Limit theorems and infinitely divisible distributions. Part III

Functional limit theorems.

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Summary: In the third part of this work we prove the sufficiency part of the main result proved in the second part with a different method. We make a good coupling of the independent random variables for whose sums we want to prove a limit theorem with independent random variables with so-called associated distributions to the distributions of the original random variables. This coupling shows that the sums of the original random variables and the sums of the random variables with these associated distributions have the same limit behaviour. This proof helps us to understand better the picture behind the limit theorem we discuss. Beside this, its method enables us to prove a functional limit theorem version of this result.

1. Introduction. Formulation of the results.

In Theorem 1 of Part II of the work *Limit theorems and infinitely divisible distributions* we gave the necessary and sufficient condition for the convergence in distribution of the appropriate normalized sums of the random variables in a triangular array which satisfies the uniform smallness condition. We recall the sufficiency part of this result and give a new proof of this result which is based on a good coupling. Then, by applying this coupling argument again, we also prove a functional limit theorem version of this result. This functional limit theorem, whose exact formulation will be given later, states that under natural weak conditions not only the sums of the random variables in a row have a limit distribution, but also the distributions of the random broken lines, made from the partial sums in a natural way converge in distribution to a probability measure in the space of functions.

We will investigate the following result of Part II.

Theorem 1. *Let $\xi_{k,j}$, be a triangular array with distribution functions $F_{k,j}$, $k = 1,2,\dots$, $1 \le j \le n_k$, which satisfies the uniform smallness condition. Let us introduce the canonical measures*

$$M_k(\,dx) = \sum_{j=1}^{n_k} x^2 F_{k,j}(\,dx), \quad k = 1,2,\dots, \tag{1.1}$$

fix some number $a > 0$, and define the function

$$\tau(x) = \tau_a(x) = \begin{cases} x & \textit{if } |x| \le a \\ a & \textit{if } x \ge a \\ -a & \textit{if } x \le -a \end{cases}. \tag{1.2}$$

Let us assume that the random variables $\xi_{k,j}$ satisfy the identity $E\tau(\xi_{k,j})=0$ for all indices $k=1,2,\dots$, $1\le j\le n_k$, and that the measures M_k converge to a canonical measure M_0 on the real line.

Then the sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ converge in distribution to a distribution function whose characteristic function $\varphi(t)$, or more explicitly its (existing) logarithm, is given by the formula

$$\log\varphi(t)=\int_{-\infty}^{\infty}\frac{e^{itu}-1-it\tau(u)}{u^2}M_0(\,du), \tag{1.3}$$

where the canonical measure M_0 is the limit of the canonical measures M_k, and the function τ was defined in formula (1.2).

To understand better the above result we recall some notions. A σ-finite measure M on the real line is called canonical if

$$M([-a,a])<\infty, \quad \text{and} \quad \int_{\{|x|>a\}}\frac{1}{x^2}M(\,dx)<\infty$$

for all real numbers $a>0$. A sequence of canonical measures M_k, $k=1,2,\dots$, (weakly) converges to a canonical measure M_0 if

$$\begin{aligned}\lim_{k\to\infty}M_k^+(x)&=\lim_{k\to\infty}\int_x^{\infty}\frac{1}{u^2}M_k(\,du)=M_0^+(x)=\int_x^{\infty}\frac{1}{u^2}M_0(\,du),\\ \lim_{k\to\infty}M_k^-(x)&=\lim_{k\to\infty}\int_{-\infty}^{-x}\frac{1}{u^2}M_k(\,du)=M_0^-(x)=\int_{-\infty}^{-x}\frac{1}{u^2}M_0(\,du),\end{aligned} \tag{1.4}$$

for all such numbers $x>0$ where the function $M_0^+(\cdot)$ or $M_0^-(\cdot)$ is continuous, and

$$\lim_{k\to\infty}M_k\{[a,b]\}=M_0\{[a,b]\}$$

for all numbers $-<\infty<a<b<\infty$ where the limit measure M_0 is continuous. (The continuity of the measure M_0 in the points a and b means that $M_0(\{a\})=M_0(\{b\})=0$).

A triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, consists of random variables from which the random variables with fixed first index k are independent. It satisfies the uniform smallness condition if for all $\varepsilon>0$ the relation $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}P(|\xi_{k,j}|>\varepsilon)=0$ holds.

To give a complete formulation of Theorem 1, or more explicitly of formula (1.3) we have to define the integrand in the integral of formula (1.3) also in the point $u=0$. We do this by continuity arguments in the following way:

$$\left.\frac{e^{itu}-1-it\tau(u)}{u^2}\right|_{u=0}=\lim_{u\to0}\frac{e^{itu}-1-it\tau(u)}{u^2}=-\frac{t^2}{2}.$$

The above formulated Theorem 1 contains only one part of Theorem 1 in Part II of this work, the sufficiency part which stated the existence of a limit distribution if the conditions of the theorem are satisfied. Moreover, even this result is formulated in the special case when the condition $E\tau(\xi_{k,j})=0$ holds for all indices $k=1,2,\dots$ and $1\le j\le n_k$. But the general case can be reduced simply to this special case with the help of Lemma 2 in Part II. The formulation of the result is simpler in the special case considered here. Now the more complicated conditions about the behaviour of the measures M_k can be expressed as the weak convergence of the canonical measures M_k introduced in Theorem 1 to an appropriate canonical measure M_0.

In the problems of probability theory we often take from the terms of the random sum we investigate the expected value of these terms and in such a way we work with the sum of random variables with expectation zero. The condition $E\tau(\xi_{k,j})=0$ is a modified version of this property in the general case when the random variables we are working with may not have a finite expectation.

The proof of Theorem 1 applies a method essentially different from the method of Part II. We shall apply the following relatively simple result whose proof will be given at the end in the Appendix.

Theorem A. *Let S_k and T_k, $k=1,2,\dots$, be sequences of such random variables for which the sequence of differences S_k-T_k stochastically converges to zero as $k\to\infty$. If the sequence of random variables S_k converges in distribution to a distribution function F, then the sequence of random variables T_k converges in distribution to the same distribution function F.*

Also the following generalization of this statement holds. Let a separable metric space (X,ρ) be given together with two sequences of random variables S_k and T_k, $k=1,2,\dots$, on a probability space which take their values on the space (X,ρ), and their distance $\rho(S_k,T_k)$ tends to zero stochastically if $k\to\infty$. It the sequence S_k converges weakly to a measure μ on the space (X,ρ), then the sequence of random variables T_k converges weakly to the same measure μ.

In the proof of Theorem 1 we only need the first statement of Theorem A. Its second statement is formulated because that will be needed in the proof of the functional limit theorem version of Theorem 1. The proof of Theorem 1 given here will be similar to the second proof of the Poisson limit theorem in the Appendix of Part I. We shall make a good coupling of the random variables considered in Theorem 1 with independent random variables with infinitely divisible distributions. We can make this coupling in such a way that the sums of the original and the sums of the coupled random variables are close to each other, and the sums of the counting measures of the Poisson measures whose (normalized) sums determine the infinitely divisible random variables we construct in this coupling construction is also convergent. Then Theorem A enables us to reduce the proof of Theorem 1 to the proof of the convergence of the coupled random variables in distribution. This latter statement can also be proved by means of a good coupling procedure and Theorem A.

To carry out the program sketched above it is useful to decompose the measures M_k appearing in the formulation of Theorem 1 to two terms in such a way that the first term is responsible for the convergence of the Gaussian and the second term for the convergence of the Poissonian part in the limit theorem we consider. The subsequent Lemma 1 supplies such a decomposition.

Lemma 1. *Let $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, be a triangular array satisfying the uniform smallness condition. Let $F_{k,j}$ denote the distribution function of the random variable $\xi_{k,j}$, and put $G_{k,j}(\,dx)=x^2F_{k,j}(\,dx)$. Let us assume that the canonical measures $M_k=\sum\limits_{j=1}^{n_k}G_{k,j}$ converge weakly to a canonical measure M_0 as $k\to\infty$. Let us write the limit measure M_0 in the form $M_0=M_0'+M_0''$, where M_0' is the restriction of the measure M_0 to the origin, i.e. for all measurable sets $A\subset R^1$ $M_0'(A)=0$, if $0\notin A$, and $M_0'(A)=M_0(\{0\})$ if $0\in A$. Furthermore $M_0''=M_0-M_0'$. Then there exists such a sequence of numbers $\varepsilon_k>0$, $\varepsilon_k\to0$ if $k\to\infty$, for which the measures M_k', $M_k'(A)=M_k(A\cap I(\{|x|<\varepsilon_k\})$ converge weakly to the canonical measure M_0', and the canonical measures M_k'', $M_k''(A)=M_k(A\cap I\{|x|\ge\varepsilon_k\})$ converge weakly to the canonical measure M_0'', where $I(B)$ denotes the indicator function of the set B. Furthermore, $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}(1-F_{k,j}(\varepsilon_k))=0$, and $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}F_{k,j}(-\varepsilon_k)=0$, and even the relation*

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}[(1-F_{k,j}(\varepsilon_k))+F_{k,j}(-\varepsilon_k)]^2=0 \tag{1.5}$$

holds.

We describe the coupling construction which enables the reduction of Theorem 1 to the proof of the convergence of appropriate infinitely divisible distributions. Then we shall also give the ideas behind this construction and formulate Proposition 1 which tells the most important properties of this coupling construction. Because of some reason of convenience we shall work in the coupling construction not with the random variables $\xi_{k,j}$ considered in Theorem 1, but we shall construct instead new (for fixed index k independent) random variables with the same distributions.

Let the random variables $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, of the triangular array we consider satisfy the conditions of Theorem 1. For all pairs of indices $k=1,2,\dots$, $1\le j\le n_k$, let us consider the random variables $\xi_{k,j}$ and define the probability measures $\bar\nu_{k,j}$ and $\bar{\bar\nu}_{k,j}$ given by the formulas $\bar\nu_{k,j}(A)=P(\xi_{k,j}\in A|\,|\xi_{k,j}|<\varepsilon_k)$ and $\bar{\bar\nu}_{k,j}(A)=P(\xi_{k,j}\in A|\,|\xi_{k,j}|\ge\varepsilon_k)$ together with the numbers $p_{k,j}=P(|\xi_{k,j}|\ge\varepsilon_k)$ where $A\in\mathbf{R}^1$ are arbitrary measurable sets and the numbers ε_k are chosen in such a way that the results of Lemma 1 hold with them. Let $\eta_{k,j}'$, $j=1,\dots,n_k$, be $\bar\nu_{k,j}$ distributed random variables which are independent for fixed index k. Let us also consider a sequence of Poisson distributed random variables $\zeta_{k,j}$ with parameters $\bar p_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, where the number $\bar p_{k,j}$ is the solution of the equation $1-e^{-\bar p_{k,j}}=p_{k,j}$. Let us also assume that these random variables $\zeta_{k,j}$ are independent for a fixed index k, and they are also independent of the random variables $\eta_{k,j}'$. Furthermore, let $\gamma_{k,j,l}$, $k=1,2,\dots$, $1\le j\le n_k$, $l=1,2,\dots$, be random variables with distribution $\bar{\bar\nu}_{k,j}$ which are independent both from each other and the random variables defined before.

Let us define, beside the already constructed random variables $\eta'_{k,j}$ the random variables $\eta''_{k,j}=\sum\limits_{l=1}^{\zeta_{k,j}}\gamma_{k,j,l}$, $\xi'_{k,j}=\eta'_{k,j}I(\zeta_{k,j}=0)$, $\xi''_{k,j}=\gamma_{k,j,1}I(\zeta_{k,j}\ge 1)$, $\tilde\xi_{k,j}=\xi'_{k,j}+\xi''_{k,j}$, and $\eta_{k,j}=\eta'_{k,j}+\eta''_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$. We shall see that the random variables $\tilde\xi_{k,j}$ and $\eta_{k,j}$ constructed in such a way give a good coupling which satisfy Proposition 1 formulated below. This enables us to reduce the proof of Theorem 1 to the study of the sums of the random variables $\eta_{k,j}$ which is a simpler problem.

The idea behind the above construction is the following. As we shall see the random variables $\xi_{k,j}$ and $\tilde\xi_{k,j}=\xi'_{k,j}+\xi''_{k,j}$ have the same distributions. The random variables $\xi'_{k,j}$ and $\eta'_{k,j}$ are close to each other, hence they satisfy the central limit theorem with a Gaussian limit with the same expected value and variance. The reason we defined them in a slightly different way is that we wanted to achieve that the random variables $\eta'_{k,j}$ and $\eta''_{k,j}$ be independent, because this allows to study their behaviour separately. The random variables $\xi''_{k,j}$ and $\eta''_{k,j}$ are also sufficiently close to each other, but this closeness has a different reason.We can observe that both probabilities $P(\eta'_{k,j}\neq 0)$ and $P(\eta''_{k,j}\neq 0)$ are small for large indices k, but we need more knowledge about their behaviour. Our construction guarantees that $\xi''_{k,j}(\omega)=\eta''_{k,j}(\omega)$ on the set $\{\omega\colon\,\zeta_{k,j}(\omega)\le 1\}$, and the set $\{\omega\colon\,\zeta_{k,j}(\omega)\ge 2\}$ has very small probability for large indices k. This fact will guarantee that the above constructed coupling is good for our purposes.

Let us also observe that the sequence of random variables $\gamma_{k,j,1},\dots,\gamma_{k,j,\zeta_{k,j}}$ (with a random number of elements) is a Poisson process with counting measure $\bar p_{k,j}\bar{\bar\nu}_{k,j}$. Hence, as we have seen in Part I the random variable $\eta''_{k,j}=\sum\limits_{l=1}^{\zeta_{k,j}}\gamma_{k,j,l}$ is infinitely divisible, and the logarithm of its characteristic function can be given by the formula

$$\log\varphi_{k,j}(t)=\log Ee^{it\eta''_{k,j}}=\int(e^{itu}-1)p_{k,j}\bar{\bar\nu}_{k,j}(\,du)=\frac{\bar p_{j,k}}{p_{j,k}}\int_{\{|u|\ge\varepsilon_k\}}\frac{e^{itu}-1}{u^2}G_{k,j}(\,du),\tag{1.6}$$

where $G_{k,j}(\,du)=u^2F_{k,j}(\,du)$ agrees with the measure $G_{k,j}$ defined in Lemma 1.

We formulate the most important properties of the above constructed coupling in the following Proposition 1.

Proposition 1. *Let the triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, satisfy the conditions of Theorem 1. Then the above coupling construction formulated after Lemma 1 has the following properties: The distribution of the random variables $\tilde\xi_{k,j}=\xi'_{k,j}+\xi''_{k,j}$ and $\xi_{k,j}$ agree. The triangular arrays $\eta'_{k,j}$ and $\eta''_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$ are independent, i.e. for a fixed index k the random vectors $\eta'_{k,j}$, $1\le j\le n_k$, and $\eta''_{k,j}$, $1\le j\le n_k$, are independent. Also the identities $P(|\xi'_{k,j}|\le\varepsilon_k)=P(|\eta'_{k,j}|\le\varepsilon_k)=1$ hold with the same sequence of numbers ε_k, $k=1,2,\dots$, which appears in Lemma 1. Furthermore,*

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}|E\xi'_{k,j}-E\eta'_{k,j}|=0,\tag{1.7}$$

and also the relations

$$
\begin{aligned}
&\sup_{1\le p\le n_k}\left|\sum_{j=1}^p(\xi'_{k,j}-E\xi'_{k,j})-(\eta'_{k,j}-E\eta'_{k,j})\right|\Rightarrow 0,\\
&\sup_{1\le p\le n_k}\left|\sum_{j=1}^p(\xi''_{k,j}-E\tau(\xi''_{k,j}))-(\eta''_{k,j}-E\tau(\eta''_{k,j}))\right|\Rightarrow 0
\end{aligned}
\tag{1.8}
$$

hold where $\Rightarrow$ denotes stochastic convergence. The logarithm of the characteristic function of the sum $\sum\limits_{j=1}^{n_k}(\eta''_{k,j}-E\tau(\eta''_{k,j}))$ can be expressed by means of a canonical measure $\bar M''_k$ close to the canonical measure M''_k as

$$
\log E\exp\left\{it\left(\sum_{j=1}^{n_k}(\eta''_{k,j}-E\tau(\eta''_{k,j}))\right)\right\}=\int\frac{e^{itu}-1-it\tau(u)}{u^2}\bar M''_k(\,du), \tag{1.9}
$$

where $\bar M''_k(\,du)=M''_k(\,du)+\sum\limits_{j=1}^{n_k}\frac{\bar p_{k,j}-p_{k,j}}{p_{k,j}}G'_{k,j}(\,du)$, and $G'_{k,j}$ is the restriction of the measure $G_{k,j}$ to the set $\mathbf{R}^1\setminus[-\varepsilon_k,\varepsilon_k]$.

The triangular array $\eta'_{k,j}-E\eta'_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, satisfies the central limit theorem with a Gaussian limit which has expectation zero and variance $M_0(\{0\})$.

We formulate a result about the convergence of infinitely divisible distributions which enables us to complete the proof of Theorem 1. Let us recall the following result discussed in Part I of this work. If M is a canonical measure on the real line, ξ_n, $n=1,2,\dots$, is a Poisson process on the real line with counting measure $\mu(du)=\frac{M(\,du)}{u^2}$, then the appropriate regularized version of the sum $\eta=\eta_M=\sum\limits_{n=1}^{\infty}\xi_n-E\left(\sum\limits_{n=1}^{\infty}\tau(\xi_n)\right)$ of the points of the Poisson process, where $\tau(\cdot)$ is the function introduced in formula (1.2) is convergent, and the random variable η_M has infinitely divisible distribution. More precisely, we can define the regularized sum by the formula

$$
\eta_M=\lim_{L\to\infty}\left(\left(\sum_{n\colon\,|\xi_n|>2^{-L}}\xi_n\right)-E\left(\sum_{n\colon\,|\xi_n|>2^{-L}}\tau(\xi_n)\right)\right).
$$

and this limit is convergent with probability one. The distribution of the so defined random variable η_M has such a characteristic function whose logarithm equals

$$
\log\varphi(t)=\log\varphi_M(t)=\int\frac{e^{itu}-1-it\tau(u)}{u^2}M(\,du). \tag{1.10}
$$

We shall call the random variable η_M defined in such a way the infinitely divisible random variable determined by the Poisson process $\xi_1,\xi_2,\dots$ with counting measure μ. Now we formulate the following result.

