In several papers a result called Segal's lemma appears. This result is Lemma 1.4 in I. E. Segal's paper Construction of nonlinear local quantum processes I. in Annals of Math. 92 (1970) 462–481. Here I describe a slightly modified and more detailed formulation and proof of this result. I present a slighly different version of Segal's proof. I finish this note with the original formulation and proof of Segal's result.

Segal's lemma. Let (M_i,μ_i) $(i=1,\dots,n)$ be separable measure spaces, and let $\mathbf{T}_i$ be an integral operator on $L_p(M_i,\mu_i)$ with positive kernel K_i, i.e. let

$$(\mathbf{T}_i f)(x_i)=\int_{M_i} K_i(x_i,y_i)f(y_i)\mu_i(\,dy_i),\qquad x_i\in M_i,\ y_i\in M_i,\quad i=1,\dots,n,$$

where $f(\cdot)$ is a function in $L_p(M_i,\mu_i)$. Let $\mathbf{T}_i$ be a contraction from $L_p(M_i,\mu_i)$ to $L_q(M_i,\mu_i)$, for certain given $1\le p<\infty$ and $1\le q<\infty$ for each i. Then the algebraic tensor product $\mathbf{T}_1\times\cdots\times\mathbf{T}_n$ is a contraction from $L_p(M_1\times\cdots\times M_n,\mu_1\times\cdots\times\mu_n)$ to $L_q(M_1\times\cdots\times M_n,\mu_1\times\cdots\times\mu_n)$ i.e.

$$\begin{aligned}\left(\int_{M_1\times\cdots\times M_n} |(\mathbf{T}_1\times\cdots\times\mathbf{T}_n f)(x_1,\dots,x_n))|^q\,\mu_1(\,dx_1)\dots\mu_n(\,dx_n)\right)^{1/q}&\\ \le\left(\int_{M_1\times\cdots\times M_n} |f(y_1,\dots,y_n)|^p\,\mu_1(\,dy_1)\dots\mu_n(\,dy_n)\right)^{1/p}&\end{aligned}$$

for all $f(y_1,\dots,y_n)\in L_p(M_1\times\cdots\times M_n,\mu_1\times\cdots\times\mu_n)$, where

$$\begin{aligned}&(\mathbf{T}_1\times\cdots\times\mathbf{T}_n f)(x_1,\dots,x_n)\\ &\quad=\int_{M_1\times\cdots\times M_n} K_1(x_1,y_1)\cdots K_n(x_n,y_n)\,f(y_1,\dots,y_n)\mu_1(\,dy_1)\dots\mu_n(\,dy_n).\end{aligned}$$

Proof. It is enough to prove the estimate of the Lemma for $n=2$, because then the lemma follows for general n by simple induction. It suffices to prove this inequality only for functions f with the additional property $f(y_1,y_2)\ge0$ for all $(y_1,y_2)\in M_1\times M_2$. Because of this additional condition we can omit the absolute value in the subsequent calculations, since we are working with non-negative functions.

Let us fix a function $f(y_1,y_2)\in L_p(M_1\times M_2,\mu_1\times\mu_2)$ with $y_1\in M_1$ and $y_2\in M_2$ such that $f(y_1,y_2)\ge0$ for all $(y_1,y_2)\in M_1\times M_2$. Then the function $f(y_1,y_2)$ with a fixed point $y_1\in M_1$ is a function in $L_p(M_2,\mu_2)$ for almost all $y_1\in M_1$. Hence we can define for all $y_1\in M_1$ the $L_p(M_2,\mu_2)$ valued measurable function $F(y_1)$ on M_1 by the formula $(F(y_1))(y_2)=f(y_1,y_2)$ for all $y_1\in M_1$ and $y_2\in M_2$, and $F(y_1)$ has the norm $\|F(y_1)\|_p=\left(\int_{M_2} f(y_1,y_2)^p\mu_2(\,dy_2)\right)^{1/p}$. The function $\|F(y_1)\|_p$, $y_1\in M_1$, is in the Banach space $L_p(M_1,\mu_1)$, since $\int\|F(y_1)\|_p^p\mu_1(\,dy_1)=\int_{M_1\times M_2} f(y_1,y_2)^p\mu_1(\,dy_1)\mu_2(\,dy_2)<\infty$. We also define the $L_q(M_2,\mu_2)$ valued function $G(x_1)$ for all $x_1\in M_1$ by the formula $(G(x_1))(y_2)=\int_{M_1}K_1(x_1,y_1)f(y_1,y_2)\mu_1(\,dy_1)$. Then

$$(G(x_1))(y_2)=\int_{M_1}K(x_1,y_1)(F(y_1))(y_2)\mu_1(\,dy_1)=\left(\int_{M_1}K_1(x_1,y_1)F(y_1)\mu_1(\,dy_1)\right)(y_2),$$

i.e. $G(x_1) = \int_{M_1} K_1(x_1, y_1) F(y_1) \mu_1(\,dy_1)$ for all $x_1 \in M_1$.

