of the previous argument. The main difference is that in this case we start our construction with a good approximation of the set A_1 (and not of A) with the union of some rectangles. Then we take these rectangles $D(j)$ together with their reflection $-D(j)$, and apply the same procedure as before to get the proof of the second part of *Statement A*. There is still a small additional modification in this construction. We choose the rectangles $D'(k)$ in our construction with such a little diameter that guarantees that if one of these rectangles intersects the set A_1, another one intersects the set $-A_1$, then they are disjoint.

Let me finally remark that we got such an approximation of a function $f \in \bar{\mathscr{H}}_G^n$ with elementary functions which are adapted to such a regular system $\mathscr{D}$, whose elements satisfy the property $G(\partial \Delta_j) = 0$ for all $\Delta_j \in \mathscr{D}$, where $\partial \Delta$ denotes the boundary of the set Δ. I made this remark, because this means that we have such an approximation in Lemma 3.3 which also satisfies the extra property needed in the proof of Lemma 6.3.