Proposition 2. *Let M_k, $k=1,2,\dots$, be a sequence of canonical measures which converges weakly to a canonical measure M_0. Let us also assume that the relations $M_0(\{0\})=0$ and $M_k(\{0\})=0$, $k=1,2,\dots$, hold. Put $\mu_k(du)=\frac{M_k(du)}{u^2}$, $k=0,1,2,\dots$. Then we can define Poisson processes $\xi_{k,1},\xi_{k,2},\dots$ with counting measure μ_k and Poisson processes $\bar\xi_{k,1},\bar\xi_{k,2},\dots$ with counting measure μ_0 in such a way that the random variables η_k with infinitely divisible distribution determined by the Poisson processes $\xi_{k,1},\xi_{k,2},\dots$ (introduced e.g. before formula 1.10) and the random variables $\bar\eta_k$ with infinitely divisible distribution determined by the Poisson processes $\bar\xi_{k.1},\bar\xi_{k,2},\dots$ satisfy the relation $\eta_k-\bar\eta_k\Rightarrow 0$ where $\Rightarrow$ denotes stochastic convergence. (Let us remark that the distributions of the random variables $\bar\eta_k$ do not depend on the index k.)*

Remark: Proposition 2 and Theorem A together imply that if the measures M_k, $k=0,1,2,\dots$, satisfy the conditions of Proposition 2, then the distributions of the random variables η_k defined in Proposition 2 converge in distribution to the distribution function whose characteristic function is given in formula (1.3). We also remark that with the help of some additional work a stronger version of Proposition 2 could also be proved. It is possible to make such a construction in which $\bar\eta_k=\bar\eta$, that is these random variables (and the Poisson processes determing them) do not depend on the index k. Further, it can be achieved that also the relation $\eta_k-\bar\eta_k\to 0$ hold with probability one. But for our purposes the weaker result formulated in Proposition 2 is as good as its above mentioned stronger version.

We show that Propositions 1 and 2 together with Theorem A imply Theorem 1. Let us consider the random variables $\eta'_{k,j}$, $\eta''_{k,j}$, and $\eta_{k,j}=\eta'_{k,j}+\eta''_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, defined in the coupling construction given after Lemma 1 together with the sums $T_k=\sum\limits_{j=1}^{n_k}(\eta_{k,j}-E\tau(\eta_{k,j}))$, $T'_k=\sum\limits_{j=1}^{n_k}(\eta'_{k,j}-E\eta'_{k,j})$ and $T''_k=\sum\limits_{j=1}^{n_k}(\eta''_{k,j}-E\tau(\eta''_{k,j}))$ defined with their help. First we claim that the sums T_k converge in distribution to an infinitely divisible distribution function whose characteristic function has a logarithm given by formula (1.3). Indeed, $T_k=T'_k+T''_k$, the random variables T'_k and T''_k are independent, and the random variables T'_k converge in distribution to the Gaussian distribution with expectation zero and variance $M_0(\{0\})$ by Proposition 1. On the other hand, the random variables T''_k converge in distribution to an infinitely divisible distribution determined by the canonical measure M''_0 because of Proposition 2, formula (1.9) and the convergence of the canonical measure $\bar M''_k$ to the canonical measure M''_0. (The measure M''_0 is the restriction of the measure M_0 to the set $\mathbf{R}^1\setminus\{0\}$.) The convergence of $\bar M''_k$ to $M''-0$ follows from the form of the measures $\bar M''_k$, the convergence of the measures M''_k to $M''-0$ by Lemma 1 and the convergence of the measures $\sum\limits_{j=1}^{n_k}\frac{\bar p_{k,j}-p_{k,j}}{p_{k,j}}G_{k,j}(\,du)$ to the measure identically zero on the real line. The last convergence holds, since $\frac{\bar p_{k,j}-p_{k,j}}{p_{k,j}}\ge 0$, and $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}\frac{\bar p_{k,j}-p_{k,j}}{p_{k,j}}\to 0$. Indeed, by the identity $1-e^{-\bar p_{k,j}}=p_{k,j}$ defining the quantity $\bar p_{k.j}$ we have $\bar p_{k,j}=-\log(1-p_{k,j})$. Hence we have $p_{k,j}\le\bar p_{k,j}\le p_{k,j}+p^2_{k,j}$ for large indices k. (Here we exploit that for a large index k all numbers $p_{k,j}$ are very small. Hence the relations formulated for the quantities $p_{k,j}$ and $\bar p_{k,j}$ really hold.

Finally Proposition 1 and Theorem A enable us to prove Theorem 1, i.e. the statement that the random sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$, or what is equivalent to it, the random sums $\tilde S_k=\sum\limits_{j=1}^{n_k}\tilde\xi_{k,j}=\sum\limits_{j=1}^{n_k}\left(\tilde\xi_{k,j}-E\tau(\tilde\xi_{k,j})\right)$ of the random variables $\tilde\xi_{k,j}$ defined in the coupling construction described after Lemma 1 satisfy the statement of Theorem 1. Indeed, formula (1.8) implies that $\tilde S_k-T_k\Rightarrow 0$ where $\Rightarrow$ denotes stochastic convergence. Hence the random variables S_k or $\tilde S_k$ converge to the same distribution function as the random variables T_k, and this implies Theorem 1.

Let us make some comments about the proof of Theorem 1 explained in this paper. This approach also may explain that although Theorem 1 supplies many cases when the normalized sums of independent random variables have a limit distribution, the central limit theorem, i.e. the case when the limit is Gaussian deserves its name, the limit theorems with a Gaussian limit really play a central role in the theory of limit theorems for sums of independent random variables. Let us observe that in the coupling construction applied in this proof we approximated each term which contributes to the non-Gaussian part of the limit individually by a random variable with an infinitely divisible distribution. Then we showed the sum of the errors caused by these approximations is negligible. This fact can be interpreted in such a way that limit theorems with a non-Gaussian limit must have a very special form, in a certain sense the distributions of the terms in the sum must resemble to the limit distribution. The picture in the case of the central limit theorem is quite different. In this case, — and this is one of the most remarkable facts in probability theory — the distribution of the individual terms in the sum may be quite general, the distribution of the sum "forgets" the distribution of the individual terms. In such a case a term by term approximation of the summands independently of each other, — and this was done in the coupling construction for the non-Gaussian part — would cause a non-negligible error.

The method of proof given here and in Part II was quite different. Here we applied the so-called coupling method and proved the result by means of a probabilistic argument. In the proof of Part II the characteristic function technique, a useful method of analysis was applied. Nevertheless, it may be useful to understand that these two approaches are not so far from each other as it may seem at first sight. The coupling argument is also present in a hidden way also in the proof by means of characteristic functions.

Indeed, let us consider a triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$ which satisfies the uniform smallness condition. Let $\varphi_{k,j}(t)$ denote the characteristic function of the random variable $\xi_{k,j}$. Put $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$. If the sums S_k have a limit distribution then the relation

$$\lim_{k\to\infty}\prod_{j=1}^{n_k}\varphi_{k,j}(t)=\psi(t)$$

holds with an appropriate characteristic function $\psi(t)$. We have seen the (non-trivial) fact that in the last formula logarithm can be taken, i.e. this relation is equivalent to the formula

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}\log\varphi_{k,j}(t)=\log\psi(t)$$

Another important step of the proof was to show that since the random variables $\xi_{k,j}$ are relatively small the replacement of the term $\log\varphi_{k,j}(t)$ by $\varphi_{k,j}(t)-1$ is allowed, i.e. we can write

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}(\varphi_{k,j}(t)-1)=\log\psi(t).$$

Let us also observe that, as we have seen in Part I, the function $\varphi_{k,j}(t)-1$ is the logarithm of the characteristic function of the infinitely divisible random variable which is determined by the Poisson process whose counting measure is the distribution function $F_{k,j}$ of the random variable $\xi_{k,j}$. In such a way the replacement of the function $\log\varphi_{k,j}(t)$ by $\varphi_{k,j}(t)-1$ corresponds to the coupling construction made in this part of the work.

Finally, we remark that the above coupling method enables us to approximate not only the sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ by means of random variables with infinitely divisible distribution, but under some natural conditions the partial sums $S_{k,l}=\sum\limits_{j=1}^{l}\xi_{k,j}$, $1\le k\le n_k$, can be approximated simultaneously by means of sums of independent random variables with indivisible distribution. This argument leads to the investigation of the so-called functional limit theorems, a result which deserves a more detailed discussion.

A.) Functional limit theorems for general triangular arrays.

An interesting and important result of probability theory, called the functional central limit theorem or invariance principle in the literature, states that if a triangular array satisfies the central limit theorem, then the broken line processes made from the partial sums of these random variables in a natural way converge weakly to a Wiener process. More explicitly, let $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, be a triangular array such that $E\xi_{k,j}=0$, $k=1,2,\dots$, $1\le j\le n_k$, $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}E\xi^2_{j,k}=1$, and the Lindeberg condition $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}E\xi^2_{j,k}I(|\xi_{j.k}|>\varepsilon)=0$ holds for all $\varepsilon>0$. Define the partial sums $S_{k,l}=\sum\limits_{j=1}^{l}\xi_{k,j}$, $k=1,2,\dots$, $1\le l\le n_k$, the numbers $u_{k,0}=0$ and $u_{k,l}=\frac{1}{D_k}\sum\limits_{j=1}^{l}E\xi^2_{k,j}$, $1\le l\le n_k$, in the interval $[0,1]$ where $D_k=\sum\limits_{j=1}^{n_k}E\xi^2_{k,j}$. Then we can define with the help of these quantities the random broken lines $S_k(t)$, $0\le t\le 1$, in such a way that $S_k(0)=0$, $S_k(u_{k,l})=S_{k,l}$, $1\le l\le n_k$, and the functions $S_k(t)$ are linear in all intervals $[u_{k,l-1},u_{k,l}]$, $1\le l\le n_k$. The functional central limit theorem states that the distributions of the stochastic processes $S_k(t)$, $0\le t\le 1$, considered as $C([0,1])$

valued random variables, converge weakly to the distribution of a Wiener process. Let us emphasize that this result states in particular that the necessary and sufficient condition of the central limit theorem implies at the same time a stronger result.

The question arises whether the general limit theorems considered in this work have a similar functional limit version. There is a positive answer to this question. The limit theorem formulated in Theorem 1 holds if and only if an appropriately defined sequence of canonical measures on the real line converges to a canonical measure M_0. We shall show that we can introduce and define an appropriate sequence of canonical measures on the strip $\mathbf{R}^1 \times [0,1]$ which are closely related to the canonical measures considered in Theorem 1, and their convergence implies a functional limit theorem for the appropriately defined random broken lines made from the sums of partial sums of the random variables in the triangular array. We shall formulate such a result in Theorem 2. But to do this first we have to introduce some definitions and notations.

To formulate Theorem 2 first we introduce the appropriate function space where we shall work. This space is called the $D([0,1])$ space in the literature.

We say that a function $x(t)$, $0 \le t \le 1$, is a cadlag function (continue à droite, limite à gauche) if the function $x(t)$ is continuous from the right, and it also has a left-hand side limit in all points. The space $D([0,1])$ consists of the cadlag functions in the interval $[0,1]$, and an appropriate distance is introduced in it. A possible definition of this metric is the distance $d(\cdot,\cdot)$ defined in the following way: Let $x,y \in D([0,1])$ be two cadlag functions and $\varepsilon > 0$ a real number. The relation $d(x,y) \le \varepsilon$ holds if there exists a strictly monotone function $\lambda(t)$ which is a homeomorphism of the interval $[0,1]$ into itself, $\sup\limits_{0 \le t \le 1} |\lambda(t) - t| \le \varepsilon$, and $\sup\limits_{0 \le t \le 1} |y(t) - x(\lambda(t))| \le \varepsilon$.

The space $D([0,1])$ is a separable metric space with the above distance, but it is not a complete metric space. The property that two cadlag functions $x(\cdot)$ and $y(\cdot)$ are close to each other with respect to the metric $d(\cdot,\cdot)$ means that although these two functions may be far from each other with respect to the supremum norm, but they can put close to each other with respect to the this norm if the argument of one of these functions is slightly perturbed in an appropriate way. We introduced the space $D([0,1])$ because we need this notion in the formulation of Theorem 2. Here we only formulate the results about this space but omit the proofs. All of them can be found in P. Billingsley's book "Convergence of probability measures". We had to introduce this notion, because the possible limit processes in Theorem 2, — the Poisson process is a typical example, — do not have continuous trajectories, hence we have to work in a different function space.

Let us remark that the above metric is not the only possible good metric which can be introduced in the space $D([0,1])$. For instance the following metric $d_0(\cdot,\cdot)$ in the space $D([0,1])$ is often applied in the literature. Let $x(\cdot)$, $y(\cdot)$ be two cadlag functions. We say that $d_0(x,y) \le \varepsilon$ if there exists such a homeomorphism $\lambda(\cdot)\colon\ [0,1] \to [0,1]$ of the interval $[0,1]$ into itself for which $\lambda(0) = 0$, $\sup\limits_{t \ne s} \log \left| \frac{\lambda(t)-\lambda(s)}{t-s} \right| \le \varepsilon$, and $|x(t) - y(\lambda(t))| \le \varepsilon$ for all numbers $t \in [0,1]$. It can be proved that the metrics $d(\cdot,\cdot)$ and $d_0(\cdot,\cdot)$ define the same topology on the space $D([0,1])$. This means that sequences of probability measures on the space $D([0,1])$ simultaneously converge or do not converge with respect to these metrics. Hence they are equivalent for our purposes. The essential difference between these two metrics is that the space $D([0,1])$ is a separable *complete* metric space with respect to the metric $d_0(\cdot,\cdot)$, but this relation does not hold for the metric $d_0(\cdot,\cdot)$. Some proofs are simpler in complete metric space, and this is the main reason why the metric $d_0(\cdot,\cdot)$ is applied in several cases.

Let us remark that in the formulation of Theorem A we only assumed that the metric space (X,ρ) we have considered is separable, but did not demand that it has to be complete. This makes possible to apply the metric $d(\cdot,\cdot)$ in subsequent proofs, and this simplifies certain arguments.

Now we define the stochastic processes and canonical measures which will appear in Theorem 2. Let a triangular array $\xi_{k,j}$, $1\le j\le n_k$, be given, and let $F_{k,j}$ denote the distribution function of the random variable $\xi_{k,j}$. Let us define the partial sums

$$S_{k,0}=0,\quad S_{k,l}=\sum_{j=1}^{l}\xi_{k,j},\quad 1\le l\le n_k \tag{1.11}$$

Then let us fix for all numbers $k=1,2,\dots$ an appropriate sequence of numbers $0=u_{k,0}\le u_{k,1}\le u_{k,2}\le\cdots\le u_{k,n_k}=1$ and define the random cadlag functions in the interval $[0,1]$ as

$$\begin{gathered} S_k(t)=S_k(t,u_{k,0},\dots,u_{k,n_k})=S_{k,l-1},\quad \text{if } u_{k,l-1}\le t<u_{k,l},\ \ 1\le l\le n_k, \\ S_k(1)=S_{k,n_k}, \end{gathered} \tag{1.12}$$

$k=1,2,\dots$, with their help. (These numbers $0=u_{k,0}\le u_{k,1}\le u_{k,2}\le\cdots\le u_{k,n_k}=1$ are needed to define the appropriate scaling in the definition of the random cadlag functions. We cannot give them in such an explicit way as in the functional central limit theorem.) Furthermore, let us define certain σ-finite measures N_k on the direct product of the real line and the interval $[0,1]$, on the set $\mathbf{R}^1\times[0,1]$, in the following way: Let $0=u_{k,0}\le u_{k,1}\le u_{k,2}\le\cdots\le u_{k,n_k}=1$ be the same sequence of numbers which appeared in formula (1.12).

$$\text{The measure } N_k(\cdot) \text{ is concentrated on the set } \mathbf{R}^1\times\bigcup_{l=1}^{n_k}\{u_{k,l}\}, \tag{1.13}$$

and the restriction of the measures $N_k(\cdot)$ to the lines $\{(t,u)\colon\ t\in\mathbf{R}^1, u=u_{k,l}\}$, $1\le l\le n_k$, equals the measure $x^2F_{k,l}(dx)$, i.e.

$$N_k(B\times\{u_{k,l}\})=\int_B x^2F_{k,l}(\,dx),\quad \text{if } 1\le l\le n_k,\ \text{and } B\subset\mathbf{R}^1 \text{ is a measurable set.} \tag{1.14}$$

Let us define the notion of canonical measures on the strip $\mathbf{R}^1\times[0,1]$ and their convergence. The measures N_k defined in formulas (1.13) and (1.14) also satisfy the properties of canonical measures.

The definition of canonical measures and their convergence. *We call a σ-finite measure $N(\cdot)$ on the strip $\mathbf{R}^1\times[0,1]$ canonical if for all numbers $s>0$*

$$N([-s,s]\times[0,1])<\infty, \quad \textit{and} \quad \int_{\{(u,v)\colon\, |u|>s,\ 0\le v\le 1\}} \frac{N(\,du,\,dv)}{u^2}<\infty.$$

Let a sequence N_k, $k=0,1,2,\dots$, of canonical measures be given on the strip $\mathbf{R}^1\times[0,1]$. We say that this sequence of canonical measures N_k converges (weakly) to a canonical measure N_0 on the strip $\mathbf{R}^1\times[0,1]$ as $k\to\infty$ if for all such numbers $0\le a\le b\le 1$ which are points of continuity of the limit measure N_0, i.e. for which $\lim\limits_{\varepsilon\to0} N_0([-R,R]\times[a-\varepsilon,a+\varepsilon])=0$, $\lim\limits_{\varepsilon\to0} N_0([-R,R]\times[b-\varepsilon,b+\varepsilon])=0$ and $\lim\limits_{\varepsilon\to0}\int_{|u|>R,\,|v-a|<\varepsilon}\frac{N_0(du,dv)}{u^2}=0$, $\lim\limits_{\varepsilon\to0}\int_{|u|>R,\,|v-b|<\varepsilon}\frac{N_0(du,dv)}{u^2}=0$ for all numbers $R>0$, the canonical measures $M_{k,a,b}$, $k=1,2,\dots$, on the real line defined by the formula $M_{k,a,b}(B)=N_k(B\times[a,b])$ for all measurable sets $B\in\mathbf{R}^1$ converge to the canonical measure $M_{0,a,b}(B)$ on the real line, where $M_{0,a,b}(B)=N_0(B\times[a,b])$, for all measurable sets $B\in\mathbf{R}^1$.

Now we formulate Theorem 2. It says that if the canonical measures N_k on the strip $\mathbf{R}^1\times[0,1]$ defined in formulas (1.13) and (1.14) converge to a canonical measure N_0 on the strip $\mathbf{R}^1\times[0,1]$ then the stochastic processes $S_k(\cdot)$ defined in formula (1.12), considered as $D([0,1])$ space valued random variables, converge weakly to a random process with cadlag trajectories which is determined by the limit canonical measure N_0 in a natural way.