An upper bound will be given on $\|G(x_1)\|_p = \left(\int_{M_2} [(G(x_1))(y_2)]^p \mu_2(\,dy_2)\right)^{1/p}$ for a fixed point $x_1 \in M_1$. In this estimate the following result will be applied. If (M, μ) is a separable measure space, $\mathbf{X}$ is a Banach space, $H(y_1)$, $y_1 \in M$, is a measurable non-negative function on M, $U(y_1)$, $y_1 \in M$, is an $\mathbf{X}$ valued measurable function on M, then $\left\|\int_M H(y_1) U(y_1) \mu(\,dy_1)\right\| \le \int_M H(y_1) \|(U(y_1))\| \mu(\,dy_1)$.

With the choice $(M, \mu) = (M_1, \mu_1)$, $\mathbf{X} = L_p(M_2, \mu_2)$, $H(y_1) = K_1(x_1, y_1)$, $U(y_1) = F(y_1)$ for all $y_1 \in M_1$ this result yields that

$$
\begin{aligned}
\|G(x_1)\|_p &= \left\| \int_{M_1} K_1(x_1, y_1) F(y_1)\, \mu_1(dy_1) \right\|_p \\
&\le \int_{M_1} K_1(x_1, y_1) \|F(y_1)\|_p \mu_1(\,dy_1) = (\mathbf{T}_1(\|F\|_p)))(x_1).
\end{aligned}
$$

This inequality together with the contraction property of $\mathbf{T}_1$ yields the following estimate on the $L_q(M_1, \mu_1)$-norm of the function $\|G(x_1)\|_p$ with arguments $x_1 \in M_1$ by the $L_p(M_1, \mu_1)$-norm of the function $\|F(y_1)\|_p$ with arguments $y_1 \in M_1$.

$$
\| \|G(\cdot)\|_p \|_q \le \|\mathbf{T}_1(\|F(\cdot)\|_p)\|_q \le \| \|F(\cdot)\|_p \|_p .
$$

Let us also observe that since $\|F(y_1)\|_p = \left(\int_{M_2} |f(y_1, y_2)|^p \mu_2(\,dy_2)\right)^{1/p}$ for almost all $y_1 \in M_1$

$$
\begin{aligned}
\| \|F(\cdot)\|_p \|_p &= \left(\int_{M_1} \left(\int_{M_2} |f(y_1, y_2)|^p \mu_2(\,dy_2) \right) \mu_1(\,dy_1) \right)^{1/p} \\
&= \left(\int_{M_1 \times M_2} |f(y_1, y_2)|^p \mu_1(\,dy_1) \mu_2(\,dy_2) \right)^{1/p} = \|f(y_1, y_2)\|_p.
\end{aligned}
$$

Define the function $u(x_1, y_2) = \int_{M_1} K_1(x_1, y_1) f(y_1, y_2) \mu_1(\,dy_1)$ which equals $(G(x_1))(y_2)$ for all $x_1 \in M_1$ and $y_2 \in M_2$, and let us estimate the number

$$
S(u(\cdot)) = \left(\int_{M_1 \times M_2} \left| \left(\int_{M_2} K_2(x_2, y_2) u(x_1, y_2) \mu_2(\,dy_2) \right) \right|^q \mu_1(\,dx_1) \mu_2(\,dx_2) \right)^{1/q} .
$$

The application of the contraction property of $\mathbf{T}_2$ for $f(x_2) = u(x_1, x_2)$ with a fixed $x_1 \in M_1$ yields the estimate

$$
\begin{aligned}
&\int_{M_2} \left| \left(\int_{M_2} K_2(x_2, y_2) u(x_1, y_2) \mu_2(\,dy_2) \right) \right|^q \mu_2(\,dx_2) \le \left(\int_{M_2} |u(x_1, x_2)|^p \mu_2(\,dx_2) \right)^{q/p} \\
&\quad = \left(\int_{M_2} |(G(x_1))(x_2)|^p \mu_2(\,dx_2) \right)^{q/p} = \|G(x_1)\|_p^q.
\end{aligned}
$$

