Extremal Subgraphs of Random Graphs

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ABSTRACT

We shall prove that if L is a 3-chromatic (so called "forbidden") graph, and

—R^n is a random graph on n vertices, whose edges are chosen independently, with probability p, and
—B^n is a bipartite subgraph of R^n of maximum size,
—F^n is an L-free subgraph of R^n of maximum size,

then (in some sense) F^n and B^n are very near to each other: almost surely they have almost the same number of edges, and one can delete $O_p(1)$ edges from F^n to obtain a bipartite graph. Moreover, with $p = \frac{1}{2}$ and L any odd cycle, F^n is almost surely bipartite.

Notation. Below we restrict our consideration to simple graphs: loops and multiple edges are excluded. We shall denote the number of edges of a graph G by $e(G)$, the number of vertices by $v(G)$, but superscripts will also denote the number of vertices; $G^n, R^n, T^{n,d}$ will always be graphs on n vertices. The set of vertices of a graph G is denoted by $V(G)$. The subgraph of a graph F spanned by a subset A will be denoted by $G_F(A)$, or simply by $G(A)$. The chromatic number of a graph G will be denoted by $\chi(G)$. The number of edges of a graph will be called "the size of the graph." If X and Y are disjoint vertex sets in a graph G, then $e_G(X,Y)$ denotes the number of edges joining X to Y, and $d_G(X,Y)$ denotes the edge-density between them:

$$d_G(X,Y) = \frac{e_G(X,Y)}{|X| \cdot |Y|}.$$

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In the case when X is just $\{x\}$, we shall write simply $e(x,Y)$. The number of edges in a subgraph spanned by a set X of vertices of G will be denoted by $e_G(X)$. We shall say that X is "completely joined" to Y if every vertex of X is joined to every vertex of Y. The set of neighbors of a vertex x will be denoted by $N_G(x)$. Sometimes we omit the subscript G.

Special Graphs. K_q will denote the complete graph on q vertices, $T^{n,d}$ is the so-called Turán graph with n vertices and d classes: n vertices are partitioned into d classes as uniformly as possible and two vertices are joined iff they belong to different classes. $K(n_1,\dots,n_d)$ denotes the complete d-partite graph with n_i vertices in its ith class, $i = 1,2,\dots,d$.

INTRODUCTION

Given a "forbidden graph" L, the corresponding Turán type extremal graph problem asks for determining the maximum number of edges a graph G^n can have without containing L (as a not necessarily induced subgraph). The maximum will be denoted by $\mathrm{ext}(n, L)$. A fairly extensive theory developed around extremal graph problems of this type (see [3], [24]).

The main problems we shall discuss in this paper will concern analogous problems for random graphs. This means that instead of trying to find a maximum size L-free subgraph of K_n we pick a random graph R^n and try to find a maximum size L-free subgraph F^n of this R^n. In the classical theory it turns out [15], that if $\chi(L) = q + 1$, then asymptotically the best graph is the q-chromatic Turán graph $T^{n,q}$:

$$\mathrm{ext}(n,L) = e(T^{n,q}) + o(n^2).$$

In case L is 3-chromatic, taking the bipartite subgraph of K_n of maximum size (i.e., the Turán graph $T^{n,2}$ with $[\frac{1}{4}n^2]$ edges) we get an asymptotically extremal graph (which is often not only asymptotically but exactly) extremal.

Motivated by this, for any fixed 3-chromatic "forbidden subgraph" L, we will determine the maximum number of edges an L-free subgraph of the random graph R^n can have. Our solution will be as follows. Take a random graph R^n, a maximum size bipartite subgraph B^n in it, a maximum size L-free subgraph $F^n \subseteq R^n$. We will show that they are very near to each other: one can delete a few edges of F^n to obtain a bipartite graph. This will give us the maximum number of edges the L-free subgraphs of R^n can have, and also will give a sufficiently good description of the structure of a "random extremal graph" F^n.

What is the connection between these two subgraphs? B^n is L-free, since $\chi(B^n) = 2$, and $\chi(L) = 3$. Thus $e(B^n) \le e(F^n)$. However, F^n could have many more edges. Our main results will show that this is not the case.

Above we have given one interpretation of our theorems to be formulated below. There is another natural interpretation. Many results of random graph theory suggest that if a random graph does not contain triangles, then we may be almost sure that it is bipartite. One such result is the Erdős–Kleitman–Rothschild theorem [12], asserting that the number of K_q-free graphs on n vertices and the number of $q-1$-chromatic graphs on n vertices are in logarithm asymptotically equal.

Our results also suggest that for random graphs being triangle-free is almost the same as being bipartite, and the same holds for any forbidden 3-chromatic L. (Moreover, analogous results hold for higher chromatic numbers.) Still, these results are not obvious: we shall give some examples of similar situations, where either the analogous theorem does not hold, or where we cannot prove it.

Random Graphs. In this paper we shall always use the binomial model for random graphs. We shall always fix a probability $p \in (0,1)$, independent of n and denote by $R^n \in \mathbf{G}(p)$ the fact that R^n is a random graph generated in "binomial way," that is, each edge is chosen with probability p and independently. The expected number of edges of R^n is $p\binom{n}{2}$, the variance is $p(1-p)\binom{n}{2}$ and for any fixed graph H^n the probability that $R^n = H^n$ depends only on $E = e(H^n)$: it is

$$p^E(1-p)^{\binom{n}{2}-E}.$$

We shall say "almost surely" if we mean that "with probability tending to 1, as $n \to \infty$."

Lower Bounds. If H is an arbitrary fixed graph on n vertices, then the expected number of edges common to H and R^n is $p \cdot e(H)$. Further, the standard deviation of this event is $O(\sqrt{p} \cdot n)$. Therefore for an arbitrary $\omega(n) \to \infty$ the number of edges common to H and R^n is between $p \cdot e(H) + \omega(n) \cdot n$ and $p \cdot e(H) - \omega(n)$, almost surely.

Let $\chi(L) = 3$. If we take an arbitrary random graph $R^n \in \mathbf{G}(p)$, and B^n is a bipartite subgraph of it, then it contains no L. So, if F^n is an L-free subgraph of maximum size, then $e(B^n) \le e(F^n)$. Since R^n almost surely contains a bipartite B^n of $\frac{1}{4}pn^2 + o(n^2)$ edges, we have

$$e(F^n) > \frac{1}{4}pn^2 - o(n^2), \tag{1}$$

almost surely. (We shall see that almost surely,

$$e(F^n) > \frac{1}{4}pn^2 + cn^{3/2} \tag{2}$$

for some constant $c > 0$.)