Theorem 2. *Let $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, be a triangular array satisfying the uniform smallness condition and such that $E\tau(\xi_{k,j})=0$ far all $k=1,2,\dots$, $1\le j\le n_k$ with the function $\tau(x)=\tau_a(x)$ defined in formula (1.2) where $a>0$ is some fixed number. Let us fix for all numbers $k=1,2,\dots$ a sequence of numbers $0=u_{k,0}\le u_{k,1}\le u_{k,2}\le\cdots\le u_{k,n_k}=1$ satisfying the relation $\lim\limits_{k\to\infty}\sup\limits_{1\le l\le n_k}|u_{k,l}-u_{k,l-1}|=0$, and let us consider the canonical measures M_k defined on the strip $\mathbf{R}^1\times[0,1]$ defined by formulas (1.13) and (1.14) with the above sequences of numbers $0=u_{k,0}\le u_{k,1}\le u_{k,2}\le\cdots\le u_{k,n_k}=1$ and the distribution functions $F_{k,l}$ of the random variables $\xi_{k,l}$, $k=1,2,\dots$, $1\le l\le n_k$. Let us assume that these canonical measures N_k on the strip $\mathbf{R}^1\times[0,1]$ converge weakly to a canonical measure N_0 which satisfies the relation $N_0(\mathbf{R}^1\times\{0\})=0$. Furthermore, let us also assume that*

- *a.) The function $\lambda(t)=N_0(\{0\}\times[0,t])$ defined with the help of the canonical measure N_0 is continuous in the interval $[0,1]$.*
- *b.) For all numbers $b>0$ the function*

$$\nu_b(t)=\int_{\{(x,y)\colon\, |x|>b,\ 0\le y\le t\}}\frac{N_0(dx,dy)}{x^2},\quad 0\le t\le 1,$$

defined with the help of the limit canonical measure N_0 is continuous in the interval $[0,1]$.

Then the stochastic processes $S_k(t)$, $0\le t\le 1$, defined in formula (1.12) considered as $D([0,1])$ valued random variables converge weakly to a stochastic process $S(t)$, $0\le t\le 1$, which can be considered as a $D([0,1])$ valued random variable. This process $S(t)$ is a stochastic process with cadlag trajectories, it has independent increments, and it is determined by a Poisson field on $\mathbf{R}^1\times[0,1]$ with counting measure $\frac{N_0(du,dv)}{u^2}$ in a natural way. The distribution of this stochastic process can be described in the following way. We have $S_0\equiv 0$, and since $S(t)$ is a process with independent increments, it is enough to give the distribution of the increments $S(v)-S(u)$, $0\le u\le v\le 1$. The distribution function of such a difference is an infinitely divisible distribution determined by the canonical measure $\frac{M_{0,u,v}(dx)}{x^2}$ with $M_{0,u,v}(dx)=N_0(dx\times(u,v])$. The characteristic function $\varphi_{u,v}(\cdot)$ of the random variable $S(v)-S(u)$ has a logarithm which is given by the following modified version of formula (1.3):

$$\log\varphi_{u,v}(t)=\int_{-\infty}^{\infty}\frac{e^{itx}-1-it\tau(x)}{x^2}M_{0,u,v}(\,dx). \tag{1.15}$$

Remark 1. We showed in Lemma 2 in Section 5 of Part I that a stochastic process with the properties demanded for the limit stochastic process appearing in Theorem 2 really exists. The Poisson process with counting measure $\frac{N_0(du,dv)}{u^2}$ defines with the help of appropriate regularized sums investigated in Part I a stochastic process $S(t)$, $0\le t\le 1$, with the required properties. In our discussion we sometimes regard a stochastic process with cadlag trajectories as a random variables $D([0,1])$. This is legitimate, but to justify our right to do this we have to solve a non-trivial measure theoretical problem. We have to show that the measurability of a stochastic process with cadlag trajectories in the usual way is equivalent to its measurability as a function in the space $D([0,1])$ with respect to the σ-algebra determined by the topology of the space $D([0,1])$. The proof of this result can be found for instance in Billingsley's book *Convergence of Probability Measures* Theorem 14.5.

Remark 2. The conditions a.) and b.) imposed for the limit canonical measure N_0 can be formulated in a unified way as $N_0(\mathbf{R}^1\times\{t\})=0$ for all $0\le t\le 1$. This condition cannot be dropped. We show this with the help of an example explained in a rather sketchy way. Let us consider a triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le k$, (i.e. $n_k=k$) which is defined in the following way. Consider a sequence of numbers ε_k, $k=1,2,\dots$, such that $\lim\limits_{k\to\infty}\varepsilon_k=0$ and $\lim\limits_{k\to\infty}k\varepsilon_k=\infty$. Let $\xi_{k,j}=\zeta_{k,j}+\eta_{k,j}$ if $(\frac12-2\varepsilon_k)k<j\le(\frac12-\varepsilon_k)k$ or $(\frac12-4\varepsilon_k)k<j\le(\frac12-3\varepsilon_k)k$ and $\xi_{k,j}=\zeta_{k,j}$ if $j\in[1,k]\setminus(((\frac12-2\varepsilon_k)k,\frac12-\varepsilon_k)k]\cup((\frac12-2\varepsilon_k)k,\frac12-\varepsilon_k)k])$ where the $\zeta_{k,j}$ are independent Gaussian random variables with expectation zero and variance $\frac1k$, and $\eta_{k,j}$ are independent random variables, $P(\eta_{k,j}=1)=1-P(\eta_{k,j}=0)=\frac{\varepsilon_k}{2k}$ which are also independent of the random variables $\zeta_{k,j}$. Define the numbers $u_{k,l}$ appearing in the formulation of Theorem 2 by the formula $u_{k,l}=\frac lk$, $0\le l\le k$. (Actually, it was not necessary to take a Gaussian part $\zeta_{k,j}$ in this example, we only introduced it to make the choice of the numbers $u_{k,l}$ more natural.) It is not difficult to see that in this case the canonical measures N_k introduced before

the formulation of Theorem 2 converge to the canonical measure $N_0 = N_0' + N_0''$, where N_0' is the Lebesgue measure on the set $\{0\} \times [0,1]$, and the measure N_0'' is concentrated in the point $(1, \frac{1}{2})$, $N_0''(\{(1, \frac{1}{2})\}) = 1$. We show that, as the limit measure N_0 does not satisfies condition b.), the processes $S_k(t)$ do not converge to the processes determined by the measure N_0. To see this observe that if we disregard the Gaussian part of this example, i.e. we consider the partial sums $S''_{k,l} = \sum\limits_{j=1}^{l} \eta_{k,j}$, $1 \le l \le k$, the stochastic processes $S''_k(t) = S_{k,l}$ if $u_{k,l} \le t < t_k$, and the candidate for the limit process is the process determined by the measure N_0''. This limit process $S_0(t)$, $0 \le t \le 1$, is defined as $S_0(t) = 0$ if $0 \le t < \frac{1}{2}$ and $S_0(t) = \eta$ if $\frac{1}{2} \le t \le 1$ where η is a Poissonian random variable with parameter 1. The distributions of the processes $S_k(\cdot)$ do not converge to the distribution of the process $S_0(\cdot)$. Indeed, the random variables $S_k(\frac{1}{2} - 4\varepsilon_k)$, $S_k(\frac{1}{2} - 3\varepsilon_k)$, $S_k(\frac{1}{2} - \varepsilon_k)$ take three different integer values with a positive probability, while the process $S''_0(\cdot)$ can take at most two values with probability one. This excludes the convergence of the distributions of the processes $S_k(\cdot)$ to those of the process $S_0(\cdot)$ in the $D([0,1])$ space. Since the Gaussian part of the processes $S_k(\cdot)$ have very small fluctuation in the interval $[\frac{1}{2} - 4\varepsilon_k, \frac{1}{2}]$ the original example also yields a counter-example.

The proof of Theorem 2 will be similar to that of Theorem 1. We can split the stochastic processes $S_k(\cdot)$ with the help of Proposition 1 and Lemma 1 to two parts, one of them responsible for the Gaussian the other one for the Poissonian part of the limit. The convergence to the Gaussian part of the limit follows from the functional central limit theorem. Then the converge to the Poissonian part of the limit process can be investigated by a refined version of the coupling argument applied in the proof of Theorem 1. Let us remark that the expression $\sup\limits_{1 \le j \le n_k}$ appeared in formula (1.8) because it was appropriate in this form in the proof of Theorem 2.

2. The proof of Theorem 1.

In this section we prove the results applied in the proof of Theorem 1, Lemma 1 and Propositions 1 and 2.

The proof of Lemma 1. Let us choose a monotone decreasing sequence of positive numbers η_p, $p=1,2,\dots$, such that $\lim\limits_{p\to\infty}\eta_p=0$ and the numbers $\pm\eta_p$ are points of continuity of the measure M_0. Then $\lim\limits_{p\to\infty} M_0((-\eta_p,\eta_p))=M_0(\{0\})$. Beside this, there exists a threshold index $k_0(p)$ for all numbers $p=1,2,\dots$ such that for all $k\ge k_0(p)$ the inequality $|M_k((-\eta_p,\eta_p))-M_0(-\eta_p,\eta_p))|\le\frac1p$ holds, and if the origin is a point of continuity of the measure M_0, i.e. $M_0(\{0\})=0$, then also the inequality $|M_k((0,\eta_p))-M_0(0,\eta_p))|\le\frac1p$ holds. We may also assume that $|M_0^\pm(\eta_p)-M_k^\pm(\eta_p)|\le\frac1p$ if $k\ge k_0(p)$ where $M_0^\pm(\cdot)$ and the functions $M_k^\pm(\cdot)$ are the functions defined in formula (1.4). Because of the uniform smallness condition we can guarantee that

$$\sup_{1\le j\le n_k}(1-F_{k,j}(\eta_p))+\sup_{1\le j\le n_k}F_{k,j}(-\eta_p)\le\frac{1}{p(M_0^+(\eta_p)+M_0^-(\eta_p))+2}$$

if $k\ge k_0(p)$, and the threshold index $k_0(p)$ is chosen sufficiently large. We may also assume that the sequence of threshold indices $k_0(p)$, $p=1,2,\dots$, is monotone increasing. Put $\varepsilon_k=\eta_p$ if $k_0(p)\le k<k_0(p+1)$. With such a choice of the numbers ε_k the statements of Lemma 1 hold. Indeed,

$$\lim_{k\to\infty}M_k'([a,b])=\lim_{k\to\infty}M_k'([-\eta_{k_0(p)},\eta_{k_0(p)}])=M_0(\{0\})=M_0'([a,b])$$

if the interval $[a,b]$ contains the origin in its interior, and $\lim\limits_{k\to\infty}M_k'([a,b])=0=M_0'([a,b])$ if the interval $[a,b]$ does not contain the origin in its interior and the points a and b are points of continuity of the measure M_0 (in particular also in the case if 0 is a point of continuity of the measure M_0, and $a=0$ or $b=0$). These relations together with the fact that the measures M_k', $k=1,2,\dots$, and M_0' are concentrated in a finite interval $[-A,A]$ imply that the measures M_k' weakly converge to the measure M_0'. As the sequence of measures M_k converges (weakly) to the measure M_0 and the sequence of measures M_k' converges (weakly) to the measure M_0' also the measures $M_k''=M_k-M_k'$ converge (weakly) to the measure $M_0''=M_0-M_0'$. Furthermore,

$$\begin{aligned}
&\sum_{j=1}^{n_k}[(1-F_{k,j}(\varepsilon_k))+F_{k,j}(-\varepsilon_k)]^2\\
&\qquad\le\sup_{1\le j\le n_k}[(1-F_{k,j}(\varepsilon_k))+F_{k,j}(-\varepsilon_k)]\sum_{j=1}^{n_k}[(1-F_{k,j}(\varepsilon_k))+F_{k,j}(-\varepsilon_k)]\\
&\qquad=\sup_{1\le j\le n_k}[(1-F_{k,j}(\varepsilon_k))+F_{k,j}(-\varepsilon_k)](M_k^+(\varepsilon_k)+M_k^-(\varepsilon_k))\\
&\qquad\le\frac{1}{p(M_0^+(\eta_p)+M_0^-(\eta_p))+2}\left(M_0^+(\varepsilon_k)+M_0^-(\varepsilon_k)+\frac2p\right)\le\frac1p
\end{aligned}$$

if $k\ge k_0(p)$. From here we get formula (1.5) by taking the limit procedure $k\to\infty$.

Proof of Proposition 1. As $\tilde\xi_{k,j}=\eta'_{k,j}I(\zeta_{k,j}=0)+\gamma_{k,j,1}I(\zeta_{k,j}\ge1)$, and the random variables $\zeta_{k,j}$ are independent of the other random variables, hence

$$\begin{aligned}P(\tilde\xi_{k,j}\in A)&=P(\tilde\xi_{k,j}\in A|\zeta_{k,j}=0)P(\zeta_{k,j}=0)+P(\tilde\xi_{k,j}\in A|\zeta_{k,j}\ge1)P(\zeta_{k,j}\ge1)\\&=P(\eta'_{k,j}\in A)P(\zeta_{k,j}=0)+P(\gamma_{k,j,1}\in A)P(\zeta_{k,j}\ge1)\\&=\bar\nu(A)(1-p_{k,j})+\bar{\bar\nu}(A)p_{k,j}\\&=P(\xi_{k,j}\in A\cap\{x\colon |x|<\varepsilon_k\})+P(\xi_{k,j}\in A\cap\{x\colon |x|\ge\varepsilon_k\})\\&=P(\xi_{k,j}\in A),\end{aligned}$$

for all measurable sets $A\subset\mathbf{R}^1$, that is the random variables $\tilde\xi_{k,j}$ and $\xi_{k,j}$ have the same distribution. The coupling construction also implies that for a fixed index k the random variables $\tilde\xi_{k,j}$, $1\le j\le n_k$, are independent. This construction also implies that for a fixed k the random variables $\eta'_{k,j}$, $\eta''_{k,j}$, $1\le j\le n_k$, are independent, and $P(|\xi'_{k,j}|\le\varepsilon_k)=P(|\eta'_{k,j}|\le\varepsilon_k)=1$. To prove the relation $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}|E\xi'_{k,j}-E\eta'_{k,j}|=0$ let us observe that

$$\begin{aligned}|E\xi'_{k,j}-E\eta'_{k,j}|&=(1-P(\zeta_{k,j}=0))|E\eta'_{k,j}|=\frac{1-P(\zeta_{k,j}=0)}{P(\zeta_{k,j}=0)}|E\xi'_{k,j}|\\&=\frac{P(|\xi_{k,j}|\ge\varepsilon_k)}{P(|\xi_{k,j}|<\varepsilon_k)}|E\tau(\xi''_{k,j})|\le 2P(|\xi_{k,j}|\ge\varepsilon_k)|E\tau(\xi''_{k,j})|\end{aligned}$$

if $k\ge k_0$ with an appropriate constant k_0, since $0=E\tau(\tilde\xi_{k,j})=E\xi'_{k,j}+E\tau(\xi''_{k,j})$, and $P(\zeta_{k,j}\ge1)=1-e^{-\bar p_{k,j}}=p_{k,j}=P(|\xi_{k,j}|\ge\varepsilon_k)$, where $\bar p_{k,j}$ is the solution of the equation $p_{k,j}=1-e^{-\bar p_{k,j}}$, $p_{k,j}=P(|\xi_{k,j}|\ge\varepsilon_k)$, and $p_{k,j}\le\frac12$ if $k\ge k_0$. As $|E\tau(\xi''_{k,j})|\le aP(|\xi_{k,j}|\ge\varepsilon_k)$ this implies that $|E\xi'_{k,j}-E\eta'_{k,j}|\le 2aP(|\xi_{k,j}|\le\varepsilon_k)^2=2a\left[(1-F_{k,j}(\varepsilon_k))+F_{k,j}(-\varepsilon_k)\right]^2$. We get relation (1.7) by summing up these inequalities and applying formula (1.5).

We can prove the first relation of formula (1.8) with the help of the Kolmogorov inequality. Indeed, for all numbers $\varepsilon>0$

$$\begin{aligned}P\left(\sup_{1\le p\le n_k}\left|\sum_{j=1}^p(\xi'_{k,j}-E\xi'_{k,j})-(\eta'_{k,j}-E\eta'_{k,j})\right|>\varepsilon\right)&\le\frac{\sum\limits_{j=1}^{n_k}\operatorname{Var}(\xi'_{k,j}-\eta'_{k,j})}{\varepsilon^2}\\=\frac{\sum\limits_{j=1}^{n_k}(1-P(\zeta_{k,j}=0))^2\operatorname{Var}\eta'_{k,j}}{\varepsilon^2}&\le\frac1{\varepsilon^2}\sup_{1\le j\le n_k}(1-P(\zeta_{k,j}=0))^2\sum_{j=1}^{n_k}E{\eta'_{k,j}}^2\\=\frac1{\varepsilon^2}M_k([-\varepsilon_k,\varepsilon_k])\sup_{1\le j\le n_k}P(|\xi_{k,j}|\ge\varepsilon_k)^2\to0\quad&\text{if } k\to\infty,\end{aligned}$$

because $\limsup\limits_{k\to\infty} M_k([-\varepsilon_k,\varepsilon_k]) \le M_0(\{0\}) < \infty$ in the construction of Lemma 1, and $\sup\limits_{1\le j\le n_k} (P(|\xi_{k,j}|\ge\varepsilon_k)^2 \to 0$ if $k\to\infty$.

To prove the second statement of formula (1.8) first we show that

$$E\tau(\eta''_{k,j}) = \frac{\bar p_{k,j}}{p_{k,j}} \int_{\{|u|\ge\varepsilon_k\}} \tau(u) F_{k,j}(\,du) = \frac{\bar p_{k,j}}{p_{k,j}} E\tau(\xi''_{k,j}), \tag{2.1}$$

where $p_{k,j} = P(|\xi_{k,j}|\ge\varepsilon_k) = [(1-F_{k,j}(\varepsilon_k)+F_{k,j}(-\varepsilon_k)]$, and $\bar p_{k,j}$ is the solution of the equation $1-e^{-\bar p_{k,j}} = p_{k,j}$. Indeed, by exploiting that if $\eta_1,\eta_2,\dots,$ are independent, identically distributed random variables, τ is a random variable taking non-negative integer values which is independent of the random variables η_j, then $E(\eta_1+\cdots+\eta_\tau) = E\tau E\eta_1$. Further, since $E\zeta_{k,j} = \bar p_{k,j} = 1-e^{-p_{k,j}}$ we get that

$$\begin{aligned} E\left(\sum_{l=1}^{\zeta_{k,j}} \gamma_{k,j,l} I(|\gamma_{k,j,l}|\le a)\right) &= \frac{E\zeta_{k,j}}{p_{k,j}} \int_{\{\varepsilon_k\le|u|\le a\}} \tau(u)F_{k,j}(\,du) \\ &= \frac{\bar p_{k,j}}{p_{k,j}} \int_{\{\varepsilon_k\le|u|\le a\}} \tau(u)F_{k,j}(\,du), \end{aligned}$$

and similarly

$$\begin{aligned} E\left(\sum_{l=1}^{\zeta_{k,j}} I(|\gamma_{k,j,l}|\ge a)\right) &= \bar{\bar\nu}_{k,j}((-\infty,-a]\cup[a,\infty))E\zeta_{k,j} \\ &= \frac{\bar p_{k,j}}{p_{k,j}} \left[1-F_{k,j}(a)+F(-a_{k,j})\right]. \end{aligned}$$

Since

$$E\tau(\eta''_{k,j}) = E\left(\sum_{l=1}^{\zeta_{k,j}} \gamma_{k,j,l} I(|\gamma_{k,j,l}|\le a)\right) + aE\left(\sum_{l=1}^{\zeta_{k,j}} I(|\gamma_{k,j,l}|\ge a)\right),$$

the above two identities imply formula (2.1).