Hence

$$S(u(\cdot)) \leq \left(\int_{M_1} \|G(x_1)\|_p^q \mu_1(\,dx_1)\right)^{1/q} = \|\,\|G(\cdot)\|_p\|_q \leq \|\,\|F(\cdot)\|_p\|_p = \|f(y_1, y_2\|_p.$$

On the other hand,

$$S(u(\cdot)) = \left(\int_{M_1\times M_2} \left|\left(\int_{M_1\times M_2} K_2(x_2,y_2)K_1(x_1,y_1)f(y_1,y_2)\mu_1(\,dy_1)\mu_2(\,dy_2)\right)\right|^q \right.$$
$$\left.\mu_1(\,dx_1)\mu_2(\,dx_2)\right)^{1/q} = \|(\mathbf{T}_1 \times \mathbf{T}_2)f(y_1,y_2)\|_q,$$

and the statement of the lemma holds.

Here I write down Segal's result in its original form. I present both its formulation and its proof.

LEMMA 1.4. Let M_i $(i = 1,\dots,n)$ be separable measure spaces, and let $\mathbf{T}_i$ be an integral operator on $L_p(M_i))$ with positive kernel K_i, which is a contraction from $L_p(M_i)$ to $L_q(M_i)$, for certain given p and q for each i. Then the algebraic tensor product $\mathbf{T}_1 \times \cdots \times \mathbf{T}_n$ is a contraction from $L_p(M_1 \times \cdots \times M_n)$ to $L_q(M_1 \times \dots M_n)$.

PROOF. It suffices by associativity to treat the case $n = 2$. Now if $\mathbf{B}$ is any separable Banach space, and if $L_p(M_1, \mathbf{B})$ denotes the space of all strongly measurable $\mathbf{B}$-valued functions F on M_1 which are p-th power integrable, with the norm $\|F\| = \left(\int \|F(x)\|^p\,dx\right)^{1/p}$, then the operator $\mathbf{T}_1'\colon F \to G$, where $G(x) = \int K_1(x,y)F(y)\,dy$, exists and is a contraction from $L_p(M_1,\mathbf{B})$ to $L_q(M_1,\mathbf{B})$. For the mapping $y \to K_1(x,y)F(y)$ is easily seen to be strongly measurable from M_1 to $\mathbf{B}$, for each x; and

$$\|G(x)\| \leq \int K_1(x,y)\|F(y)\|\,dy,$$

i.e., $\|G(\cdot)\| \leq \mathbf{T}_1(\|F(\cdot)\|)$, so that $\|\|G(\cdot)\|_{\mathbf{B}}\|_q \leq \|\mathbf{T}_1(\|F(\cdot)\|)_{\mathbf{B}}\|_q \leq \|\|F(\cdot)\|_{\mathbf{B}}\|_p$. This shows the absolute integrability of the integral defining $G(x)$ almost everywhere, and gives the estimate $\|\mathbf{T}_1'\| \leq 1$.

In addition, the operation $\mathbf{T}_2''$ from $L_q(M_1,\mathbf{B})$ to $L_q(M_2,\mathbf{B}')$, where $\mathbf{B}'$ is another separable Banach space and $\mathbf{T}_2$ is a contraction from $\mathbf{B}$ to $\mathbf{B}'$, defined by the equation $(\mathbf{T}_2''F)(x) = \mathbf{T}_2F(x)$, $F \in L_q(M_1,\mathbf{B})$, is easily seen to be a contraction. Now taking $\mathbf{B}$ as $L_p(M_2)$ and $\mathbf{B}'$ as $L_q(M_2)$, and making the natural identifications of $L_p(M_1,\mathbf{B})$ with $L_p(M_1 \times M_2)$ and of $L_q(M_1,\mathbf{B}')$ with $L_q(M_1 \times M_2)$ which are justified by the Fubini theorem, it follows that the contraction $\mathbf{T}_2''\mathbf{T}_1'$ extends the algebraic tensor product $\mathbf{T}_1 \times \mathbf{T}_2$; the latter is therefore a contraction, as stated.