For "most" forbidden L's (where for the sake of simplicity assume that $\chi(L) = 3$), we can get a lower bound better than (1):

$$e(F^n) \geq p \cdot \text{ext}(n, L) - \omega(n) \cdot n \tag{3}$$

almost surely. Indeed, we may fix an arbitrary extremal graph S^n for L and, clearly, R^n will have $p \cdot \text{ext}(n, L) - \omega(n) \cdot n$ edges in common with the fixed S^n, almost surely, proving (3). In most cases $\text{ext}(n, L) > \frac{1}{4}n^2 + c_L n^{1+c}$, with positive constant c. Then (3) is really sharper than (1). Often $\text{ext}(n, L) > \frac{1}{4}n^2 + cn^{1+c}$ with some $c > \frac{1}{2}$. Then (3) is better even than (2). Some details can be found in later parts of this paper and a more detailed description of the corresponding extremal results in [24].

MAIN RESULTS

Below we formulate four theorems. Theorem 1 deals with the simplest case, namely, when $p = \frac{1}{2}$ and K_3 is excluded. Theorem 2 generalizes Theorem 1 to arbitrary 3-chromatic excluded graphs with "critical edges" (see the definition below). Theorem 4 describes the asymptotically extremal structure in the general case, i.e., when a 3-chromatic L is fixed, and though $L \subseteq F^n$ is not completely excluded, the graph F^n contains only a small number of copies of L. Theorem 3 yields a more accurate description of the exact extremal graphs, and may be needed for future applications.

Theorem 1. Let $p = \frac{1}{2}$. If R^n is a p-random graph and F^n is a K_3-free subgraph of R^n containing the maximum number of edges, and B^n is a bipartite subgraph of R^n having maximum number of edges, then

$$e(B^n) = e(F^n).$$

Moreover, F^n is almost surely bipartite.

Definition 1. (Critical edge) Given a k-chromatic graph L and an edge e in it, e is called critical if $L - e$ is $(k - 1)$-chromatic.

All the edges of a K_k and of any odd cycle are critical. Many theorems valid for complete graphs were generalized to arbitrary L having critical edges (see, e.g., [23]). Theorem 1 also generalizes to every k-chromatic L containing a critical edge e and to every probability $p > 0$.

Theorem 2. Let L be a fixed 3-chromatic graph with a critical edge e (i.e., $\chi(L - e) = 2$). There exists a function $f(p)$ such that if $p \in (0,1)$ is given, $R^n \in \mathbf{G}(p)$, and if B^n is a bipartite subgraph of R^n of maximum size and F^n is an L-free subgraph of the maximum size, then

$$e(B^n) \leq e(F^n) \leq e(B^n) + f(p) \tag{4}$$

almost surely, and we can delete $f(p)$ edges of F^n so that the resulting graph is already bipartite, almost surely. Furthermore, there exists a $p_0 < \frac{1}{2}$ such that if $p > p_0$, then F^n is almost surely bipartite: $e(F^n) = e(B^n)$.

The second part of Theorem 2 immediately implies Theorem 1. In connection with the first part one could ask, How large is $f(p)$ as $p \to 0$? We do not know the precise answer, just that Theorem 2 holds with $f(p) = O(p^{-4})$ or even $f(p) = O(p^{-3}\log p)$. As to the lower bound on $f(p)$, we do not know if

$$e(F^n) - e(B^n) \longrightarrow \infty \quad \text{as } p \longrightarrow 0.$$

In the second part of Theorem 2 we are not concerned with the exact value of the threshold probability p_0. Our main point is that the observed phenomenon is valid not just for $p = \frac{1}{2}$, but for some smaller (and for all the greater) values of p as well.

If $\chi(L) = 3$ but we do not assume that L has a critical edge, then we get similar results, having slightly more complicated forms. To formulate them we have to introduce the notion of the "decomposition family" of L [23].

Definition 2 (Decomposition family). Let $\chi(L) = 3$. The family $\mathbf{M}$ of all the spanned subgraphs $M \subseteq L$ such that $L - M$ is an independent set will be called the decomposition family of L.

Describing a decomposition family, it is enough to describe the minimal graphs in it.

Examples. If $L = K_3$, then $M = K_2$ is a minimum decomposition graph, and more generally, the same holds if L has a critical edge. Obviously, there are no other minimum decomposition graphs. If $L = K(t,t,t)$, then $M = K(t,t)$ is the only minimum decomposition graph. If L is the dodecahedron graph (on 20 vertices), then the graph consisting of 6 independent edges will be in the decomposition family. However, there are other minimum decomposition graphs too, e.g., if M is the union of two pentagons and 5 edges hanging from one of these pentagons, then M is also a minimum decomposition graph. We have mentioned that mostly $\text{ext}(n,L) > \frac{1}{4}n^2 + c_L n^{1+c}$, with some constants $c_L, c > 0$. Now we can tell that this occurs exactly if each decomposition graph contains a cycle.

We shall use—to simplify the form of our results—that for $\alpha > 1$,

$$\text{ext}(n,\mathbf{M}) \le \text{ext}(\alpha n,\mathbf{M}) \le (\alpha^2 + o(1)) \cdot \text{ext}(n,\mathbf{M}).$$

(The left inequality is trivial, the right one follows from [20], where—using a simple averaging argument—the authors showed that $\text{ext}(n,\mathcal{L})/\binom{n}{2}$ is decreasing.) It is known in extremal graph theory ([8],[22]) that if

$\chi(L) = 3$, then

$$\mathrm{ext}(n, L) = \frac{1}{4}n^2 + O(\mathrm{ext}(n, \mathbf{M})) + O(n). \tag{5}$$

Below we shall neglect the "ceiling signs." (In some sense (5) is sharp: putting an extremal graph $H^{n/2}$ into one class of a $T^{n,2}$ we get a graph G^n with

$$e(G^n) \geq \frac{1}{4}n^2 + \mathrm{ext}\left(\frac{1}{2}n, \mathbf{M}\right)$$

and not containing any L.) Now, taking a random R^n with edge probability p, we get almost surely

$$\geq p \cdot \left(\frac{1}{4}n^2 + \mathrm{ext}\left(\frac{1}{2}n, \mathbf{M}\right)\right) - O(n \log n)$$

edges common to R^n and G^n. Hence F^n must have at least this many edges. The next theorem asserts that it does not have essentially more edges.