To prove the second relation of formula (1.8) let us also observe that

$$\xi''_{k,j}-\eta''_{k,j} = \xi''_{k,j} - \sum_{l=1}^{\zeta_{k,j}} \gamma_{k,j,l} = I(\zeta_{k,j}\ge 2)\sum_{l=2}^{\zeta_{k,j}} \gamma_{k,j,l},$$

because by the coupling construction $\xi''_{k,j} = \eta''_{k,j} = 0$, on the set $\{\omega\colon\ \zeta_{k,j}(\omega)=0\}$ and $\xi''_{k,j} = \eta''_{k,j} = \gamma_{k,j,1}$ on the set $\{\omega\colon\ \zeta_{k,j}(\omega)\ge 1\}$. With the help of these relations we get that

$$\begin{aligned} P\left(\sum_{j=1}^p \xi''_{k,j} - \sum_{j=1}^p \eta''_{k,j} \ne 0 \text{ for some number } 1\le p\le n_k\right) &\le \sum_{j=1}^{n_k} P(\zeta_{k,j}\ge 2) \\ \le \sum_{j=1}^{n_k} \left[(1-F_{k,j}(\varepsilon_k)+F_{k,j}(-\varepsilon_k))\right]^2 \to 0 \quad \text{if } k\to\infty \end{aligned}$$

by the formula (1.5). Hence to prove the second relation of formula (1.8) it is enough to show that

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}|E\tau(\xi''_{k,j})-E\tau(\eta''_{k,j})|=0.$$

But by formula (2.1)

$$|E\tau(\xi''_{k,j})-E\tau(\eta''_{k,j})|=\left|\frac{\bar p_{k,j}-p_{k,j}}{p_{k,j}}E\tau(\xi''_{k,j})\right|\le 2ap_{k,j}^2=2aP^2(|\xi_{k,j}|\ge\varepsilon_k)$$

if $k\ge k_0$. (Observe that $|1-e^{-p_{k,j}}-p_{k,j}|<p_{k,j}^2$ and $|E\tau(\xi''_{k,j}|\le 2ap_{k,j}$.) This implies that

$$\sum_{j=1}^{n_k}|E\tau(\xi''_{k,j})-E\tau(\eta''_{k,j})|\le 2a\sum_{j=1}^{n_k}[(1-F_{k,j}(\varepsilon_k)+F_{k,j}(-\varepsilon_k)]^2\to 0\quad\text{if } k\to\infty$$

by formula (1.5). The above relations imply also the second part of formula (1.8).

Finally, we get by summing up the identities in formula (1.6) for the numbers $1\le j\le n_k$ for a fixed integer k and by applying the definition given in Lemma 1 that

$$\log E\exp\left\{it\left(\sum_{j=1}^{n_k}(\eta''_{k,j}\right)\right\}=\int\frac{e^{itu}-1}{u^2}\bar M''_k(\,du),$$

and athe summation of the identity (2.1) in the variable j yields the identity

$$\sum_{j=1}^{n_k}E\tau(\eta''_{k,j}))=\int\frac{\tau(u)}{u^2}\bar M''_k(\,du).$$

These relations imply formula (1.9). The random variables $\eta'_{k,j}-E\eta'_{k,j}$, $1\le j\le n_k$, are independent, $|\eta'_k-E\eta'_k|\le\varepsilon_k$, hence the sums $T'_k=\sum\limits_{j=1}^{n_k}(\eta'_{k,j}-E\eta_{k,j})$ satisfy the central limit theorem. Beside this, $ET'_k=0$, and we complete the proof of Lemma 1 if we show that $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}\operatorname{Var}\eta'_{k,j}=M_0(\{0\})$. This follows from the identity

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}\left(E\eta'_{k,j}\right)^2=0,\quad \lim_{k\to\infty}\sum_{j=1}^{n_k}\left|E{\eta'_{k,j}}^2-E{\xi'_{k,j}}^2\right|=0 \tag{2.2}$$

to be proved below, since $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}E{\xi'_{k,j}}^2=M_0(\{0\})$.

We get similarly to the proof of formula (1.7) that

$$\left|E\eta'_{k,j}\right|=\frac{|E\xi'_{k,j}|}{P(\zeta_{k,j}=0)}=\frac{|E\tau(\xi''_{k,j})|}{P(|\xi_{k,j}|<\varepsilon_k)}\le 2|E\tau(\xi''_{k,j})|\le 2aP(|\xi_{k,j}|>\varepsilon_k)$$

if $k\ge k_0$ with an appropriate constant k_0. Then formula (1.5) implies the first statement of formula (2.2). On the other hand,

$$\left|E{\xi'_{k,j}}^2-E{\eta'_{k,j}}^2\right|=(1-P(\zeta_{k,j}=0))E{\eta'_{k,j}}^2=\frac{1-P(\zeta_{k,j}=0)}{P(\zeta_{k,j}=0)}E{\xi'_{k,j}}^2$$
$$\le 2P(|\xi_{k,j}|\ge\varepsilon_k)E{\xi'_{k,j}}^2.$$

This inequality together with the Schwarz inequality imply that

$$\sum_{j=1}^{n_k}\left|E{\xi'_{k,j}}^2-E{\eta'_{k,j}}^2\right|\le\left(\sum_{j=1}^{n_k}4P^2(|\xi_{k,j}|\ge\varepsilon_k)\cdot\sum_{j=1}^{n_k}\left(E{\xi'_{k,j}}^2\right)^2\right)^{1/2}\to 0,\quad\text{if } k\to\infty,$$

since $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}P^2(|\xi_{k,j}|\ge\varepsilon_k)=0$ by formula (1.5), and

$$\sum_{j=1}^{n_k}\left(E{\xi'_{k,j}}^2\right)^2\le\text{const.}\sum_{j=1}^{n_k}E{\xi'_{k,j}}^2\le\text{const.}$$

with an appropriate constant for all numbers $k\ge1$.

The proof of Proposition 2. As the canonical measures M_k converge to the canonical measure M_0 and $M_0(\{0\})=0$ for all numbers $\varepsilon>0$ there exist such numbers $\delta=\delta(\varepsilon)>0$, $R=R(\varepsilon)$ and threshold index $\bar n=\bar n(\varepsilon)$ for which

$$M_k((-\delta,\delta))<\varepsilon^3,\quad\text{and}\quad\int_{\{u\colon |u|>R\}}\frac{1}{u^2}M_k(\,du)<\varepsilon\quad\text{if } k\ge\bar n(\varepsilon).\tag{2.3}$$

We may also assume that the numbers $\pm\delta=\pm\delta(\varepsilon)$ and $\pm R=\pm R(\varepsilon)$ are points of continuity of the measure M_0.

Let us introduce the measures $\mu_k(\,dx)=\frac{M_k(dx)}{x^2}$, $k=1,2,\dots$ and $\mu_0(\,dx)=\frac{M_0(dx)}{x^2}$ on the real line. Let us choose such numbers $\delta=x_1<x_2<\cdots<x_s=R$ with an appropriate index s for which $\pm x_l$ are points of continuity of the measure M_0, $1\le l\le s$, and $\frac{\varepsilon^4}{2L}<x_l-x_{l-1}<\frac{\varepsilon^4}{L}$, $1<l\le s$ with $L=\sup\limits_{k\ge0}\mu_k\left((-R,-\delta)\cup(\delta,R)\right)$. Actually the above defined sequence $\delta=x_1<x_2<\cdots<x_s=R$ also depends on the number ε although we have not indicated this dependence.

Let us consider the sequence of numbers $\varepsilon_j=2^{-j}$, $j=1,2,\dots,$ We shall choose an appropriate sequence of numbers $n_j=n_j(\varepsilon_j)\ge\bar n(\varepsilon_j)$, $j=1,2,\dots$ and construct the random variables η_k and $\bar\eta_k$ with the help of the same sequence of numbers $\delta=x_1<x_2<\cdots<x_s=R$ considered in the previous paragraph (which depends on $\varepsilon=\varepsilon_j$) for all indices $n_j\le k\le n_{j+1}$. We shall show that in the case of a good choice of the sequence n_j and a good construction of the random variables η_k and $\bar\eta_k$, $k=1,2,\dots$ we can satisfy the statement Proposition 2.

We shall construct the random variables η_k and $\bar\eta_k$ with infinitely divisible distributions by first constructing two Poisson processes $\xi_{k,1},\xi_{k,2},\dots$ and $\bar\xi_{k,2},\bar\xi_{k,2},\dots$ with counting measures μ_k and μ_0 respectively. Then we define the random variables η_k and $\bar\eta_k$ as the regularized sums of these Poisson processes described in Part I. More explicitly, we put

$$\begin{aligned}
\eta_k(\omega)&=\lim_{N\to\infty}\left(\sum_{p\colon\, |\xi_{k,p}(\omega)|\ge 2^{-N}}\xi_{k,p}(\omega)-E\left(\sum_{p\colon\, |\xi_{k,p}(\omega)|\ge 2^{-N}}\tau(\xi_{k,p}(\omega))\right)\right),\\
\bar\eta_k(\omega)&=\lim_{N\to\infty}\left(\sum_{p\colon\, |\bar\xi_{k,p}(\omega)|\ge 2^{-N}}\bar\xi_{k,p}(\omega)-E\left(\sum_{p\colon\, |\bar\xi_{k,p}(\omega)|\ge 2^{-N}}\tau(\bar\xi_{k,p}(\omega))\right)\right),
\end{aligned}\tag{2.4}$$

where the function $\tau(x)=\tau_a(x)$ was defined in formula (1.2). In Part I we have seen that the limits in formula (2,4) exist with probability 1, and the random variables η_k and $\bar\eta_k$ they define have the prescribed distributions.

To construct the Poisson processes $\xi_{k,1},\xi_{k,2},\dots$ and $\bar\xi_{k,2},\bar\xi_{k,2},\dots$ with counting measures μ_k and μ_0 respectively first we construct some Poisson distributed random variables $\zeta^{\pm}_{k,l}$ and $\bar\zeta^{\pm}_{k,l}$, $1\le l<s$, from which the random variable $\zeta^{+}_{k,l}$ and $\bar\zeta^{+}_{k,l}$. $l=1,2,\dots$, tell us that the Poisson processes $\xi_{k,n}$ and $\bar\xi_{k,n}$, $n=1,2,\dots$, how many points have in the interval $[x_l,x_{l+1})$. Similarly, the random variables a $\zeta^{-}_{k,l}$ and $\bar\zeta^{-}_{k,l}$ tell that these Poisson processes how many points have in the intervals $[-x_{l+1},-x_l)$.

To construct the above random variables and $\zeta^{\pm}_{k,l}$ and $\bar\zeta^{\pm}_{k,l}$ let us first define independent Poisson random variables $\alpha^{\pm}_{k,l}$, $\beta^{\pm}_{k,l}$, $1\le l<s$, with Poisson distribution such that the distribution of $\alpha^{+}_{k,l}$ has parameter $\min(\mu_k((x_l,x_{l+1})),\mu((x_l,x_{l+1})))$ and the distribution of $\beta^{+}_{k,l}$ has parameter

$$\max(\mu_k((x_l,x_{l+1})),\mu_0((x_l,x_{l_1})))-\min(\mu_k((x_l,x_{l+1}),\mu_0(x_l,x_{l+1}))).$$

Similarly, let the distribution of $\alpha^{-}_{k,l}$ have parameter

$$\min(\mu_k((-x_{l+1},-x_l)),\mu_0((-x_{l+1},-x_l))),$$

and let the distribution of $\beta^{-}_{k,l}$ have parameter

$$\begin{aligned}
&\max(\mu_k((-x_{l+1},-x_l)),\mu_0((-x_{l+1},-x_l)))\\
&\qquad-\min(\mu_k((-x_{l+1},-x_l)),\mu_0((-x_{l+1},-x_l))).
\end{aligned}$$

If $\mu_k((x_l,x_{l+1}))\le\mu_0((x_l,x_{l+1}))$, then put $\zeta^{+}_{k,l}=\alpha^{+}_{k,l}$, $\bar\zeta^{+}_{k,l}=\alpha^{+}_{k,l}+\beta^{+}_{k,l}$, and if $\mu_k((x_l,x_{l+1}))>\mu_0((x_l,x_{l+1}))$, then put $\bar\zeta^{+}_{k,l}=\alpha^{+}_{k,l}$, $\zeta^{+}_{k,l}=\alpha^{+}_{k,l}+\beta^{+}_{k,l}$, $1\le l<s$. Let us define similarly the random variables $\zeta^{-}_{k,l}$ and $\bar\zeta^{-}_{k,l}$ only in this case we replace the interval (x_l,x_{l+1}) by the interval $(-x_{l+1},-x_l)$ and the random variables $\alpha^{+}_{k,l}$ and $\beta^{+}_{k,l}$ by the random variables $\alpha^{-}_{k,l}$ and $\beta^{-}_{k,l}$.

The random variables $\zeta^+_{k,l}$ and $\zeta^-_{k,l}$ are Poisson distributed, and their parameters are $\mu_k((x_l,x_{l+1}))$ and $\mu_k((-x_{l+1},-x_l))$. Similarly, the random variables $\bar\zeta^+_{k,l}$ and $\bar\zeta^-_{k,l}$ are Poisson distributed with parameters $\mu_0((x_l,x_{l+1}))$ and $\mu_0((-x_{l+1},-x_l))$. This means that the distributions of the above constructed random variables agree with the distributions of the number of points in the appropriate intervals of the Poisson processes to be constructed. This property makes possible the application of these random variables in the construction. Let us also observe that these random variables also satisfy the identity

$$\begin{aligned} E\left|\zeta^+_{k,l}-\bar\zeta^+_{k,l}\right| &= \left|E\left(\zeta^+_{k,l}-\bar\zeta^+_{k,l}\right)\right| = \left|\mu_k((x_l,x_{l+1}))-\mu_k((x_l,x_{l+1}))\right|, \\ E\left|\zeta^-_{k,l}-\bar\zeta^-_{k,l}\right| &= \left|E\left(\zeta^-_{k,l}-\bar\zeta^-_{k,l}\right)\right| = \left|\mu_k((-x_{l+1},-x_l))-\mu_k((-x_{l+1},-x_l))\right|. \end{aligned} \tag{2.5}$$

Now we turn to the construction of the Poisson processes $\xi_{k,n}$ and $\bar\xi_{k,n}$, $n=1,2,\dots$, with the help of the above constructed random variables $\zeta^\pm_{k,l}$ and $\bar\zeta^\pm_{k,l}$. Let us throw $\zeta^+_{k,l}$ number of points to the interval (x_l,x_{l+1}) and $\zeta^-_{k,l}$ number of points to the interval $(-x_{l+1},x_l)$ independently of each other so that these points fall into a set $A\subset(x_l,x_{l+1})$ or $A\subset(x_l,x_{l+1})$ with probability $\frac{\mu(A)}{\mu((x_l,x_{l+1}))}$ and $\frac{\mu(A)}{\mu((x_{-l+1},-x_l))}$, $1\le l\le s$, respectively. Similarly, let us consider the Poisson distributed random variable $\zeta_{k,0}$ with parameter $\mu_k(-x_1,x_1)$ and the Poisson distributed random variable $\zeta^+_{k,s}+\zeta^-_{k,s}$ with parameter $\mu_k((-\infty,-x_s)\cup(x_s,\infty))$, and let us throw $\zeta_{k,0}$ number of points to the interval $(-x_1,x_1)$ so that a point falls into a set $A\subset(-x_1,x_1)$ with probability $\frac{\mu_k(A)}{\mu_k((-x_1,x_1))}$, and let us throw $\zeta^+_{k,s}+\zeta^-_{k,s}$ number of points to the set $(-\infty,-x_s)\cup(x_s,\infty)$ so that a point falls into a set $A\subset(-\infty,-x_s)\cup(x_s,\infty)$ with probability $\frac{\mu_k(A)}{\mu_k((-\infty,-x_s)\cup(x_s,\infty))}$. Let the above considered random variables $\zeta_{k,l}$ together with all point throws made in the above construction be independent of each other. Then the union of the points thrown to different intervals is a Poisson process $\xi_{k,1},\xi_{k,2},\dots$ with counting measure μ_k. We can construct similarly a Poisson process $\bar\xi_{k,1},\bar\xi_{k,2},\dots$ with counting measure μ_0 on the real line. Only in this case we replace the Poisson distributed random variables $\zeta^\pm_{k,l}$, $0\le l\le s$, by the Poisson distributed random variables $\bar\zeta^\pm_{k,l}$, $0\le l\le s$, whose parameters can be given similarly, only the measure μ_k is replaced by the measure μ_0.

In such a way we have constructed the underlying Poisson processes and the random variables η_k and $\bar\eta_k$ determined by them. (Only the threshold index $n_j=n_j(\varepsilon_j)$ is still not fixed.) We want to show that the above construction satisfies the stochastic convergence $\eta_k-\bar\eta_k\Rightarrow 0$. The proof of this statement will be based on the observations that the underlying Poisson processes have almost the same number of points in the intervals (x_{l-1},x_l), (the relation (2.5) expresses such a fact). Beside this, these intervals are very small, hence the precise position of the points falling to them has a very small influence on the value of the random variables η_k and $\bar\eta_k$. To simplify further notations let us denote by B_l the interval (x_l,x_{l+1}) if $1\le l<s$ and the interval (x_{l-1},x_l) if $-1\ge l>-s$.

First we consider the contribution of those points of the Poisson processes to the random variables η_k and $\bar\eta_k$ which take a large value, more explicitly whose absolute values are larger than R with the number R introduced in formula (2.3). If $k\ge n_j\ge \bar n(\varepsilon_j)$, then the measure M_k satisfies relation (2.3) with $\varepsilon_j=2^{-j}$. This relation also holds if the measures M_k are replaced by the limit measure M_0. By the second relation of formula (2.3) the parameters of the Poisson distributed random variables $\zeta_{k,s}=\zeta^+_{k,s}+\zeta^-_{k,s}$ and $\bar\zeta_{k,s}=\bar\zeta^+_{k,s}+\bar\zeta^-_{k,s}$ are less than 2^{-j}. Hence the probability of the event that the corresponding Poisson processes $\xi_{k,n}$ and $\bar\xi_{k,n}$, $n=1,2,\dots$, contain no point such that $|\xi_{k,n}|>R$ or $|\bar\xi_{k,n}|>R$ is greater than $1-2\cdot 2^{-j}$. Furthermore, the expected number of the points of the Poisson processes with absolute value larger than R is less than $2\cdot 2^{-j}$, and $|\tau(x)|\le a$ for all points $x\in R^1$. The above relations imply that

$$
\begin{aligned}
&P\left(\left|\sum_{\{n\colon\, |\xi_{k,n}|\ge R\}}\xi_{k,n}-E\left(\sum_{\{n\colon\, |\xi_{k,n}|\ge R\}}\tau(\xi_{k,n})\right)\right|\le 2a\cdot 2^{-j}\right)<2\cdot 2^{-j}\quad\text{if } k\ge n_j,\\
&P\left(\left|\sum_{\{n\colon\, |\bar\xi_{k,n}|\ge R\}}\bar\xi_{k,n}-E\left(\sum_{\{n\colon\, |\bar\xi_{k,n}|\ge R\}}\tau(\bar\xi_{k,n})\right)\right|\le 2a\cdot 2^{-j}\right)<2\cdot 2^{-j}\quad\text{if } k\ge n_j.
\end{aligned}\tag{2.6}
$$

For a Poisson process ξ_n, $n=1,2,\dots$, in the interval $[a,b]$ with a finite counting measure μ the variance of the random sum $\sum\limits_n\xi_n$ equals $\int_a^b u^2\mu(\,du)$, (see e.g. Lemma 1 in Part I.) Hence the Chebishev inequality and the first part formula (2.3) imply that for all sufficiently large $\delta>0$ (observe that $\tau(x)=\tau_a(x)=x$ if $|x|\le\delta$.)

$$
\begin{aligned}
&P\left(\left|\sum_{\{l\colon\, 2^{-N}\le|\xi_{k,n}|\le\delta\}}\xi_{k,n}-E\left(\sum_{\{l\colon\, 2^{-N}\le|\xi_{k,n}|\le\delta\}}\tau(\xi_{k,n})\right)\right|\ge 2^{-j}\right)\\
&\qquad\le 2^{2j}\operatorname{Var}\left(\sum_{\{l\colon\, 2^{-N}\le|\xi_{k,p}|\le\delta\}}\xi_{k,n}\right)\le 2^{2j}\varepsilon_j^3=2^{-j}\\
&P\left(\left|\sum_{\{l\colon\, 2^{-N}\le|\bar\xi_{k,n}|\le\delta\}}\bar\xi_{k,n}-E\left(\sum_{\{l\colon\, 2^{-N}\le|\bar\xi_{k,n}|\le\delta\}}\tau(\bar\xi_{k,n})\right)\right|\ge 2^{-j}\right)\\
&\qquad\le 2^{2j}\operatorname{Var}\left(\sum_{\{n\colon\, 2^{-N}\le|\bar\xi_{k,n}|\le\delta\}}\bar\xi_{k,n}\right)\le 2^{2j}\varepsilon_j^3=2^{-j}\\
&\qquad\qquad\text{if } k\ge n_j \text{ and } 2^{-N}<\delta.
\end{aligned}\tag{2.7}
$$

Further we claim that if the indices n_j are chosen sufficiently large, then

$$
\begin{aligned}
&P\Biggl(\sum_{l\colon\, 1\le|l|<s}\left(\sum_{n\colon\, \xi_{k,n}\in B_l}\xi_{k,n}-E\left(\sum_{n\colon\, \xi_{k,p}\in B_l}\tau(\xi_{k,n})\right)\right)\\
&\qquad-\left(\sum_{n\colon\, \xi_{k,n}\in B_l}\bar\xi_{k,n}-E\left(\sum_{p\colon\, \xi_{k,n}\in B_l}\tau(\bar\xi_{k,n})\right)\right)>2^{-j}\Biggr)<2^{-j}\quad\text{if } k\ge n_j
\end{aligned}\tag{2.8}
$$

First we show with the help of formulas (2.4), (2.6), (2.7) and (2.8) that $\eta_k - \bar\eta_k \Rightarrow 0$. Indeed, by summing up formulas (2.6), (2.7) and (2.8) we get that of all integers N such that $2^{-N} < \delta$

$$P\left(\left(\sum_{n\colon |\xi_{k,n}|\ge 2^{-N}} \xi_{k,n} - E\left(\sum_{n\colon |\xi_{k,n}|\ge 2^{-N}} \tau(\xi_{k,n}) \right)\right) - \left(\sum_{n\colon |\bar\xi_{k,n}|\ge 2^{-N}} \bar\xi_{k,n} - E\left(\sum_{n\colon |\xi_{k,n}|\ge 2^{-N}} \tau(\bar\xi_{k,n}) \right)\right) > (4a+3)\cdot 2^{-j} \right) < 5\cdot 2^{-j} \tag{2.9}$$

if $k \ge n_j$.

Let us consider the lim inf of the events whose probabilities were investigated in formula (2.9) in the variable N. (We recall that $\liminf\limits_{N\to\infty} A_N = \bigcup\limits_{N=1}^{\infty} \left(\bigcap\limits_{L=N}^{\infty} A_L \right)$.) By applying formula (2.4) we get that $P\left(|\eta_k - \bar\eta_k| > (4a+3)\cdot 2^{-j}\right) < 5\cdot 2^{-j}$ if $k \ge n_j$. Hence formula(2.8) implies the relation $\eta_k - \bar\eta_k \Rightarrow 0$ as we claimed. To give the still missing proof of formula (2.8) first we show that

$$\begin{aligned} &P\left(\sum_{l\colon 1\le |l|<s} \left(\sum_{n\colon \xi_{k,n}\in B_l} \xi_{k,n} - E\left(\sum_{n\colon \xi_{k,n}\in B_l} \tau(\xi_{k,n}) \right) - \zeta_{k,l}x_l + \tau(x_l)\mu_k(B_l) \right) > 2^{-2j} \right) \\ &\qquad\qquad < 2\cdot 2^{-2j} \quad \text{and} \\ &P\left(\sum_{l\colon 1\le |l|<s} \left(\sum_{n\colon \bar\xi_{k,n}\in B_l} \bar\xi_{k,n} - E\left(\sum_{n\colon \xi_{k,n}\in B_l} \tau(\bar\xi_{k,n}) \right) - \bar\zeta_{k,l}x_l + \tau(x_l)\mu_0(B_l) \right) > 2^{-2j} \right) \\ &\qquad\qquad < 2\cdot 2^{-2j} \end{aligned} \tag{2.10}$$

if $k \ge n_j$.

The first inequality of formula (2.10) bounds the error we commit if those points of the Poisson processes $\xi_{k,n}$, $n = 1, 2, \dots$, for which $\xi_{k,n} \in B_l$ are replaced by the end-point x_l of the interval B_l, and sum up these errors for all such points of the Poisson process for which $\delta \le |\xi_{k,l}| \le R$. As

$$|\xi_{k,n} - x_l| \le \sup_{1\le |l|\le s} |x_{l+1} - x_l| \le \frac{\varepsilon_j^4}{L},$$

$$\left| E\left(\sum_{n\colon \xi_{k,n}\in B_l} \tau(\xi^{(l)}_{k,n}) \right) - \tau(x_l)\mu_k(B_l) \right| \le \sup_{1\le |l|\le s} |x_{l+1} - x_l| \mu_k(B_l) \le \frac{\varepsilon_j^4}{L} \mu_k(B_l)$$

if $k\ge n_j$, the first inequality of relation (2.10) follows from the relations

$$\begin{aligned}\sum_{l\colon 1\le|l|<s}&\left|E\left(\sum_{n\colon \xi_{k,n}\in B_l}\tau(\xi_{k,n}^{(l)})\right)-\tau(x_l)\mu_k(B_l)\right|\\ &\le\sum_{l\colon 1\le|l|<s}\frac{\varepsilon_j^4}{L}\mu_k(B_l)=\varepsilon_j^4\frac{\mu_k((-R,R)\setminus(-\delta,\delta))}{L}\le\varepsilon_j^4\end{aligned}$$

and

$$\begin{aligned}P&\left(\sum_{l\colon 1\le|l|<s}\left|\sum_{n\colon \xi_{k,n}\in B_l}\xi_{k,n}-\zeta_{k,l}x_l\right|>\varepsilon_j^2\right)\\ &\le P\left(\#\{n\colon\ \delta<|\xi_{k,n}|<R\}>\frac{L}{2\varepsilon_j^2}\right)\le\frac{2\varepsilon_j^2E\left(\#\{n\colon\ \delta<|\xi_{k,n}|<R\}\right)}{L}\\ &=\frac{2\varepsilon_j^2\mu_k\left((-R,-\delta)\cup(\delta,R)\right)}{2L}\le 2\varepsilon_j^2=2\cdot 2^{-2j}\quad\text{if } k\ge n_j.\end{aligned}$$

The second inequality of formula (2.10) can be proved similarly, only in this case we have to consider the Poisson process $\bar\xi_{k,n}$, $n=1,2,\dots$, instead of the Poisson process $\xi_{k,n}$, $n=1,2,\dots$, and the measure μ_k has to be replaced by the measure μ_0.

By formula (2.10) to complete the proof of formula (2.8) hence of Proposition 2 it is enough to show that if the threshold indices $n_j\ge\bar n_j$ are chosen sufficiently large, then

$$P\left(\sum_{l\colon 1\le|l|<s}\left|x_l(\zeta_{k,l}-\bar\zeta_{k,l})-\tau(x_l)(\mu_k(B_l)-\mu_0(B_l))\right|>2^{-2j}\right)\le\frac{2^{-j}}{2}\quad\text{if } k\ge n_j \tag{2.11}$$

or to prove the following slightly stronger inequality:

$$\sum_{l\colon 1\le|l|<s}\left(|x_l|\,E|\zeta_{k,l}-\bar\zeta_{k,l}|+|\tau(x_l)|\,|\mu_k(B_l)-\mu_0(B_l)|\right)<2^{-4j}\quad\text{if } k\ge n_j. \tag{2.12}$$

But formula (2.5) implies that $E|\zeta_{k,l}-\bar\zeta_{k,l}|=C_j|\mu_k(B_l)-\mu_0(B_l)|$. Beside this, the identities $\lim\limits_{k\to\infty}\mu_k(B_l)=\mu_0(B_l)$ hold because of the (weak) convergence of the canonical measures M_k and M_0. As the sum in formula (2.12) contains only finitely many terms (the number of terms depends only on the index j), and each term tends to zero as $k\to\infty$ by the above observations the expression at the left-hand side of formula (2.12) can be made an arbitrary small positive number by choosing the threshold index n_j sufficiently large.

3. The functional limit theorem. The proof of Theorem 2.

First we shall show with the help of Lemma 1 and Proposition 1 that also the stochastic processes $S_k(t)$, $0\le t\le 1$, appearing in Theorem 2 can be split to two parts, one of them responsible for the convergence of the Gaussian and one of them responsible for the convergence of the Poissonian part. The convergence of the Gaussian part can be deduced from the functional central limit theorem and the convergence of the Poissonian part can be reduced to a simpler statement about the convergence of infinitely divisible processes. This will be the content of Part A in Section 3. In Part B we prove the convergence of the Poissonian part with the help of two Propositions. The first of them, Proposition 3, enables us to discretize the time parameter of the stochastic processes we investigate. Proposition 4 gives a good coupling of infinitely divisible processes whose canonical measures (on the strip $\mathbf{R}^1\times[0,1]$) are close to each other. It can be considered as a generalization of Theorem 2 where the coupling of infinitely divisible processes is considered instead of the coupling of infinitely divisible random variables. Finally in Part C we prove Propositions 3 and 4, and this completes the proof of Theorem 2.

A.) Poisson approximation. Separation of the normal and Poisson part of the limit process.

We reduce the proof of Theorem 2 to that of a simpler statement with the help of Lemma 1, Proposition 1 and the coupling construction described after Lemma 1.

Let us consider the random variables $\xi'_{k,j}$, $\xi''_{k,j}$, $\eta'_{k,j}$, $\eta''_{k,j}$, $\tilde\xi_{k,j}=\xi'_{k,j}+\xi''_{k,j}$ and $\eta_{k,j}=\eta'_{k,j}+\eta''_{k,j}$, $1\le j\le n_k$, introduced in the coupling construction described after Lemma 1 together with the partial sums $S'_{k,l}=\sum\limits_{j=1}^{l}\xi'_{k,j}$, $S''_{k,l}=\sum\limits_{j=1}^{l}\xi''_{k,j}$, $\tilde S_{k,l}=S'_{k,l}+S''_{k,l}$, $T'_{k,l}=\sum\limits_{j=1}^{l}(\eta'_{k,j}-E\eta'_{k,j})$, $T''_{k,l}=\sum\limits_{j=1}^{l}(\eta''_{k,j}-E\tau(\eta''_{k,j}))$, $T_{k,l}=T'_{k,l}+T''_{k,l}$, $1\le l\le n_k$. Let us also introduce the stochastic processes $S'_k(t)$, $S''_k(t)$, $\tilde S_k(t)$, $T'_k(t)$, $T''_k(t)$ and $T_k(t)$, $0\le t\le 1$, with cadlag function trajectories which we define similarly to the stochastic process $S(t)$, $0\le t\le 1$, introduced in formula (1.12) with the difference that we replace the random variables $S_{k,l}$ in formula (1.12) by the random variables $S'_{k,l}$, $S''_{k,l}$, $\tilde S_{k,l}$ and $T'_{k,l}$, $T''_{k,l}$, $T_{k,l}$. It follows from formula (1.8) and the identity $E\xi'_{k,j}+E\tau(\xi''_{k,j})=E\tau(\tilde\xi_{k,j})=0$ that

$$\sup_{0\le t\le 1}|T_k(t)-\tilde S_k(t)|\Rightarrow 0,\quad \text{if } k\to\infty, \tag{3.1}$$

where $\Rightarrow$ denotes stochastic convergence. The distributions of the stochastic processes $S_k(t)$ and $\tilde S_k(t)$ agree. Furthermore, if $x_k(\cdot)$ and $y_k(\cdot)$ are such functions in the space $D([0,1])$ for which $\lim\limits_{k\to\infty}\sup\limits_{0\le t\le 1}|x_k(t)-y_k(t)|=0$ then also the relation $\lim\limits_{k\to\infty}d(x_k(\cdot),y_k(\cdot))=0$ holds with the metric $d(\cdot,\cdot)$ introduced at page 10 to metrize the space $D([0,1])$. Thus by Theorem A and formula (3.1) to prove Theorem 2 it is enough to show that the stochastic processes $T_k(t)$, $0\le t\le 1$, converge weakly to the limit stochastic process described in Theorem 2.

The identity $T_k(t)=T'_k(t)+T''_k(t)$ holds, and the stochastic processes $T'_k(t)$ and $T''_k(t)$, $1\le t\le 1$, appearing in this formula are independent. We shall show that the stochastic processes $T'_k(\cdot)$ converge to the Gaussian and the stochastic processes $T''_k(\cdot)$ converge to the Poisson component of the limit process as $k\to\infty$. To prove these statements first we have to clarify how the convergence of the canonical measures N_k on the strip $\mathbf{R}^1\times[0,1]$ to a canonical measure N_0 is reflected in the behaviour of the canonical measures corresponding to the stochastic processes $T'_k(\cdot)$ and $T''_k(\cdot)$.

Let us consider a sequence of positive numbers ε_k with which Lemma 1 holds. Let us define, by using the notation of Lemma 1, the measures N'_k on the interval $[0,1]$ concentrated in the points $0\le u_{k,1}\le u_{k,2}\le\cdots\le u_{k,n_k}=1$ introduced in the formulation of Theorem 2, for which $N'_k(u_{k,l})=G_{k,l}(\varepsilon_k)-G_{k,l}(-\varepsilon_k)$, $k=1,2,\dots$, where $G_{k,l}(\,dx)=x^2F_{k,l}(\,dx)$, similarly to the formulation of Lemma 1. Let us also define the canonical measures N''_k on the strip $\mathbf{R}^1\times[0,1]$ which are concentrated on the union of the lines $\mathbf{R}^1\times u_{k,l}$, $1\le l\le n_k$, and

$$N''_k(B\times\{u_{k,l}\})=\int_{B\cap\{u\colon\,|u|\ge\varepsilon_k\}}u^2F_{k,l}(\,du),\quad 1\le l\le n_k,\quad k=1,2,\dots.$$

We claim that under the conditions of Theorem 2 the measures N'_k weakly converge to the measure N'_0 defined by the relation $N'_0(B)=N_0(\{0\}\times B)$ if $B\subset[0,1]$. Beside this, the canonical measures N''_k on the strip $\mathbf{R}^1\times[0,1]$ converge weakly to the canonical measure N''_0 defined by the relation $N''_0(B)=N_0(B\setminus(\{0\}\times[0,1]))$ if $B\subset\mathbf{R}^1\times[0,1]$.

To prove the above statements let us observe that if $B\subset[0,1]$ is a set with boundary ∂B such that $\lambda(\partial B)=0$ and $C\subset\mathbf{R}^1$ is a bounded set such that $M_0(\partial B)=0$, then under the conditions of Theorem 2 $\lim\limits_{k\to\infty}N_k(C\times B)=N_0(C\times B)$. Furthermore, $\lim\limits_{k\to\infty}N'_k([0,1])=M(\{0\})$ by Lemma 1. First we show that for an arbitrary set $B\subset[0,1]$ whose boundary satisfies the relation $\lambda(\partial B)=0$, $\limsup\limits_{k\to\infty}N'_k(B)\le N_0(\{0\}\times B)=N'_0(B)$. Indeed, for all numbers $\delta>0$ there exists an interval $[-\eta,\eta]$ with some $\eta>0$ such that $\pm\eta$ is a point of continuity of the measure M and $M_0([-\eta,\eta])\le M_0(\{0\})+\delta$, and this implies that $N_0([-\eta,\eta]\times B)\le N_0(\{0\}\times B)+\delta$. Since $\varepsilon_k\to0$ if $k\to\infty$, it follows from the above facts that $\limsup\limits_{k\to\infty}N'_k(B)\le\lim\limits_{k\to\infty}N_k(B\times[-\eta,\eta])=N_0([-\eta,\eta]\times B)\le N_0(\{0\}\times B)+\delta$. As this relation holds for all numbers $\delta>0$, hence $\limsup\limits_{k\to\infty}N'_k(B)\le N'_0(B)$. By applying this inequality for both sets B and $[0,1]\setminus B$ we get that

$$\begin{aligned}N'_0([0,1])=&M_0(\{0\})=\lim_{k\to\infty}N'_k([0,1])\le\limsup_{k\to\infty}N'_k(B)+\limsup_{k\to\infty}N'_k([0,1]\setminus B)\\ &\le N'_0(B)+N'_0([0,1]\setminus B))=N'_0([0,1]).\end{aligned}$$

This series of inequalities may be valid only if also the identity $\lim\limits_{k\to\infty}N'_k(B)=N'_0(B)$ holds.

Let us also introduce the canonical measures $\bar N'_k$, $k=1,2,\dots$ and $\bar N'_0$ on the strip $\mathbf{R}^1\times[0,1]$ with the help of the formulas $\bar N'_k(A)=N_k(A\cap[-\varepsilon_k,\varepsilon_k])$, $\bar N'_0(A)=$

$N_0(A \cap \{0\} \times [0,1])$, $A \subset \mathbf{R}^1 \cap [0,1]$. (In these formulas we lifted the measures N'_k, $k = 1,2,\dots$, to the strip $\mathbf{R}^1 \times [0,1]$.) Then the convergence of the measures N'_k to the measure N'_0 implies the convergence of the canonical measures $\bar N'_k$ to the canonical measure $\bar N'_0$. Furthermore, $N''_k = N_k - \bar N'_k$, $k = 1,2,\dots$, and the convergence of the canonical measures N''_k on the strip $\mathbf{R}^1 \times [0,1]$ follows from the facts that the canonical measures N_k converge to the canonical measure N_0 and the canonical measures converge to the canonical measure $\bar N'_k$.