Theorem 3. Let L be a given 3-chromatic graph. Let $p \in (0,1)$ be fixed and let $R^n \in \mathbf{G}(p)$. If B^n is a bipartite subgraph of R^n of maximum size and F^n is an L-free subgraph of maximum size, then almost surely

$$e(B^n) \leq e(F^n) \leq e(B^n) + 2\,\mathrm{ext}(n, \mathbf{M}) + O(n),$$

and we can delete $O(\mathrm{ext}(n, \mathbf{M})) + O(n)$ edges of F^n so that the resulting graph is already bipartite, almost surely.

Examples. In the proof of Theorem 2 we shall use the Kövári–T. Sós–Turán theorem [21] according to which (for $r \leq s$)

$$\mathrm{ext}(n, K(r,s)) \leq \frac{1}{2}\sqrt[r]{s-1}\, n^{2-1/r} + O(n). \tag{6}$$

Equation (6) is sharp for $r = 2,3$ (see [14],[5]). For $L = K(r,r,r)$ this yields $\mathrm{ext}(n, \mathbf{M}) < c_2 n^{2-1/r}$. If L is the dodecahedron graph, then $\mathrm{ext}(n, \mathbf{M}) = 5n + O(1)$.

Theorem 3 is meant to be "applied" primarily when $\frac{1}{n}\mathrm{ext}(n, \mathbf{M}) \to \infty$. The extreme case $\mathrm{ext}(n, \mathbf{M}) = 0$ is described by Theorem 2.

In many cases (a) excluding some L, or (b) assuming that there are only few copies of L in the considered G^n (or now in F^n), has the same effect in the results. This is the case e.g. in the Erdös–Kleitman–Rothschild theorem, or in the case of the Erdös–Simonovits theorem, or of the Ajtai–Erdös–Komlós–Szemerédi results [1]. And this is the case in our theorems, too.

Theorem 4. Let L be a fixed 3-chromatic graph. Let R^n be a p-random graph, B^n a maximum size bipartite subgraph of it, and F^n a subgraph of R^n with $o(n^{v(L)})$ subgraphs isomorphic to L and with

$$e(F^n) > e(B^n) - o(n^2)$$

edges. Then with probability tending to 1, there exists a partition $[A/B]$ of $V(G^n)$ into two sets A and B with $|A| = \frac{n}{2} + o(n)$ and $|B| = \frac{n}{2} + o(n)$ such that

$$e_F(A) = o(n^2), \qquad e_F(B) = o(n^2), \quad \text{and} \quad e_F(A, B) = \frac{1}{4}pn^2 + o(n^2).$$

All the results of this paper generalize to r-chromatic graphs as well. The formulation and proofs of the theorems are almost the same, though the results for $r > 3$ have more complicated forms. Hence we restrict our considerations to the case $r = 3$.

Remark. (A Third Interpretation). An alternative interpretation of the above theorems is that if $\chi(L) \geq 3$, then taking first a random graph R^n and then a maximal L-free subgraph $F^n \subseteq R^n$ is almost the same as taking first an extremal graph S^n and then taking the edges of S^n with probability p.

SOME RELATED EXAMPLES

One could think for a moment that theorems stating that "B^n and F^n are very near to each other" must have some deeper reason, and therefore there must be a much more general and more precise theorem in this field. The following three observations are to convince the reader that this is not quite so.

The first construction shows that there are random graphs of specific structure in which the maximum size triangle-free subgraphs and the maximum size bipartite subgraphs are far from each other.

Construction. Let us divide n vertices into 5 (almost) equal groups $C_1, \ldots, C_5$. For $i = 1, \ldots, 5$ join a vertex x in C_i to a vertex y in C_{i+1} with probability p. (By definition, $C_6 = C_1$.) Denote the resulting graph by Q^n. Now, for all the other pairs (x, y) join them with probability $q = \frac{p}{20}$. then we obtain a random graph R^n, in which we have to delete at least $\frac{1}{25}pn^2 + o(n^2)$

edges to make it bipartite (because we need at least that many edge deletion to turn Q^n into a bipartite graph). On the other hand, Q^n is triangle-free; therefore deleting all the other edges of R^n we can turn R^n into a triangle-free graph by just deleting $\leq q\binom{n}{2} + o(n^2)$ edges: in this case the maximum size triangle-free subgraph has definitely more edges than the maximum size bipartite subgraph.

The Path Theorem. By a theorem of Erdös and Gallai [11], if G^n contains no path P^m, then

$$e(G^n) < \frac{m-2}{2}n,$$

and the union of $n/(m-1)$ vertex disjoint K_{m-1} is asymptotically optimal. As Erdös pointed out, R^n contains at least $n/(m-1) - o(n/(m-1))$ vertex disjoint copies of K_{m-1}. Hence for $L = P^m$, $e(F^n)$ is asymptotically equal to $\mathrm{ext}(n, L)$ (instead of being around $p \cdot \mathrm{ext}(n, L)$).

This shows that in general the "third interpretation"—given at the end of the last paragraph—does not necessarily hold.

The C_4 Problem. Let $L = C_4$. Take an arbitrary fixed $p \in (0,1)$ and a p-random graph R^n. Let F^n be a C_4-free subgraph of maximum size. Clearly,

$$e(F^n) < \mathrm{ext}(n, C_4).$$

On the other hand, if S^n is a C_4-extremal graph on n vertices, then a p-random graph will contain at least

$$(p + o(1)) \cdot \mathrm{ext}(n, C_4)$$

edges of this S^n, showing that

$$e(F^n) \geq (p + o(1)) \cdot \mathrm{ext}(n, C^4).$$

Here we have a big gap (a factor of p) between the lower and upper bounds on $e(F^n)$ and "finding the truth" seems to be difficult.

LEMMATA

Our strategy is to get "self-improving information" on the structure of F^n: prove some estimates and then use them to obtain better estimates. Partly this is why we prove the theorems in reverse order. In the proofs we shall often delete the phrase "almost surely."

We shall estimate the "tail of the binomial distribution" by Chernoff's bounds [6].