Let us define the random variables $T'_{k,l} = \sum\limits_{j=1}^{l} \eta'_{k,j}$, $1 \le l \le n_k$, $T'_{k,0} = 0$ and the stochastic processes $T'_k(t)$, $T'_k(t) = T'_{k,l}$ if $u_{k,l-1} \le t < u_{k,l}$, $T_{k,l}(1) = T'_{k,n_k}$. Let us also consider the continuous function $\lambda(t) = N_0(\{0\} \times [0,t])$, $0 \le t \le 1$ introduced in Part a) of Theorem 2. We claim that the stochastic processes $T'_k(t)$, $0 \le t \le 1$, weakly converge to the stochastic process $W(\lambda(t))$, $0 \le t \le 1$, as $k \to \infty$, where $W(t)$, $0 \le t \le M(\{0\})$, is a standard Wiener process.

To prove the above statement let us introduce the numbers $\bar u_{k,0} = 0$, $\bar u_{k,l} = \frac{1}{U_k} \sum\limits_{j=1}^{l} \operatorname{Var} \eta'_{k,j}$, $1 \le l \le n_k$, $k = 1,2,\dots$, where $U_k = \sum\limits_{j=1}^{n_k} \operatorname{Var} \eta'_{k,j}$, together with the stochastic processes $\bar T'_k(\cdot)$ which will be defined similarly to the stochastic processes $T'_k(\cdot)$ with the only difference that the numbers $u_{k,l}$ are replaced by the numbers $\bar u_{k,l}$ in the definition. Then we can state on the basis of the functional central limit theorem that the stochastic processes $\bar T'_k(t)$, $0 \le t \le 1$, converge weakly to a standard Wiener process $W(t)$ as $k \to \infty$.

Let us define the monotone increasing and continuous functions $\lambda_k(t)$, $k = 1,2,\dots$, so that $\lambda_k(u_{k,l}) = \bar u_{k,l}$, $0 \le l \le k_n$, and the function $\lambda_k(\cdot)$ is linear in the intervals $[u_{k,l-1}, u_{k,l}]$, $1 \le k \le n_k$. We shall show that

$$\lim_{k\to\infty} U_k = M_0(\{0\}), \qquad \lim_{k\to\infty} \sup_{0\le t\le 1} \left| \lambda_k(t) - \frac{\lambda(t)}{M_0(\{0\})} \right| = 0, \tag{3.2}$$

and this implies that the stochastic processes $T'_k(t)$, $0 \le t \le 1$ weakly converge to the stochastic process $\sqrt{M_0(\{0\})}\, W\left(\frac{\lambda(t)}{M_0(\{0\})}\right)$ whose distribution agrees with the distribution of the stochastic process $W(\lambda(t))$, $0 \le t \le 1$.

Indeed, Lemma 1 and formula (2.2) imply that

$$\lim_{k\to\infty} U_k = \lim_{k\to\infty} \sum_{j=1}^{n_k} E{\xi'_{k,j}}^2 = M(\{0\}).$$

We get similarly that $\lim\limits_{k\to\infty} \sup\limits_{1\le l\le n_k} \left| \bar u_{k,l} U_k - \sum\limits_{j=1}^{l} E{\xi'_{k,j}}^2 \right| = 0$, that is

$$\lim_{k\to\infty} \sup_{1\le l\le n_k} |\bar u_{k,l} M_0(\{0\}) - N_k([-\varepsilon_k, \varepsilon_k]) \times [0, u_{k,l}])| = 0.$$

The monotone functions $h_k(t)=N_k([-\varepsilon_k,\varepsilon_k])\times[0,t])$ converge to the monotone, continuous function $\lambda(t)=N_0(\{0\}\times[0,t])$ for all numbers $0\le t\le 1$, and because the above functions are monotone, and the limit function is continuous the above convergence is uniform. Hence $\lim\limits_{k\to\infty}\sup\limits_{1\le l\le n_k}|\bar u_{k,l}M_0(\{0\})-\lambda(u_{k,l})|=0$. Thus

$$\lim_{k\to\infty}\sup_{1\le l\le n_k}\left|\lambda_k(u_{k,l})-\frac{\lambda(u_{k,l})}{M_0(\{0\})}\right|=0.$$

This implies relation (3.2).

To prove the weak convergence of the stochastic processes $T'_k(t)=\frac{1}{\sqrt{U_k}}\bar T'_k(\lambda_k(t))$ to the stochastic process $W(\lambda(t))$ it is enough to show that

$$\sqrt{U_k}\bar T'_k(\lambda_k(t))-\sqrt{U_k}\bar T'_k\left(\frac{\lambda(t)}{M_0(0)}\right)\Rightarrow 0, \tag{3.3}$$

where $\Rightarrow$ denotes stochastic convergence. Indeed, Theorem A and relation (3.3) imply the desired statement, since the stochastic processes $\sqrt{U_k}\bar T'_k\left(\frac{\lambda(t)}{M_0(0)}\right)$ converge weakly to the stochastic process $W(\lambda(t))$. On the other hand, relation (3.3) follows from relation (3.2) and the result of the general theory by which the weak convergence of the stochastic processes $\sqrt{U_k}\bar T'_k\left(\frac{\lambda(t)}{M_0(0)}\right)$ to a stochastic process with continuous trajectories follows that the distributions of these processes are uniformly tight, i.e. for all numbers $\varepsilon>0$ and $\eta>0$ there exists a number $\delta=\delta(\varepsilon,\eta)>0$ such that

$$P\left(\sup_{(s,t)\colon\,|t-s|\le\delta}\left|\sqrt{U_k}\bar T'_k\left(\frac{\lambda(t)}{M_0(0)}\right)-\sqrt{U_k}\bar T'_k\left(\frac{\lambda(s)}{M_0(0)}\right)\right|>\eta\right)\le\varepsilon$$

for all indices $k=1,2,\dots$. (The number $\delta=\delta(\varepsilon,\eta)$ does not depend on the index k.)

B.) The method of the proof. The study of the convergence of the Poissonian part.

In Part A of Section 3 we defined the representations $T_k(t)=T'_k(t)+T''_k(t)$ of the stochastic processes $T_k(\cdot)$, $1\le t\le 1$, $k=1,2,\dots$, introduced there and showed that Theorem 2 follows from the convergence of the distributions of the stochastic processes $T_k(t)$, $k=1,2,\dots$, to the process $S(t)$, defined in the formulation of Theorem 2, in the space $D([0,1])$. Beside this, the stochastic processes $T'_k(t)$ and $T''_k(t)$ are independent, and the stochastic processes $T'_k(t)$ converge weakly to a Gaussian process with independent increments whose distribution can be described similarly to the limit process given in Theorem 2 with the difference that the measure $N_0(\cdot,\cdot)$ on the strip $\mathbf{R}^1\times[0,1])$ is replaced by the measure $N'_0(A)=N_0(A\cap\{0\})$ in formula (1.15), or more explicitly in the definition of the measure $M_{0,u,v}$ appearing in this formula. Hence to complete the proof of Theorem 2 it is enough to show that the stochastic processes $T''_k(t)$ converge weakly to a stochastic process $S''_0(t)$, $0\le t\le 1$, with independent increments described by the canonical measure $N''_0(\cdot,\cdot)$ on the strip $\mathbf{R}^1\times[0,1]$, given by the formula

$N_0''(A) = N_0(A \cap (\mathbf{R}^1 \setminus \{0\}) \times [0,1])$. To simplify further discussions let us introduce the following definition.

The definition of the (time)–discretization of a stochastic process. *Let $Z(t)$, $0 \le t \le 1$, be a stochastic process on the interval $[0,1]$, and let $0 = t_0 < t_1 < t_2 < \cdots < t_s = 1$, be a monotone sequence in the interval $[0,1]$. Then the discretization of the stochastic process $Z(t)$, $0 \le t \le 1$, determined by the sequence of numbers $0 = t_0 < t_1 < t_2 < \cdots < t_s = 1$ is the stochastic process $\bar Z(t) = \bar Z_{t_0,t_1,\dots,t_s}(t)$, $0 \le t \le 1$, given by the formula*

$$\bar Z(t) = \bar Z_{t_0,t_1,\dots,t_s}(t) = Z(t_{l-1}), \quad \textit{if } t_{l-1} \le t < t_l,\ 1 \le l \le s, \quad \bar Z(1) = Z(1).$$

Let us observe that the stochastic processes $T_k''(t)$ are discretizations (in the points $0 = u_{0,k} \le u_{1,k} \le \cdots \le u_{k,n_k} = 1$,) of infinitely divisible stochastic processes determined by Poisson processes with such counting measures $\nu_k(dx,dy) = \frac{N_k(dx,dy)}{x^2}$ for which the canonical measures N_k on the strip $\mathbf{R}^1 \times [0,1]$ converge weakly to a canonical measure N_0'', and the identity $N_k(\{0\} \cap [0,1]) = 0$ holds. Hence we complete the proof of Theorem 2 if we prove the *Statement* formulated below. It can be considered as a generalization of Proposition 2 to random variables taking values in more general function spaces.

Statement. *Let N_k, $k = 0,1,2,\dots$, be a sequence of canonical measures on the strip $\mathbf{R}^1 \times [0,1]$, and assume that these canonical measures N_k converge weakly to a canonical measure N_0 if $k \to \infty$, they satisfy condition b.) of Theorem 2, and $N_k(\{0\} \times [0,1]) = 0$, $k = 0,1,2,\dots$. (This latter condition means that neither the processes determined by the measures N_k, $k = 1,2,\dots$, nor the process determined by the limit measure N_0 have a Gaussian component.) Let us define the canonical measures $\nu_k(dx,dy) = \frac{N_k(dx,dy)}{y^2}$, $k = 0,1,2,\dots$, and consider Poisson fields $X_n(k) = (X_n^{(1)}(k), X_n^{(2)}(k))$, $X_n(k) \in \mathbf{R}^1 \times [0,1]$, $n = 1,2,\dots$, $k = 1,2,\dots$, on the strip $\mathbf{R}^1 \times [0,1]$ with canonical measures $\nu_k(dx,dy)$. Let us consider the infinitely divisible stochastic processes $T_k(t)$, $0 \le t \le 1$, $k = 0,1,2,\dots$, determined by these stochastic fields which can be considered as $D([0,1])$ space valued random variables. The distributions of the stochastic processes $T_k(t)$, $k = 1,2,\dots$, converge to the distribution of the stochastic process $T_0(t)$ in the space $D([0,1])$ as $k \to \infty$.*

Beside this, let us have for all numbers $k = 1,2,\dots$ a partition $0 = u_{k,0} < u_{k,1} < \cdots < u_{k,n_k} = 1$ of the interval $[0,1]$, which satisfy the condition $\sup\limits_{1 \le j \le n_k} |u_{k,j} - u_{k,j-1}| = 0$, and let us consider the discretizations $\bar T_k(t) = \bar T_{k,u_{k,0},u_{k,1},\dots,u_{k,n_k}}(t)$ of the infinitely divisible processes $T_k(t)$. The distributions of these discretizations $\bar T_k(t)$ of the stochastic processes $T_k(t)$ also converge weakly to the distribution of the stochastic process $T_0(t)$ in the space $D([0,1])$.

The missing part of Theorem 2 agrees with the second part of the *Statement* about the convergence of the stochastic processes $\bar T_{k,u_{k,0},u_{k,1},\dots,u_{k,n_k}}(\cdot)$ to the stochastic process

$T_0(\cdot)$ if the same sequences of numbers $0 = u_{k,0} < u_{k,1} < \cdots < u_{k,n_k} = 1$ are considered as in Theorem 2.

Now we formulate two propositions make some comments about them and give the proof of the *Statement* with their help. These propositions will be proved in Part C.. Before their formulation let us recall how an infinitely divisible stochastic process (with nice trajectories in the space $D([0,1])$ can be constructed by means of a Poissonian field with a counting measure which has some nice properties.

Let $X_n = (X_n^{(1)}, X_n^{(2)})$, $X_n \in \mathbf{R}^1 \times [0,1]$, $n = 1, 2, \dots$, be a Poisson field with such a counting measure ν on the strip $\mathbf{R}^1 \times [0,1]$ for which $\nu(\mathbf{R}^1 \setminus [-b,b] \times [0,1]) < \infty$ and $\int_{(x,y)\colon |x| \le b} x^2 \nu(\,dx, dy) < \infty$ for all numbers $b > 0$. Then this Poisson field determines an infinitely divisible stochastic process $T(t)$, $0 \le t \le 1$, defined in the following way: Let us choose an appropriate sequence of numbers A_L, $L = 1, 2, \dots$, $\lim\limits_{L\to\infty} A_L = 0$, and put

$$T^{(L)}(t) = \sum_{n\colon |X_n^{(1)}| > A_L,\, 0 \le X_n^{(2)} \le t} X_n^{(1)} - E\left(\sum_{n\colon |X_n^{(1)}| > A_L,\, 0 \le X_n^{(2)} \le t} \tau\left(X_n^{(1)}\right) \right), \quad 0 \le t \le 1,$$

$L = 1, 2, \dots$, where the function $\tau(\cdot)$ was defined in formula (1.2). It is proven in Lemma 2 in Section 5 of Part I that if the sequence A_L tends to zero sufficiently fast, then the limit $T(t) = \lim\limits_{L\to\infty} T^{(L)}(t)$, $0 \le t \le 1$, exists with probability 1, where the limit is taken in supremum norm in the interval $[0,1]$. Beside this, the trajectories of the so constructed stochastic process $T(t)$ is an infinitely divisible stochastic process with cadlag trajectories. If N_0 is a canonical measure on the strip $\mathbf{R}^1 \times [0,1]$, then the distribution of the increments of the above defined stochastic process $T(t)$, $0 \le t \le 1$, determined by a Poisson field with counting measure $\nu_0(dx, dy) = \frac{N_0(dx,dy)}{x^2}$ is described by formula (1.15). We shall call the stochastic process $T(t)$, $0 \le t \le 1$, constructed in the above way the infinitely divisible process determined by the Poisson field $X_n = (X_n^{(1)}, X_n^{(2)})$, $X_n \in \mathbf{R}^1 \times [0,1]$, $n = 1, 2, \dots$.

Proposition 3. *Let a canonical measure N_0 be given on the strip $\mathbf{R}^1 \times [0,1]$ such that $N_0(\{0\} \times [0,1]) = 0$. Let us also assume that the measure N_0 satisfies Condition b.) of Theorem 2. Let us consider a Poisson field $X_n = (X_n^{(1)}, X_n^{(2)})$, $X_n \in \mathbf{R}^1 \times [0,1]$, $n = 1, 2, \dots$, on the strip $R^1 \times [0,1])$ with counting measure $\nu_0(dx, dy) = \frac{N_0(dx,dy)}{x^2}$, and let $T_0(t)$, $0 \le t \le 1$, be the infinitely divisible process determined by this Poisson field. For all numbers $\varepsilon > 0$ and $\eta > 0$ there exists a number $= \delta(\varepsilon, \eta)$ such that for all sequences of number $0 = t_0 < t_1 < \cdots < t_s = 1$ for which the inequality $\sup\limits_{1 \le l \le s} |t_l - t_{l-1}| < \delta$ holds the stochastic process $T_0(t)$ and its discretization $\bar T_0(t) = \bar T_{0,t_0,t_1,\dots,t_s}(t)$ satisfies the inequality*

$$P\left(d(T_0(\cdot), \bar T_0(\cdot)) > \eta\right) < \varepsilon, \tag{3.4}$$

where $d(\cdot,\cdot)$ denotes the (simpler, not complete) metric introduced in the space $D([0,1])$.

Let N_k, $k=1,2,\dots$, be canonical measures on the strip $\mathbf{R}^1\times[0,1]$ which satisfy the relation $N_k(\{0\}\times[0,1])=0$ and which converge to the canonical measure N_0 considered in the previous paragraph. Let us define the measures $\nu_k(dx,dy)=\frac{N_k(dx,dy)}{x^2}$, $k=1,2,\dots$ and consider Poisson fields $X_n(k)=(X_n^{(1)}(k),X_n^{(2)}(k))$, $X_n(k)\in\mathbf{R}^1\times[0,1]$, $n=1,2,\dots$, $k=1,2,\dots$, on the strip $R^1\times[0,1])$ with counting measures $\nu_k(dx,dy)$. Let $T_k(t)$, $0\le t\le 1$, $k=1,2,\dots$, denote the infinitely divisible stochastic processes determined by these Poissonian fields. Given some numbers $\varepsilon>0$ and $\eta>0$ there exists a number $\delta=\delta(\varepsilon,\eta)$ and a threshold index $k_0=k_0(\eta,\varepsilon)$ such that for all indices $k\ge k_0$ and sequences of numbers $0=t_0<t_1<\dots<t_s=1$ for which the inequality $\sup\limits_{1\le l\le s}|t_l-t_{l-1}|<\delta$ holds the stochastic processes $T_k(t)$ and their discretizations $\bar T_k(t)=\bar T_{k,t_0,t_1,\dots,t_s}(t)$ satisfy the inequality

$$P\left(d(T_k(\cdot),\bar T_k(\cdot))>\eta\right)<\varepsilon \quad \textit{if } k\ge k_0, \tag{3.5}$$

where $d(\cdot,\cdot)$ is the same metric in the space $D([0,1])$ as that in formula (3.4).

Let us emphasize that the threshold index $k_0=k_0(\eta,\varepsilon)$ in formula (3.5) can be chosen independently of the (sufficiently dense) sequence of numbers $0=t_0<t_1<\dots<t_s=1$.

Proposition 4. *Let a sequence of canonical measures N_k, $k=1,2,\dots$, be given on the strip $\mathbf{R}^1\times[0,1]$ which converges to a canonical measure N_0 as $k\to\infty$, and $N_k(\{0\}\times[0,1])=0$ for all numbers $k=0,1,2,\dots$. Furthermore, let us fix a finite, monotone increasing sequence $0=t_0<t_1<t_2<\dots<t_s=1$ on the interval $[0,1]$. Then for all indices $k=1,2,\dots$ a Poisson field $X_n(k)=(X_n^{(1)}(k),X_n^{(2)}(k))$, $X_n(k)\in\mathbf{R}^1\times[0,1]$, $n=1,2,\dots$, can be constructed with canonical measures $\nu_k(\,dx,dy)=\frac{N_k(\,dx,\,dy)}{x^2}$ together with a Poisson field $X'_n(k)=({X'_n}^{(1)}(k),{X'_n}^{(2)}(k))$, $X'_n(k)\in\mathbf{R}^1\times[0,1]$, $n=1,2,\dots$, with canonical measures $\nu_0(\,dx,\,dy)=\frac{N_0(\,dx,\,dy)}{x^2}$ in such a way that the infinitely divisible stochastic processes $T_k(t)$ and $T'_k(t)$, $k=1,2,\dots$, determined by these Poisson fields, or more explicitly their discretizations, the stochastic processes $\bar T_k(t)=\bar T_{k,t_0,t_1,\dots,t_s}(t)$ and $\bar T'_k(t)=\bar T'_{k,t_0,t_1,\dots,t_s}(t)$, $0\le t\le 1$, satisfy the relation*

$$\sup_{0\le t\le 1}|\bar T_k(t)-\bar T'_k(t)|\Rightarrow 0 \quad \textit{if } k\to\infty, \tag{3.6}$$

where $\Rightarrow$ denotes stochastic convergence. (Let us remark that the distributions of the stochastic processes $T'_k(\cdot)$ and of their discretizations do not depend on the index k, since they are determined by a Poisson field with canonical measure ν_0.)