Chernoff Bound. Let p be a probability and $\varepsilon_i = 1, 0$ with probabilities p and $q = 1 - p$, independently ($i = 1, \dots, h$). Let $c > p$ be a constant and $d = 1 - c$. Define the Chernoff function as

$$I(p,c) = c \log \frac{c}{p} + d \log \frac{d}{q}.$$

Then

$$P\left(\sum_1^h \varepsilon_i > ch \right) < e^{-I(p,c)h}.$$

Analogously, if $c < p$, then

$$P\left(\sum_1^h \varepsilon_i < ch \right) < e^{-I(p,c)h}.$$

Corollary (Folklore). For fixed p, if R^n is a binomial p-random graph and for some constant $c > \frac{1}{2}$, X, Y are two disjoint vertex sets of size $> n^c$, then almost surely

$$e_R(X,Y) = (p + o(1))|X|\,|Y|.$$

The proof is trivial. The proof of Theorem 3 will consist of two parts.

—First we prove that the F^n of Theorem 3 is almost bipartite. Namely, its vertices can be partitioned into two classes A and B of roughly equal size, with $e_F(A) = o(n^2)$, $e_F(B) = o(n^2)$. This is the content of the main lemma. Actually, the main lemma is an obvious weakening of Theorem 4. The content of the Randomness lemma is that the edges joining A and B behave in a "pseudorandom way."

—In the second part we apply a finer argument to F^n, and show that if A and B are chosen in the "best" way, then the edges in A (and in B) form a graph of bounded degree, and can be represented by a bounded number of vertices. This will imply that the number of edges in A and in B is $O_p(1)$, and will immediately imply Theorem 3.

Main Lemma. Let L be a fixed 3-chromatic graph. Let R^n be a p-random graph and F^n an L-free subgraph of R^n with

$$e(F^n) > \frac{1}{4}pn^2 - o(n^2)$$

edges. Then with probability tending to 1, there exists a partition $[A/B]$ of $V(G^n)$ into two sets A and B with $|A| = \frac{n}{2} + o(n)$ and $|B| = \frac{n}{2} + o(n)$ such that

$$e_F(A) = o(n^2), \qquad e_F(B) = o(n^2).$$

To prove this lemma we need the regularization lemma of Szemerédi [25].

Regularity Condition. Given a graph G^n and two disjoint vertex sets in it, X and Y, we shall call the pair (X, Y) η-regular if for every subset $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| > \eta|X|$ and $|Y'| > \eta|Y|$,

$$|d(X',Y') - d(X,Y)| < \eta.$$

The next lemma asserts that given an $\eta > 0$, the vertex-set of every G^n can be partitioned into a bounded number of classes so that almost all the pairs of classes will be η-regular.

Regularization Lemma [25]. For every $\eta > 0$, and integer k_0 there exists a k_η such that for every G^n $V(G^n)$ can be partitioned into sets $V_0, V_1, \ldots, V_k$—for some $k_0 < k < k_\eta$—so that each $|V_i| < \eta n$, $|V_i| = m$ (is the same) for every $i > 0$, and for all but at most $\eta \cdot \binom{k}{2}$ pairs (i, j) for every $X \subseteq V_i$ and $Y \subseteq V_j$, satisfying $|X|, |Y| > \eta m$, we have

$$|d(X,Y) - d(V_i,V_j)| < \eta.$$

(One interpretation of this lemma is that all the graphs of positive edge density can be approximated by random graphs. The role of V_0 is to make possible that all the other classes be exactly of the same size, and the role of k_0 is to make the classes V_i sufficiently small, so that we could forget about the edges inside those classes.)

To prove the main lemma we shall also use the following theorem of Erdös and Simonovits, (formulated in [7,8] and [22] in a much more general way):

Stability Lemma [7,22]. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $e(F^k) > (\frac{1}{4} - \delta)k^2$ and $K_3 \not\subseteq F^k$, then F^k can be obtained from the Turán graph $T^{k,2}$ by changing $<\varepsilon k^2$ edges in it.

Proof of the Main Lemma. Let $v = v(L)$. Let $\varepsilon > 0$ be fixed and apply the Szemerédi Regularization lemma to F^n, with $\eta = \varepsilon^v$, obtaining a partition $V_0, V_1, \ldots, V_k$ of the vertices as described above. Now we define a new graph H^k, called the "reduced graph," in which the vertices are the classes $V_i, i = 1, 2, \ldots, k$, and two classes V_i and V_j are joined by a "reduced edge" if (V_i, V_j) is a regular pair and

$$d(V_i,V_j) > 2\varepsilon.$$

Let $m = |V_i|, (i > 0)$. In (a) we shall show that $K_3 \not\subseteq H^k$. In (b) we shall show that all but $4\varepsilon n^2$ of the edges of F^n correspond to edges of H^k, i.e., join vertices of classes that are joined in the reduced graph. This will imply that

$$e(F^n) < p \cdot e(H^k)m^2 + 5\varepsilon n^2. \tag{7}$$

(a) First we show that if H^k contained a K_3, then F^n would contain a $K(v, v, v)$. Indeed, assume that the vertices of this K_3 are the classes W_1, W_2, and W_3. We choose recursively the vertices $x_1, \dots, x_j \in W_1$ and the sequence of nested subsets $W_{j,2} \subseteq W_{j-1,2} \subseteq W_2, W_{j,3} \subseteq W_{j-1,3} \subseteq W_3$ so that $|W_{j,2}|, |W_{j,3}| > \varepsilon^j m$ and each of the vertices $x_1, \dots, x_j$ is joined to each $y \in W_{j,2} \cup W_{j,3}$. By the assertion of the Szemerédi lemma, this can be done: for $j < v$,

$$d(W_1 - \{x_1, \dots, x_j\}, W_{j,2}) > \varepsilon \quad \text{and} \quad |W_{j,2}| > \varepsilon^v m.$$

Let $W_j^* \subseteq W_1$ be the set of those vertices that are joined to at least $\varepsilon^{j+1}m$ vertices of $|W_{j,2}|$. By the regularity, $|W_j^*| > \frac{1}{2}m$. By the same argument, there is an $x_{j+1} \in W_1 - \{x_1, \dots, x_j\}$ joined also to $W_{j,3}$ by $>\varepsilon^{j+1}m$ edges. Let

$$W_{j+1,2} = N_F(x_{j+1}) \cap W_{j,2} \quad \text{and} \quad W_{j+1,3} = N_F(x_{j+1}) \cap W_{j,3}.$$

So we can fix the nested sequence of subsets in W_2 and W_3 as stated above. Further,

$$d(W_{v,2}, W_{v,3}) > \varepsilon.$$

Hence, if n is sufficiently large, then (by the Kővári–T. Sós–Turán theorem) we can find a $K(v, v)$ between $W_{v,2}$ and $W_{v,3}$. They and $\{x_1, \dots, x_v\}$ form a $K(v, v, v) \subseteq F^n$. This contradiction proves that H^k contains no triangles.

(More generally, let L be an arbitrary fixed graph of v vertices and t be a positive integer. If one applies Szemerédi's regularization lemma to an arbitrary graph F^n with $\eta = \varepsilon^{vt}$ and constructs H^k as above, then the following "blowing up" principle holds. Let $L \circ I^t$ be the graph obtained by replacing each vertex of L by a t-tuple and joining every vertex of a t-tuple to every one of an other t-tuple iff the original vertices of L were joined. Now, if n is large and $L \subseteq H^k$, then $L \circ I^t \subseteq F^n$. The proof goes by induction on vt, see, e.g., [18].)

(b) Since $K_3 \not\subseteq H^k$, we can apply Turán's theorem:

$$e(H^k) \leq \frac{1}{4}k^2. \tag{8}$$

By the corollary, each "reduced edge" of H^k corresponds to at most $pm^2 + o(m^2)$ edges of F^n. This yields

$$\frac{1}{4}pm^2k^2 + o(n^2) = \frac{1}{4}pn^2 + o(n^2)$$

edges. The remaining edges

—either join vertices of the same V_i,
—or a vertex of V_0 to some other vertex,
—or correspond to a low-density $d(V_i, V_j)$,
—or to a non-regular pair (V_i, V_j).

We can estimate the number of edges joining vertices of the same classes by

$$\frac{n}{m}\binom{m}{2} \leq \frac{1}{2}nm \leq \frac{1}{2}\varepsilon n^2.$$

The number of edges represented by V_0 is $\leq |V_0|n \leq (\varepsilon n)n = \varepsilon n^2$. Clearly, the low-density pairs (V_i, V_j) contribute $<2\varepsilon\binom{n}{2} \leq \varepsilon n^2$ edges. Finally, the nonregular pairs give at most $\varepsilon\binom{k}{2}m^2 \leq \frac{1}{2}\varepsilon n^2$ edges. This proves (7).

Comparing (1) and (7) we get that in (8) we must have almost equality:

$$e(H^k) > \left[\frac{1}{4}k^2\right] - \frac{8}{p}\varepsilon k^2.$$

Since $K_3 \not\subseteq H^k$, we can apply the stability lemma to H^k: there is a function $\gamma \to 0$ (if $\varepsilon \to 0$) such that $V(H^k)$ can be partitioned into two classes A_R and B_R, with $|A_R|, |B_R| \leq \frac{k}{2} + \gamma k$; further, $e_H(A_R) < \gamma k^2$, $e_H(B_R) < \gamma k^2$. Define A as the union of the sets V_i in A_R, and B as the union of the V_i's in B_R. If we delete all the edges of F^n joining sets V_i in A_R and the edges joining vertices of the same V_i for some $i = 1, \ldots, k$, and the edges not corresponding to reduced edges and the edges represented by V_0, then we deleted all the edges in A, by deleting $<(5\varepsilon + \gamma)n^2$ edges. Similarly, we can delete $<(5\varepsilon + \gamma)n^2$ edges to turn B into an independent set. This proves the main lemma, if $\varepsilon \to 0$.

PROOFS OF THE THEOREMS

Proof of Theorem 4 (Sketched). The proof of Theorem 4 is almost word by word the same as the proof of the main lemma. The only change is that now we have to show that

(*) if the reduced graph contained a K_3, then F^n would contain $cn^{e(L)}$ forbidden subgraphs L.

This would be almost trivial if the edges were picked at random, with some fixed probability $p > 0$, to join three classes V_i, V_j, and V_h. The proof is slightly more complicated but still just a standard argument if we know that three classes V_i, V_j, and V_h of H^k form a triangle in the reduced graph. One could easily give a—somewhat longer—self-contained proof of (*), using only the Szemerédi lemma, however, below we shall provide a proof based on the "theory of supersaturated graphs."

First we settle the simplest case, when $L = K_3$ (leaving some details to the reader). We can argue as follows. We call a vertex $x \in V_i$ typical, if it is joined to V_j by at least $(d(V_i, V_h) - \varepsilon^v)m$ edges and the analogous statement for V_h also holds. By the regularity, all but $2\varepsilon^v m$ points of V_i are "typical." For a "typical" vertex $x \in V_i$

$$|N_F(x) \cap V_j| > \frac{1}{2}pm \quad \text{and} \quad |N_F(x) \cap V_h| > \frac{1}{2}pm.$$

By the regularity,

$$e(N_F(x) \cap V_j, N_F(x) \cap V_h) > 2\varepsilon\left(\frac{1}{2}pm\right)^2 - \varepsilon^v m^2.$$

Hence a "typical" x is contained in at least $\frac{1}{4}\varepsilon p^2 m^2 > c_1 n^2$ triangles: there are at least $c_2 n^3$ copies of K_3 in F^n.

We shall apply the following Corollary 2 of [16].

Theorem on Supersaturated Hypergraphs. Let $K_h^h(t, \ldots, t)$ be the h-uniform hypergraph obtained by taking h disjoint t-tuples $X_1, \ldots, X_h$ and all those h-tuples that contain one vertex from each X_i. For every $c > 0$ there exists a $c' > 0$ such that if an h-uniform hypergraph contains at least cn^h hyperedges, then it contains at least $c'n^{ht}$ copies of $K_h^h(t, \ldots, t)$.

Put $h = 3$ and take a t for which $K(t, t, t) \supseteq L$. Apply the above theorem to the 3-uniform hypergraph of the triangles of F^n. We have at least $c_2 n^3$ triangles, therefore at least $c_3 n^{3t}$ copies of $K_3^3(t, t, t)$. Therefore we have at least $c_3 n^{3t}$ copies of $K(t, t, t)$ in F^n. Each L is contained in at most $c_4 n^{3t - v(L)}$ copies of $K(t, t, t)$. Thus we must have at least $c_5 n^{v(L)}$ copies of L in F^n. (These type of arguments are standard in papers on supersaturated graphs, see, e.g., [16], [5]).

This contradicts the assumption of Theorem 4. Hence $K_3 \not\subseteq H^k$. From here the proof is the same as above.