Let a sequence of canonical measures N_k, $k=1,2,\dots$, be given on the strip $\mathbf{R}^1\times[0,1]$ which satisfies the conditions of the *Statement.* Then with the help of Propositions 3 and 4 for all numbers $\varepsilon>0$ and $\eta>0$ a partition $0=t_0<t_1<t_2<\dots<t_s=1$ of the interval $[0,1]$ can be given together with two sequences of Poisson fields $X_n(k)=({X_n}^{(1)}(k),{X_n}^{(2)}(k))$ and $X'_n(k)=({X'_n}^{(1)}(k),{X'_n}^{(2)}(k))$, $n=1,2,\dots$,

$k=1,2,\dots$ on the strip $\mathbf{R}^1\times[0,1]$) with counting measures $\nu_k(dx,dy)=\frac{N_k(dx,dy)}{x^2}$ and $\nu_0(dx,dy)=\frac{N_0(dx,dy)}{x^2}$ respectively which satisfy the following property. The infinitely divisible stochastic processes $T_k(t)$ and $T'_k(t)$, $0\le t\le 1$, determined by the Poisson fields $X_n(k)$ and $X'_n(k)$, $n=1,2,\dots$, $k=1,2,\dots$, and their discretizations, the stochastic processes $\bar T_k(\cdot)=\bar T_{k,t_0,t_1,\dots,t_s}(\cdot)$ and $\bar T'_k(\cdot)=\bar T'_{k,t_0,t_1,\dots,t_s}(\cdot)$, satisfy the relations

$$
\begin{aligned}
P\left(d(T'_k(\cdot),\bar T'_{k,t_0,\dots,t_s}(\cdot))>\eta\right)&<\varepsilon\quad\text{for all numbers } k\ge 1\\
P\left(d(T_k(\cdot),\bar T_{k,t_0,\dots,t_s}(\cdot))>\eta\right)&<\varepsilon,\quad\text{if } k\ge k_0\\
P\left(\sup_{0\le t\le 1}\left|\bar T_{k,t_0,\dots,t_s}(t)-\bar T'_{k,t_0,\dots,t_s}(t)\right|>\eta\right)&<\varepsilon,\quad\text{if } k\ge k_0,
\end{aligned}
\tag{3.7}
$$

where $k_0=k_0(\varepsilon,\eta)$ is an appropriate threshold index.

Indeed, by Proposition 3 the first two relations of formula (3.7) hold for an appropriate partition $0=t_0<t_1<t_2<\cdots<t_s=1$ of the interval for all indices $k\ge k_0$ if $k_0=\bar k_0(\varepsilon,\eta)$ is sufficiently large. The validity of these relations does not depend on the way the Poisson fields $X_n(k)$ and $X'_n(k)$, $n=1,2,\dots$, $k=1,2,\dots$, with counting measures ν_k and ν_0 are constructed. Then we can guarantee, because of Proposition 4, with an appropriate construction that also the third relation of formula (3.7) holds. (In this step we may increase the threshold index k_0 if it is needed.)

Let us apply formula (3.7) with numbers $\varepsilon_j=\eta_j=\frac1j$. Then we can see that there exists a monotone sequence of positive integers $k_0\left(\frac1j\right)$ and for all $j=1,2,\dots$ a sequence of numbers $0=t_0^{(j)}<t_1^{(j)}<t_2^{(j)}<\cdots<t_{s_j}^{(j)}=1$ can be given which satisfy formula (3.7) for all $j=1,2,\dots$ with the choice $\varepsilon=\eta=\frac1j$ where the condition $k\ge k_0$ is replaced by the condition $k_0\left(\frac1j\right)\le k\le k_0\left(\frac1{j+1}\right)$, and we write $0=t_0^{(j)}<t_1^{(j)}<t_2^{(j)}<\cdots<t_{s_j}^{(j)}=1$ instead of $0=t_0<t_1<t_2<\cdots<t_s=1$, i.e. the partition of the interval $[0,1]$ we consider may depend on the index j. Hence under the conditions of the *Statement* two sequences of Poisson fields $X_n(k)$ and $X'_n(k)$, $n=1,2,\dots$, $k=1,2,\dots$, can be constructed on the strip $\mathbf{R}^1\times[0,1]$ with canonical measures ν_k and ν_0 in such a way that the infinitely divisible stochastic processes $T_k(t)$ and $T'_k(t)$, $k=1,2,\dots$, $0\le t\le 1$, determined by them, together with their discretizations given by appropriate sequences of numbers $0=t_0^{(k)}<t_1^{(k)}<t_2^{(k)}<\cdots<t_{s_k}^{(k)}=1$, satisfy the relations

$$d\left(T'_k(\cdot),\bar T'_{k,t_0^{(k)},\dots,t_{s_k}^{(k)}}(\cdot)\right)\Rightarrow 0\quad\text{if } k\to\infty,\tag{3.8a}$$

$$d\left(T_k(\cdot),\bar T_{k,t_0^{(k)},\dots,t_{s_k}^{(k)}}(\cdot)\right)\Rightarrow 0\quad\text{if } k\to\infty,\tag{3.8b}$$

$$\sup_{0\le t\le 1}\left|\bar T_{k,t_0^{(k)},\dots,t_{s_k}^{(k)}}(t)-\bar T'_{k,t_0^{(k)},\dots,t_s^{(k)}}(t)\right|\Rightarrow 0\quad\text{if } k\to\infty.\tag{3.8c}$$

By Theorem A and relation (3.8a) the distributions of the stochastic processes $\bar T'_{k,t_0^{(k)},\dots,t_{s_k}^{(k)}}(\cdot)$ converge to the distribution of the stochastic process $T_0(\cdot)$ in the space

$D([0,1])$. (Let us recall that the distributions of the stochastic processes $T'_k(\cdot)$ and $T_0(\cdot)$ agree.) Then by relation (3.8c) and Theorem A the distributions of the stochastic processes $\bar T_{k,t_0^{(k)},\dots,t_{s_k}^{(k)}}(\cdot)$, and after this by relation (3.8b) and Theorem A the distributions of the stochastic processes $T_k(\cdot)$ converge to the distribution of the stochastic process $T_0(\cdot)$ in the space $D([0,1])$. Thus we have proved the first part of the *Statement*. After this, the second part of the *Statement* follows from Theorem A and the second part of Proposition 3. Indeed, by this result

$$d\left(T_k(\cdot)\,,\bar T_{k,u_{k,0}^{(k)},\dots,u_{k,n_k}^{(k)}}(\cdot)\right)\Rightarrow 0 \quad \text{if } k\to\infty.$$

(We exploit in this step of the proof that the threshold index k_0 in formula (3.5) does not depend on the choice of the sufficiently dense partition $0=t_0<t_1<\cdot<t_s$ of the interval $[0,1]$.)

Let us remark that in Proposition 3 we have estimated the distance of a stochastic process and its discretization in the metric $d(\cdot,\cdot)$ introduced in the space $D([0,1])$ and not in the supremum norm. We had to do so, since if the original stochastic processes have jumps, and we have to work with such stochastic processes, then these processes are far from their discretizations in the supremum norm. On the other hand, if the points of jumps are not too dense, which means that small intervals contain only at least one jump, then under general conditions a stochastic process and its sufficiently dense discretization are close to each other in the metric $d(\cdot,\cdot)$. In the next Lemma 2 we give an estimate for the $d(\cdot,\cdot)$ distance of two (simple) functions. It can help us to estimate the $d(\cdot,\cdot)$ distance of stochastic processes. In particular, it will be useful in the proof of Proposition 3.

Lemma 2. *Let $x(t)$ and $y(t)$, $0\le t\le 1$, be two cadlag functions on the interval $[0,1]$ with $p<\infty$ numbers of jumps. (We assume that two functions have the same number of jumps.) Let us also assume that the values of the functions $x(\cdot)$ and $y(\cdot)$ agree after the j-th jump, $0\le j\le p$. Let there exist a finite monotone sequence of numbers $0=t_0<t_1<\dots<t_s=1$ such that $\inf\limits_{1\le l\le s}|t_l-t_{l-1}|\le\delta$ with some number $\delta>0$, and the function $x(\cdot)$ is constant in all intervals $[t_{l-1},t_l)$, $1\le l\le s-1$. Furthermore, we assume that if the j-th point of jumps of the function $x(\cdot)$ is the point t_{l_j} with some $l_j\ge j$, then the j-th jump of the other function $y(\cdot)$ is in a point of the interval $(t_{l_{j-1}},t_{l_j}]$, $1\le j\le p$. Then the inequality $d(x(\cdot),y(\cdot))<\delta$ holds.*

The proof of Lemma 2. Let $u_1,\dots,u_p$ be the points of jumps of the function $y(\cdot)$. Let us define the following homeomorphism $\lambda(\cdot)$ of the interval $[0,1]$ onto itself: $\lambda(u_j)=t_{l_j}$, $1\le j\le p$, $\lambda(0)=0$, $\lambda(1)=1$, and let the function $\lambda(\cdot)$ be linear in the intervals $[u_{j-1},\,u_j]$, $1\le j\le p$, and $[0,u_1]$, $[u_p,1]$. (The number t_{l_j} is the j-th point of jump of the function $x(\cdot)$.) Then $y(\lambda(\cdot))=x(\cdot)$, and $\sup\limits_{0\le t\le 1}|\lambda(t)-t|\le\delta$. Hence $d(x(\cdot),y(\cdot))\le\delta$, as we have claimed.

C.) The proof of Propositions 3 and 4.

The proof of Proposition 3. Let us choose a number $\alpha>0$ such that the numbers $\pm\alpha$ are points of continuity of the canonical measure M_0 on the real line, and $M_0(-\alpha,\alpha])<\frac{\varepsilon\eta^2}{8}$, where $M_0(B)=N_0(B\times[0,1])$, $B\in\mathbf{R}^1$. Let us also introduce the canonical measures $N_0''(\cdot)=N_{0,\alpha}(\cdot)$ and $N_{0,A_L}'(\cdot)=N_{0,A_L,\alpha}'(\cdot)$ on the strip $\mathbf{R}^1\times[0,1]$ defined by the formulas $N_0''(B)=N_0(B\cap\{(x,y)\colon\,|x|\ge\alpha\})$, $N_{0,A_L}'(B)=N_0(B\cap\{(x,y)\colon\,A_L\le|x|<\alpha\})$ if $B\in\mathbf{R}^1\cap[0,1]$, and the numbers A_L, $A_L>0$, are chosen in such a way that they can be applied in the regularized sums which define the stochastic process $T_0(t)$ by means of a Poissonian field. Let us also introduce the measures $\nu_0''(\,dx,\,dy)=\frac{N_0''(\,dx,\,dy)}{x^2}$ and $\nu_{0,A_L}'(\,dx,\,dy)=\frac{N_{0,A_L}'(\,dx,\,dy)}{x^2}$ and consider a Poisson fields $X_n''=(X_n''^{(1)},X_n''^{(2)})$, $n=1,2,\dots$, with counting measure ν_0'' and the infinitely divisible field

$$T_0''(t)=\sum_{n\colon\,X_n''^{(1)}>\alpha,\;0\le X_n''^{(2)}\le t}X_n''^{(1)}-E\left(\sum_{n\colon\,X_n''^{(1)}>\alpha,\;0\le X_n''^{(2)}\le t}\tau\left(X_n''^{(1)}\right)\right),\quad 0\le t\le1,$$

determined by it. Let us consider similarly a Poisson field $X_{n,A_L}'=({X_{n,A_L}'}^{(1)},{X_{n,A_L}'}^{(2)})$, $n=1,2,\dots$ with counting measure ν_{0,A_L}' and the infinitely divisible stochastic process

$$T_{0,A_L}'(t)=\sum_{n\colon\,A_L\le {X_n'}^{(1)}<\alpha,\;0\le {X_n'}^{(2)}\le t}{X_n'}^{(1)}-E\left(\sum_{n\colon\,A_L\le {X_n'}^{(1)}<\alpha,\;0\le {X_n'}^{(2)}\le t}\tau\left({X_n'}^{(1)}\right)\right),$$

$0\le t\le1$, determined by this Poisson field. Put $T_{0,A_L}(t)=T_{0,A_L}'(t)+T_0''(t)$, $0\le t\le1$. The stochastic processes $T_{0,A_L}(\cdot)$ converge with probability 1 to the stochastic process $T_0(\cdot)$ in the supremum norm. Hence to prove formula (3.4) it is enough to show that

$$P\left(d(T_{0,A_L}(\cdot),\bar T_{0,A_L,t_0,\cdots,t_s}(\cdot))>\eta\right)<\varepsilon\quad\text{for all numbers } L\ge L_0,\tag{3.9}$$

where L_0 is an appropriate number, and $\bar T_{0,A_L,t_0,\cdot,t_s}(\cdot))$ denotes the discretization of the stochastic process $T_{0,A_L}(\cdot)$. Indeed,

$$\left\{\omega\colon\,d(T_0,(\cdot,\omega),\bar T_0(\cdot,\omega))>\eta\right\}\subset\liminf_{L\to\infty}\left\{\omega\colon\,d(T_{0,A_L}(\cdot,\omega),\bar T_{0,A_L,t_0,\cdots,t_s}(\cdot,\omega))>\eta\right\},$$

hence formula (3.9) implies formula (3.4).

We can write

$$P\left(d(T_{0,A_L}(\cdot),\bar T_{0,A_L,t_1,\dots,t_s}(\cdot))>\eta\right)\le P\left(\sup_{0\le t\le1}|T_{0,A_L}'(t)|>\frac{\eta}{2}\right)+P\left(d(T_0''(\cdot),\bar T_{0,t_1,\dots,t_s}''(\cdot))>\frac{\eta}{2}\right),$$

hence to prove formula (3.9) it is enough to show that

$$P\left(\sup_{0\le t\le1}|T_{0,A_L}'(t)|>\frac{\eta}{2}\right)<\frac{\varepsilon}{2},\quad\text{if } L\ge L_0,\tag{3.10}$$

and

$$P\left(d(T''_0(\cdot),\bar T''_{0,t_1,\dots,t_s}(\cdot))>\frac{\eta}{2}\right)<\frac{\varepsilon}{2},\quad \text{if } \sup_{1\le l\le s}|t_l-t_{l-1}|<\delta \tag{3.11}$$

with some appropriate number $\delta>0$.

As $T'_{0,A_L}(t)$ is a stochastic process with independent increments, and its trajectories are cadlag functions hence we can write with the help of the Kolmogorov inequality that

$$P\left(\sup_{0\le t\le 1}|T'_{0,A_L}(t)|>\frac{\eta}{2}\right)\le\frac{4ET'_{0,A_L}(1)^2}{\eta^2}=\frac{4}{\eta^2}\int u^2\mu'_{0,A_L}(\,du)\le\frac{4}{\eta^2}M_0([-\alpha,\alpha])<\varepsilon,$$

where the measure μ'_{0,A_L} is defined by the relation $\mu'_{0,A_L}(B)=\nu'_{0,A_L}(B\times[0,1])$ for all measurable sets $B\subset\mathbf{R}^1$. Hence inequality (3.10) holds.

To prove formula (3.11) let us first introduce the function $\lambda_0(t)=\lambda_{0,\alpha}(t)$ defined by the formula $\lambda_0(t)=\nu''_0(\mathbf{R}^1\times[0,t])=\int_{\{(x,y)\colon\,|x|>\alpha,\ 0\le|y|<t\}}\frac{N_0(\,dx,dy)}{x^2}$, $0\le t\le 1$. Let us remark that because of Condition b.) of Theorem 2 the function $\lambda_0(\cdot)$ is continuous in the interval $[0,1]$. We claim that thee exists some number $\delta>0$ such that for all partitions $0=t_0<t_1<\cdots<t_s=1$ of the interval $[0,1]$

$$\sum_{l=1}^s(\lambda_0(t_l)-\lambda_0(t_{l-1}))^2<\varepsilon\quad\text{if } |t_l-t_{l-1}|<\delta \text{ for all numbers } 1\le l\le s. \tag{3.12}$$

Indeed, as the function $\lambda_0(\cdot)$ is uniformly continuous, hence there exists a number $\delta>0$ such that $|\lambda_0(t)-\lambda_0(s)|<\frac{\varepsilon}{\lambda(1)}$ if $|t-s|<\delta$. Furthermore, the function $\lambda_0(\cdot)$ is monotone increasing. Hence $\sum\limits_{l=1}^s(\lambda_0(t_l)-\lambda_0(t_{l-1}))^2\le\sup\limits_{1\le l\le s}|\lambda_0(t_l)-\lambda_0(t_{l-1})|\sum\limits_{l=1}^s|\lambda_0(t_l)-\lambda_0(t_{l-1})|<\varepsilon$ if $|t_l-t_{l-1}|<\delta$ for all numbers $1\le l\le s$, i.e. formula (3.12) is valid.

A Poisson distributed random variable with parameter λ takes a value more than or equal to two with probability $1-e^{-\lambda}-\lambda e^{-\lambda}\le\frac{\lambda^2}{2}$. Hence the probability of the event that a Poisson field $X''_n=(X''^{(1)}_n,X''^{(2)}_n)$, $n=1,2,\dots$, on the strip $\mathbf{R}^1\times[0,1]$ with counting measure ν''_0 contains at least two such points X''_{n_1} and X''_{n_2} whose second coordinates $X''^{\,(2)}_{n_1}$ and $X''^{\,(2)}_{n_2}$ are in an interval $[s,t]$, $0\le s<t\le1$, is less than $\frac12\nu''(\mathbf{R}^1\times[s,t])^2=\frac12(\lambda(t)-\lambda(s))^2$. Hence by formula (3.12) for a Poisson field $X''_n=(X''^{(1)}_n,X''^{(2)}_n)$, $n=1,2,\dots$, with counting measure ν''_0 and a partition $0\le t_0<t_1<\cdots<t_s=1$ of the interval $[0,1]$ the probability of the event

$$A(t_1,\dots,t_s)=\{\omega\colon\ \#\{n\colon\ t_{l-1}\le X''^{(2)}_n(\omega)\le t_l\}\le1\quad\text{for all numbers } 1\le l\le s\}$$

can be estimated as

$$1-P\left(A(t_1,\dots,t_s)\right)=P\left(\Omega\setminus A(t_1,\dots,t_s)\right)\le\frac12\sum_{l=1}^s(\lambda_0(t_l)-\lambda_0(t_{l-1}))^2\le\frac{\varepsilon}{2} \tag{3.13}$$

if $\sup\limits_{1\le l\le s}|t_l-t_{l-1}|<\delta$ with a sufficiently small $\delta>0$.

Let us fix a number $\delta>0$ with which formula (3.13) holds together with the inequality $\delta<\frac{\eta}{2}$. Then formula (3.13) means that on a set of probability greater than $1-\frac{\varepsilon}{2}$ the cadlag functions $x(t)=\bar T''_{0,t_0,t_1,\dots,t_s}(t,\omega)$ and $y(t)=T''_0(t,\omega)$ together with the partition $0=t_0<t_1<\cdots<t_s=1$ of the interval $[0,1]$ satisfy the conditions of Lemma 2, and $d(T''_0(\cdot,\omega),\bar T''_{0,t_0,\cdots,t_s}(\cdot,\omega))\le\delta<\frac{\eta}{2}$ on a set of probability greater than $1-\frac{\varepsilon}{2}$. This fact implies relation (3.11) and as a consequence also relation (3.4).

To prove formula (3.5) it is enough to prove the following analogs of formulas (3.10) and (3.11).

$$P\left(\sup_{0\le t\le 1}|T'_{k,A_L}(t)|>\frac{\eta}{2}\right)<\frac{\varepsilon}{2},\quad\text{if } k\ge k_0 \text{ and } L\ge L_0 \tag{3.14}$$

with an appropriate threshold index $k_0=k_0(\varepsilon,\eta)$, and

$$P\left(d(T''_k(\cdot),\bar T''_{k,t_1,\dots,t_s}(\cdot))>\frac{\eta}{2}\right)<\frac{\varepsilon}{2},\quad\text{if } k\ge k_0,\ \text{and}\ \sup_{1\le l\le s}|t_l-t_{l-1}|<\delta \tag{3.15}$$

with an appropriate threshold index $k_0=k_0(\varepsilon,\eta)$ and number $\delta>0$.

Formula (3.14) can be proved similarly to formula (3.10). The only difference is that now we exploit that since the numbers $\pm\alpha$ are points of continuity of the measure M_0, and $M_0([-\alpha,\alpha])<\frac{\varepsilon\eta^2}{8}$ hence also the relation $M_k(-\alpha,\alpha)<\frac{\varepsilon\eta^2}{8}$ holds if $k\ge k_0$ with an appropriate threshold index k_0. (We define the measures M_k, $k=1,2,\dots,$ on the real line, analogously to the definition of the measure M_0, by the formula $M_k(B)=N_k(B\times[0,1])$ for all measurable sets $B\in\mathbf{R}^1$.)

Formula (3.15) can be proved similarly to formula (3.11). The only difference is that now we have to prove and apply the following analog of relation (3.12).

Let us introduce the functions $\lambda_k(t)=\lambda_{k,\alpha}(t)$ defined by the formula $\lambda_k(t)=\nu''_k(\mathbf{R}^1\times[0,t])=\int_{\{(x,y)\colon |x|>\alpha,\ 0\le|y|<t\}}\frac{N_k(\,dx,dy)}{x^2}$, $0\le t\le 1$ for all indices $k=1,2,\dots$. Then there exists number $\delta>0$ and threshold index $k_0=k_0(\delta)$ in such a way that for all partitions $0=t_0<t_1<\cdots<t_s=1$ of the interval $[0.1]$

$$\sum_{l=1}^s(\lambda_k(t_l)-\lambda_k(t_{l-1}))^2\le\varepsilon\quad\text{if } k\ge k_0 \text{ and } |t_l-t_{l-1}|<\delta \text{ for all numbers } 1\le l\le s. \tag{3.16}$$

Let us emphasize the threshold index k_0 in formula (3.16) depends only on the number $\delta>0$ and not on the partition $0=t_0<t_1<\cdots<t_s=1$ of the interval $[0,1]$.

Let us prove formula (3.16) first in the special case if the partition $0=t_0<t_1<\cdots<t_s=1$ of the interval $[0,1]$ satisfies not only the inequality $|t_l-t_{l-1}|<\delta$, but also the inequality $|t_l-t_{l-1}|\ge\frac{\delta}{2}$ for all numbers $1\le l\le s$. The convergence of the canonical measures N_k to the canonical measures N_0 and the continuity of the measure $\lambda_0(\cdot)$ imply that the monotone functions $\lambda_k(\cdot)$ converge in all points of the interval $[0,1]$ to the monotone and continuous function $\lambda_0(\cdot)$. The properties of the functions $\lambda_k(\cdot)$

also imply that the convergence $\lim\limits_{k\to\infty}\lambda_k(t)=\lambda_0(t)$, $0\le t\le 1$, is uniform. Beside this, the sum at the right-hand side of formula (3.16) contains at most $\frac{2}{\delta}$ terms. Hence formula (3.12) implies formula (3.16) with the same number δ and a sufficiently large threshold index $k_0(\delta)$.

Let us now consider a partition $0=t_0<t_1<\cdots<t_s=1$ of the interval such that $\sup\limits_{1\le s\le k}|t_l-t_{l-1}|<\frac{\delta}{2}$ where formula (3.16) holds in the special case considered in the previous paragraph with the number δ. It is not difficult to see that the sequence of numbers $0=t_0<t_1<\cdots<t_s=1$ has a subsequence $0=t_{j_0}<t_{j_1}<\cdots<t_{j_p}=1$ such that $\frac{\delta}{2}\le\left|t_{j_u}-t_{j_{u-1}}\right|<\delta$ for all numbers $1\le u\le p$. Let us take such a subsequence. Then we can write with the help of the already proven case that

$$\sum_{l=1}^{s}(\lambda_k(t_l)-\lambda_k(t_{l-1}))^2\le\sum_{u=1}^{p}\left(\lambda_k(t_{j_u})-\lambda_k(t_{j_{u-1}})\right)^2<\varepsilon$$

if $k\ge k_0(\delta)$. In such a way we have proved formula (3.16) (with the choice $\frac{\delta}{2}$ instead of the number δ.) After the proof of formulas (3.15) and (3.16) formula (3.5) can be proved in the same way as formula (3.4).

The proof of Proposition 4. The proof of Proposition 4 is based on the following observation. To determine the difference $\bar T_k(\cdot)-\bar T'_k(\cdot)$ of the discretizations of the processes $T_k(\cdot)$ and $T'_k(\cdot)$ we need not know the precise values of the Poisson fields $X_{k,n}=(X^{(1)}_{k,n},X^{(2)}_{k,n})$ and $X'_{k,n}=({X'_{k,n}}^{(1)},{X'_{k,n}}^{(2)})$, $n=1,2,\dots$, $k=1,2,\dots$, which determine these processes. The knowledge of the values of the second coordinates of these fields is not necessary, it is enough to know in which one of the intervals $[t_{l-1},t_l]$ they lie. Hence in the first step of the construction we do not decide the precise value of the Poisson fields we have to define. In such a way in the first step of the construction a coupling problem has to be handled which can be solved relatively simply with the help of Proposition 2. Then the construction can be completed by an appropriate randomization.

To work out the details first we introduce some notations. Let us define the canonical measures $\tilde N_{k,l}$, $k=0,1,2,\dots$, $1\le l\le s$, on the real line by the formula $\tilde N_{k,l}(B)=N_k(B\times(t_{l-1},t_l])$, $k=0,1,2,\dots$, $1\le l\le s$. Let us then define the canonical measures $\tilde N'_{k,l}$ on the strip $\mathbf{R}^1\times[0,1]$ by the formulas $\tilde N'_{k,l}(B\times\{t_l\})=\tilde N'_{k,l}(B)$, $B\subset\mathbf{R}^1$, $k=0,1,2,\dots$, $1\le l\le s$, i.e. let the measure $\tilde N'_{k,l}$ be the shift of the measure $\tilde N_{k,l}$ from the real line to the line $\{(x,y)\colon\ y=t_l\}$ parallel to the coordinate axis in $\mathbf{R}^2$. Let us also define the canonical measures $\tilde N_k$, $k=0,1,2,\dots$, on the strip $\mathbf{R}^1\times[0,1]$ as $\tilde N_k=\sum\limits_{l=1}^{s}\tilde N'_{k,l}$, $k=0,1,2,\dots$.

Let us define the measures $\tilde\nu_{k,l}(dx)=\frac{\tilde N_{k,l}(dx)}{x^2}$, $\tilde\nu'_{k,l}(dx)=\frac{\tilde N'_{k,l}(dx)}{x^2}$ and $\tilde\nu_k(dx,dy)=\frac{\tilde N_k(dx,dy)}{x^2}$, $k=0,1,2,\dots$, $l=1,2,\dots,s$. By Proposition 2 some Poisson fields $\xi_{k,l}(n)$, $n=1,2,\dots$, $k=1,2,\dots$, $l\le 1\le s$, can be constructed with counting measures $\tilde\nu_{k,l}$ together with some other Poisson fields $\xi'_{k,l}(n)$, $n=1,2,\dots$, $k=1,2,\dots$, $1\le l\le s$, with counting measures $\tilde\nu_{0,l}$ in such a way that the random variables $U_{k,l}$ and $U'_{k,l}$ with infinitely divisible distributions determined by these Poisson fields satisfy the relations $U'_{k,l}-U_{k,l}\Rightarrow 0$ as $k\to\infty$ for all numbers $1\le l\le s$, where $\Rightarrow$ denotes stochastic convergence. Let us also define the random variables $\tilde T_{k,j}=\sum\limits_{l=1}^{j}U_{k,l}$, $\tilde T'_{k,j}=\sum\limits_{l=1}^{j}U'_{k,l}$, $k=1,2,\dots$, $1\le j\le s$. Then also the relation $\sup\limits_{1\le j\le s}\left|\tilde T_{k,j}-\tilde T'_{k,j}\right|\Rightarrow 0$ holds as $k\to\infty$.

Now we begin the construction of Poisson fields for which the discretizations of the stochastic processes $T_k(t)$ and $T'_k(t)$ determined by them satisfy formula (3.6). Let us consider the measures $\nu_{k,l}$, $k=0,1,2,\dots$, $1\le l\le s$, which are restrictions of the measures $\nu_k(dx,dy)=\frac{N_k(dx,dy)}{x^2}$ to the strip $\mathbf{R}^1\times(t_{l-1},t_l]$. Then for all integers $k=0,1,2\dots$, $1\le l\le s$, points $x\in\mathbf{R}^1$, and measurable sets $B\in\mathbf{R}^1\times(t_{l-1},t_l]$ there exist such "conditional measures" $\nu_{k,l}(B|x)$ on the interval $(t_{l-1},t_l]$ for which $\nu_{k,l}(\,\cdot\,|x)$ is a probability measure in the strip $\mathbf{R}^1\times(t_{l-1},t_l]$ for all numbers $x\in\mathbf{R}^1$, $\nu_{k,l}(B|\,\cdot\,)$ is a measurable function on the real line for all measurable sets $B\subset\mathbf{R}^1\times(t_{l-1},t_l]$, and

$$\nu_{k,l}(B)=\int\nu_{k,l}(B|x)\tilde\nu'_{k,l}(\,dx)\quad\text{for all measurable sets } B\subset\mathbf{R}^1\times(t_{l-1},t_l]\qquad(3.17)$$

for all numbers $k=0,1,2,\dots$ and $1\le l\le s$, where $\tilde\nu_{k,l}$ is the measure defined in the previous paragraph. The existence of a "conditional measure" $\nu_{k,l}(\,\cdot\,|\,\cdot\,)$ satisfying relation (3.17) is a consequence of a classical result of probability theory about the existence of regular conditional distributions.

Now we construct Poisson fields $X_{k,n}=(X^{(1)}_{k,n},X^{(2)}_{k,n})$, $n=1,2,\dots$, $k=1,2,\dots$ with counting measures ν_k and Poisson fields $X'_{k,n}=({X'_{k,n}}^{(1)},{X'_{k,n}}^{(2)})$, $n=1,2,\dots$, $k=1,2,\dots$ with counting measures ν_0 such that the stochastic processes $T_k(t)$ and $T'_k(t)$ determined by them satisfy Proposition 4. Let us consider the already constructed Poisson fields $\xi_{k,l}(n)$, $n=1,2,\dots$, $k=1,2,\dots$, $l\le 1\le s$, with counting measures $\tilde\nu_{k,l}$ and the Poisson fields $\xi'_{k,l}(n)$, $n=1,2,\dots$, $k=1,2,\dots$, $1\le l\le s$, with counting measures $\tilde\nu_{0,l}$. For all points $\xi_{k,l}(n)$ and $\xi'_{k,l}(n)$ of these Poisson fields we shall construct random variables $\eta_{k,l}(n)$ and $\eta'_{k,l}(n)$ in a random way, and the Poisson fields we want to construct will consist of the points $(\xi_{k,l}(n),\eta_{k,l}(n))$ and $(\xi'_{k,l}(n),\eta'_{k,l}(n))$. Let us construct for all random variables $\xi_{k,l}(n)$ a $\nu_{k,l}(\,\cdot\,|\xi_{k,l}(n))$ distributed random variable $\eta_{k,l}(n)$ and for all random variables $\xi'_{k,l}(n)$ a $\nu_{0.l}(\,\cdot\,|\xi'_{k,l}(n))$ distributed random variables $\eta'_{k,l}(n)$ on the interval $[t_{l-1},t_l]$. Let us construct these random numbers independently of each other. For a fixed index k let the Poison field $X_{k,n}=\left(X^{(1)}_{k,n},X^{(2)}_{k,n}\right)$, $n=1,2,\dots$, $k=1,2,\dots$ with counting measure ν_k consist of the previously constructed pairs of points $(\xi_{k,l}(n),\eta_{k,l}(n))$, $n=1,2,\dots$, and similarly let the Poisson field $X'_{k,n}=\left({X'_{k,n}}^{(1)},{X'_{k,n}}^{(2)}\right)$, $n=1,2,\dots$, $k=1,2,\dots$ with counting measure ν_0 consist of the pairs of points $(\xi'_{k,l}(n),\eta'_{k,l}(n))$, $n=1,2,\dots$, $1\le l\le s$. We claim that the above constructed Poisson fields satisfy Proposition 4.

We shall prove that if the rectangle $B\times[t_{l-1},t_l]$ satisfies the property $\nu_{k,l}(B\times[t_{l-1},t_l])<\infty$, then the points falling to this rectangle $B\times[t_{l-1},t_l]$ define a Poisson field with counting measure $\nu_{k,l}$ on this rectangle. We shall show this for all numbers $k = 1, 2, \dots$ and $1 \le l \le s$. Beside this we claim that if the property $\nu_{0,l}(B \times [t_{l-1}, t_l]) < \infty$ holds, then the points $(\xi'_{k,l}(n), \eta'_{k,l}(n))$, $n = 1, 2, \dots$, falling to the rectangle $B \times [t_{l-1}, t_l]$ define a Poisson field with counting measure $\nu_{0,l}$ on this rectangle for all numbers $k = 1, 2, \dots$ and $1 \le l \le s$. These facts imply that we have really constructed Poisson fields with the right counting measure. These statements can be simply proved with the help of the following observation. The distributions of these point processes agree with the distributions of the point process we get in the following way: Let us choose randomly many number of points with Poisson distribution with parameter $\nu_{k,l}(B \times [t_{l-1}, t_l])$ and drop them randomly to the rectangle $B \times [t_{l-1}, t_l]$ independently of each other with distribution $\frac{\mu_{k,l}(dx,dy)}{\nu_{k,l}(B\times[t_{l-1},t_l])}$. Such constructions supply Poisson fields with the right counting measure.

Finally we remark that the above constructed Poisson fields are such that the discretizations of the infinitely divisible stochastic processes $T_k(\cdot)$ and $T'_k(\cdot)$ determined by them satisfy the inequalities $\bar T_{k,t_0,\cdots,t_s}(t) = \tilde T_{k,j-1}$ and $\bar T'_{k,t_0,\cdots,t_s}(t) = \tilde T'_{k,j-1}$ if $t_{j-1} < t \le t_j$, $1 \le j \le s$, and $\bar T_{k,t_0,\cdots,t_s}(1) = \tilde T_{k,s}$, $\bar T'_{k,t_0,\cdots,t_s}(1) = \tilde T'_{k,s}$, $k = 1, 2, \dots,$ Hence the processes $\bar T_{k,t_0,\cdots,t_s}(\cdot)$ and $\bar T'_{k,t_0,\cdots,t_s}(\cdot)$ satisfy formula (3.7). Proposition 4 is proved.

Appendix. The proof of Theorem A.

The proof of Theorem A. It is enough to prove the statement formulated in general separable metric spaces. The weak convergence of the random variables S_k to a probability measure μ can be formulated so that the distributions μ_k of the random variables S_k satisfy the relation $\limsup\limits_{k\to\infty} \mu_k(\mathbf{F}) \le \mu(\mathbf{F})$ for all closed sets $\mathbf{F} \subset X$. We shall show that under the conditions of Theorem A the distributions $\bar\mu_k$ of the random variables T_k also satisfy the relation $\limsup\limits_{k\to\infty} \bar\mu_k(\mathbf{F}) \le \mu(\mathbf{F})$ for all closed sets $F \subset X$. (Let us remark that the characterization of the weak convergence applied in this proof is valid in all separable metric spaces. We do not have to assume that the metric space is complete. (See for instance Theorem 2.1 in the book of P. Billingsley *Convergence of Probability Measures.*)

Let us fix some number $\varepsilon > 0$. As $\mathbf{F} = \bigcap\limits_{n=1}^{\infty} \mathbf{F}_{\frac1n}$, where $\mathbf{F}_a = \{x\colon\ \rho(x, \mathbf{F}) \le a\}$, hence there exists a number $\delta = \delta(\varepsilon) > 0$ such that $\mu(\mathbf{F}) \ge \mu(\mathbf{F}_\delta) - \varepsilon$. Furthermore, the inequality $\mu_k(\mathbf{F}_\delta) < \mu(\mathbf{F}_\delta) + \varepsilon$ holds if $k \ge k_0 = k_0(\varepsilon, \delta, \mathbf{F})$. As $\rho(S_k, T_k)$ tends to zero stochastically, hence also the inequality $\bar\mu_k(\mathbf{F}) = P(T_k \in \mathbf{F}) \le P(S_k \in \mathbf{F}_\delta) + P(\rho(S_k, T_k) > \delta) \le \mu_k(\mathbf{F}_\delta) + \varepsilon$, holds if $k \ge k_0$ and the threshold index $k_0 = k_0(\varepsilon, \delta, \mathbf{F})$ is chosen sufficiently large. The above inequalities imply that $\bar\mu_k(\mathbf{F}) \le \mu_k(\mathbf{F}_\delta) + \varepsilon \le \mu(\mathbf{F}_\delta) + 2\varepsilon \le \mu(\mathbf{F}) + 3\varepsilon$ if $k \ge k_0(\varepsilon, \delta, \mathbf{F})$. The above inequality holds for all numbers $\varepsilon > 0$, and this implies that $\limsup\limits_{k\to\infty} \bar\mu_k(\mathbf{F}) \le \mu(\mathbf{F})$. Thus Theorem A is proved.