Randomness Lemma. Let us fix a probability $p \in (0, 1)$, a constant $c \in (0, 1)$, and an integer k. Then a random binomial graph R^n with edge-probability p (and vertex set V) has the following property almost surely:

Let $m > cn$ and

$$t = 2 \log \frac{en}{m} \Big/ I(p^k, cp^k) = O(1).$$

For every subset $U \subseteq V$ of m vertices there exists a set $Q = Q_U \subset V$ of at most $tk = O(1)$ vertices such that every k-tuple of $V - Q - U$ is completely joined to at least $cp^k|U|$ vertices of U.

The meaning of this lemma is that in R^n, fixing a large set U and k vertices $x_1, \ldots, x_k \in V - U$, the expected number of common neighbors of the x_i's in U is $p^k|U|$. The lemma says that though there are many sets U and k-tuples outside, still large deviations from $p^k|U|$ are highly improbable.

The constant c in the lower bound on m and in cp^k do not have to be the same.

Proof of the Randomness Lemma. Assume that R^n is a p-random graph, $c \in (0,1)$ and k are fixed. Given a set U of m elements, a k-tuple $\{x_1, \ldots, x_k\}$ will be called "violating" if U contains fewer than $cp^k m$ common neighbors of these x_i's. The expected number of common neighbors of $x_1, \ldots, x_k$ in U is $p^k m$. By Chernoff inequality, the probability that for a given U, $\{x_1, \ldots, x_k\}$ is "violating" is $<e^{-I(p^k, cp^k)m}$. Since the probabilities of the violations for disjoint sets are independent, the probability that for a fixed U, t given vertex disjoint k-tuples are "violating" is $<e^{-t \cdot I(p^k, cp^k)m}$. Since t k-tuples can be chosen in $<n^{tk}$ ways, therefore the probability there are t vertex disjoint violating k-tuples, is $<n^{tk}e^{-tI(p^k, cp^k)}m$. The m-element subset U can be chosen in

$$\binom{n}{m} < \left(\frac{ne}{m}\right)^m$$

ways, hence the probability of the existence of a U and t disjoint violating k-tuples is

$$<e^{tk \log n + m(\log(en/m) - t \cdot I(p^k, cp^k))}.$$

If $t < c \log n$, then the n^{tk} term is negligible: for

$$t = 2 \log \frac{en}{m} \Big/ I(p^k, cp^k)$$

the above probability is $o(1)$. (And therefore for $t > \log n$ it is also $o(1)$.)

Hence for each U we can find a set Q_U of size tk so that in $V(F^n) - Q_U - U$ there are no violating k-tuples. ∎

Proof of Theorem 3. (A) We start with some general remarks. If every $M \in \mathbf{M}$ contains a cycle, then there exists a $\gamma > 1$ such that $\mathrm{ext}(n, \mathbf{M}) >$

n^γ if n is sufficiently large. (See e.g., [24].) In this case the $O(n)$ terms are negligible. If $\mathbf{M}$ contains a tree or a forest, then $\operatorname{ext}(n, \mathbf{M}) = cn + o(n)$, where in some cases $c > 0$, in some others $c = 0$. In this second case one can easily see that for some α and β the decomposition family contains both a $K(1, \alpha)$ and a graph of β independent edges. One can easily see that in these cases $\operatorname{ext}(n, \mathbf{M}) = O(1)$. Below the "linear" and "sublinear" cases shall also be covered; however, the reader should primarily concentrate on the "superlinear" cases.

By the main lemma, there exists a partition $[A/B]$, such that

$$e_F(A) = o(n^2) \quad \text{and} \quad e_F(B) = o(n^2).$$

We shall call a partition $[A/B]$ optimal if $e_F(A, B)$ attains its maximum. Fix an optimal partition $[A/B]$. By the optimality, for each $x \in A$,

$$e_F(x, A) \leq e_F(x, B).$$

An edge will be called "horizontal" if it joins two vertices of the same class, and we shall call an edge of R^n "missing" if it joins A and B and is not in F^n.

Let $v = v(L)$. We fix an $\varepsilon < \frac{1}{50v}p^v$ and choose n so large that the $o(n), o(n^2)$ terms below are "negligible" compared to $\varepsilon n, \varepsilon n^2$. (Later $\varepsilon \to 0$.)

Exceptional Vertices. We apply the randomness lemma first to $U = A$, $c = \frac{1}{2}$, and to the v-tuples in B, thus obtaining that there exists a subset X_B of size $O(1)$, such that all the v-tuples of $B - X_B$ have at least $\frac{1}{4}p^v n$ common neighbors in A. Next we apply the lemma to B and the v-tuples in A, obtaining an $X_A \subset A$. The vertices in $X_a \cup X_B$ will be called exceptional.

Now we show that if t is the number of vertices in A, joined—in F^n—to A by more than εn edges, then $t = O_\varepsilon(1)$. More specifically, $t < 2v/\varepsilon^2$, if n is sufficiently large.

To prove this we shall assume that $x_1, \ldots, x_t$ are these vertices, and we shall define some configurations called "flowers" and count them in two different ways. Consider the triangles in F^n one vertex of which is an x_i, and the opposite edge joins A to B. A flower is an edge $e = (a, b)$ with v such triangles (abx_i) on it $(a \in A, b \in B)$. The edge e will be called "center-edge," the v other vertices of the triangles form the "blossoms." So first let us count the triangles (abx_i) in F^n, $a \in A, b \in B$. Each of the t vertices $x_1, \ldots, x_t$ is joined to both A and B by at least εn edges. If all the edges of R^n were present in F^n, then each x_i would be roughly in $\varepsilon^2 pn^2$ triangles. At most $o(n^2)$ edges are missing, hence we have at least $\frac{2}{3}t\varepsilon^2 pn^2$ triangles. Now we count the v-flowers. If $\sigma(e)$ denotes the number of triangles on the edge e, then

$$\sum_e \sigma(e) > \frac{2}{3} t\varepsilon^2 pn^2.$$

Clearly, e yields $\binom{\sigma(e)}{v}$ flowers. Thus the total number of flowers is

$$N = \sum_e \binom{\sigma(e)}{v}.$$

Since $e(F^n) < \frac{2}{3}pn^2$, on the average we have $>\varepsilon^2 t$ triangles per edge. We extend the definition of $\binom{x}{v}$ to all the reals:

$$\binom{x}{v} = \begin{cases} x(x-1)\dots(x-v+1)/v! & \text{if } x > v-1; \\ 0, & \text{otherwise} \end{cases}$$

One can easily see that this function is convex. This yields that the number of flowers is

$$N = \sum \binom{\sigma(e)}{v} \geq c_1 n^2 \binom{\varepsilon^2 t}{v}.$$

On the other hand,

$$N < c_2 n^{2-1/v} \binom{t}{v},$$

since the blossoms of a flower can be chosen only in $\binom{t}{v}$ ways, and for each choice we have at most $c_3 n^{2-1/v}$ center-edges: otherwise, by [21], we could find a fixed v-tuple $\{x_1, \dots, x_v\}$ and a $K(v, v)$ outside, so that all the edges of $K(v, v)$ would be center-edges of a flower with that very v-tuple (as "blossoms"). This would yield a $K(v, v, v) \subseteq F^n$, a contradiction.

Assume (indirectly) that $t \geq 2v/\varepsilon^2$. Then (9) and (10) imply

$$c_4 n^{-1/v} \geq \binom{\varepsilon^2 t}{v} \bigg/ \binom{t}{v} \geq \frac{(\varepsilon^2 t - v)^v}{t^v} \geq \left(\varepsilon^2 - \frac{1}{2}\varepsilon^2\right)^v = \frac{\varepsilon^{2v}}{2^v} \qquad (11)$$

The left-hand side of (11) converges to 0 as $n \to \infty$. Therefore for $n > n_0$ we get $t \leq 2v/\varepsilon^2$.

(B) Denote by Y_A (and respectively, by Y_B) the set of vertices that are either in X_A (in X_B) or are joined to at least εn vertices of their own class. We wish to prove that the subgraphs $M \subseteq G(A - Y_A)$ and $M \subseteq G(B - Y_B)$ (where $M \in \mathbf{M}$) can be represented by $q < O(\frac{1}{n}\text{ext}(n, \mathbf{M})) + O(1)$ vertices.

Assume that, e.g., $A - Y_A$ contains some M's. Fix in $A - Y_A$ a maximum vertex-independent set of subgraphs $M_1, \dots, M_a \in \mathbf{M}$. We shall prove that if a is large then the number e_M of edges joining A and B in R^n but missing from F^n is so large that F^n cannot be maximal. Denote by S_A the number of horizontal edges incident with the vertices of these subgraphs. By the randomness lemma, for every M_i there are $\geq \frac{1}{4}p^v n$ vertices in B joined to M_i completely (in R^n). Denote this set by B_i. ($B_i = B \cap (\cap N_R(x_i))$.) Any v vertices $z_1, \ldots, z_v \in B_i$, form an L with M_i. Hence for any $z_1, \ldots, z_v \in B_i$, at least one edge joining them in R^n to M_i is missing from F^n. In other words, all but at most $v - 1$ vertices $z \in B_i$ are joined to M_i (in R^n) by at least one "missing edge." Thus each M is incident with at least $|B_i| - v > \frac{1}{5}p^v n$ "missing edges." Therefore $e_M \geqslant a \cdot \frac{1}{5}p^v n$. At the same time, each M is incident with at most $\varepsilon v n$ horizontal edges, since it does not intersect Y_A. Hence, if the vertices of these a disjoint M's represent S_A horizontal edges, then $a > S_A/(\varepsilon v n)$. Therefore $e_M \geq a \cdot \frac{1}{5}p^v n \geq S_A/(\varepsilon v n) \cdot \frac{1}{5}p^v n \geq 5S_A$ missing edges are incident with these vertices, since $\varepsilon < \frac{1}{25v}p^v$.

If we fix a maximum family of M's in $B - Y_B$ and S_B denotes the number of horizontal edges incident with them, then—by symmetry—we may assume that $S_A \geq S_B$.

Deleting all the edges of $G(A)$ and $G(B)$ and adding all the e_M missing edges, we get a bipartite Z^n not containing L. Therefore $e(Z^n) \leq e(F^n)$. Here $e(G(A)) \leq S_A + c_6 n + \text{ext}(n, \mathbf{M})$, since deleting all the $\leq S_A + |Y_A|n$ edges incident to the M_i's and the vertices in Y_A we get a graph not containing any $M \in \mathbf{M}$. Similarly, $e(G(B)) \leq S_B + c_6 n + \text{ext}(n, \mathbf{M})$. On the other hand, $e_M > 5S_A$. Hence

$$0 \leq e(F^n) - e(Z^n) = e(G(A)) + e(G(B)) - e_M \leq 2S_A + 2c_6 n + 2\text{ext}(n, \mathbf{M}) - 5S_A. \tag{12}$$

Therefore

$$S_B \leq S_A < \text{ext}(n, \mathbf{M}) + c_6 n,$$

and

$$e_M \leq e(G(A)) + e(G(B)) \leq 4\text{ext}(n, \mathbf{M}) + 4c_6 n.$$

Consequently,

$$q < va = O(e_M/p^v n) < O\left(\frac{1}{n}\text{ext}(n, \mathbf{M})\right) + O(1). \quad \blacksquare$$

Proof of Theorem 2. We start with some obvious inequalities, valid for any random graph R^n. Clearly, for every vertex the degree $d_R(x) = pn - o(n)$, almost surely. Further, for every pair of vertices x and y, almost surely $|N_R(x) \cap N_R(y)| = p^2 n + o(n)$. Define

$$\Delta(x, y) = (N_R(x) - N_R(y)) \cup (N_R(y) - N_R(x)).$$

We know that in a random R^n,

$$|\Delta(x, y)| = 2(p - p^2)n + o(n). \tag{13}$$

Let $v = v(L)$. Let $T^{qv,q,1}$ be the graph obtained from $K_q(v, \dots, v)$ by adding one edge to it. In case when L has a critical edge, clearly, $L \subset T^{2v,2,1}$. Thus F^n contains no $T^{2v,2,1}$ either. (As a matter of fact, a $(q + 1)$-chromatic L has a critical edge iff it is contained in a $T^{qv,q,1}$.)

(A) Again, fix an $\varepsilon > 0$ and let n be so large that the $o(\dots)$ terms be negligible. We shall prove that the maximum degree $d_F(G(A)) = o(n)$. This will imply that $X_A = Y_A$ in the previous proof: in any "optimal" partition, if $x \in A$, then $|N_F(x) \cap A| \leq \varepsilon n$. (Later we shall see that $d_F(G(A)) = O(1)$.)

Assume that $x \in A$, $A' = A \cap N_F(x)$, $B' = B \cap N_F(x)$, and $|A'| \geq \varepsilon n$. By the optimality of the partition, $|B'| \geq |A'| \geq \varepsilon n$. Clearly, $e_R(A', B') \geq p(\varepsilon n)^2 + o(n^2)$. If F^n contained a $K(v, v)$ with one class in A' and the other in B', then x and this $K(v, v)$ would yield a $T^{2v+1,2,1} \subseteq F^n$. Thus $e_F(A', B') = O(n^{2-1/v}) = o(n^2)$, by [21]: almost all of these edges are missing from F^n.

Let Z^n be the graph obtained from F^n by deleting all the horizontal edges and then adding the $p(\varepsilon n)^2 + o(n^2)$ missing edges of R^n (joining A' to B'). Since $e_F(A) = o(n^2)$, $e_F(B) = o(n^2)$ and $Z^n \subseteq R^n$ is also L-free, therefore

$$0 \leq e(F^n) - e(Z^n) = e_F(A) + e_F(B) - e_F(A', B') < -\frac{1}{2}p(\varepsilon n)^2,$$

a contradiction. This proves that $d_F(G(A)) = o(n)$ and $Y_A = X_A$.

(B) What does the argument (B) of the previous proof yield now? Let M_0 be the graph with $v(M_0) = v$ and $e(M_0) = 1$. If $\mathbf{M}$ is the decomposition class of L, then $M_0 \in \mathbf{M}$. Therefore $\operatorname{ext}(n, \mathbf{M}) = 0$ (if $n \geq v$). As we saw in the previous proof, the edges in $A - X_A$ can be represented by $O(1)$ vertices. Putting these $O(1)$ vertices into X_A, and into X_B on the other side, we achieve that $A - X_A$ (and $B - X_B$) contain no edges.

(C) Let us count the number of edges in A. We know that these edges are incident with the $O_p(1)$ vertices in $X_A \cup X_B$. To prove (4) it is enough to show that the horizontal degree $|N_F(x) \cap A| = O_p(1)$ for each $x \in X_A$. Assume that we have an $x \in X_A$ and $y_1, \dots, y_t \in A \cap N_F(x)$. We apply the randomness lemma to the set $U = B \cap N_F(x)$ and the vertices (i.e., 1-tuples) y_i. Clearly, $|U| = \frac{p}{2}n + o(n)$. If t is large enough, at least one y_i is joined in R^n to $>\frac{1}{4}p^2n$ vertices of U. Since $e_M \leq e(G(A)) + e(G(B)) < (|X_A| + |X_B|)\varepsilon n < \frac{1}{8}p^2n$, assumed that ε is small enough, hence

$$|N_F(y_i) \cap U| > \frac{1}{8}p^2n = c_7 n.$$

We pick v such vertices: $z_1, \ldots, z_v \in N_F(x) \cap N_F(y_i) - X_B$. They are joined to (x, y_i) completely in F^n and are outside X_B. Hence we can apply the randomness lemma again, to this v-tuple and $U' = A - X_A$, obtaining $v - 2$ further vertices $w_1, \ldots, w_{v-2} \in A$, completely joined to $\{z_1, \ldots, z_v\}$. They yield a $T^{2v,2,1} \subseteq F^n$, a contradiction. Hence (4) is proved. Now, by (12), we know that deleting $O_p(1)$ appropriate edges we get a bipartite graph, and also we know that $e_M = O_p(1)$.

(D) Now we prove that there exists a $\frac{2}{5} < p_0 < \frac{1}{2}$ such that for every $p > p_0$, almost surely F^n is bipartite. Here we have to use a finer argument. First we sketch the proof, carried out in (i)–(iv).

On the one hand, we shall show that for an optimal partition $[A/B]$

$$e_R(A, B) > \frac{p}{4}n^2 + c_8 n^{3/2}, \tag{14}$$

that is, we have noticeably more edges across, than expected. (This is again a property of every random R^n.)

On the other hand, using that $T^{2v,2,1} \not\subseteq F^n$, we shall show that unless this partition is a 2-coloring of F^n, there must exist two vertices x and y so that, say, apart from a small error (of δn vertices), $B = \Delta(x, y)$ and $A = V(R^n) - \Delta(x, y)$. More precisely, for every $\delta > 0$, if $p > p(\delta) = \frac{1}{2} - \frac{\delta}{10}$, then

$$|\Delta(x, y) - B| + |B - \Delta(x, y)| < \delta n. \tag{15}$$

The probability of (14) for any fixed partition is exponentially small and the existence of partitions satisfying (14) is highly probable only because there are exponentially many partitions. However, the number of partitions satisfying (15) is much smaller; therefore the probability of a partition satisfying both (14) and (15) will be negligible. Hence the optimal partition will give a 2-coloring, almost surely.

(i) First we show that for the optimal partition $[A/B]$ (14) holds with some appropriate constant $c_8 > 0$. To prove this we shall make use of the following purely probability theoretical assertion.

If we fix two numbers α and β with $\alpha + \beta = 1$, and set out with $X = \emptyset$, $Y = \emptyset$, and put an element e_i with probability $p\alpha$ into the set X, and with probability $p\beta$ into Y, and into none of them with probability $1 - p$, for $i = 1, 2, \ldots, m$, then there is a constant $c(p) > 0$ such that the probability $\text{Prob}(\||X| - |Y|\| > c(p)\sqrt{m}) > c(p)$. The reason for this is that the standard deviation of the binomial distribution of m events is $\sqrt{p(1 - p)m}$. The details are left to the reader.

Let the vertices of our random graph be $x_1, x_2, \ldots, x_n$. We may regard R^n as a random graph, generated in $n - 1$ passes, where in the ith pass we decide for $j = 1, \dots, i-1$ if (x_i, x_j) is an edge or not. We build up the sets (A_i, B_i) as follows: $A_1 = B_1 = \varnothing$. In the ith step ($i \geq 2$) we check whether $e_R(x_i, B_{i-1}) < e_R(x_i, A_{i-1})$ or not. According to the result we put x_i into B_i or A_i, respectively: either

$$B_i = B_{i-1} \cup \{x\} \quad \text{and} \quad A_i = A_{i-1}$$

or

$$A_i = A_{i-1} \cup \{x\} \quad \text{and} \quad B_i = B_{i-1}.$$

Let $d_i = e_R(x, A_{i-1} \cup B_{i-1})$, and δ_i be the number of edges joining x_i to the "other" class. Then $\delta_i \geq \frac{1}{2}d_i$, and for $i > \frac{n}{2}$, with some fixed positive probability p_1 we have $\delta_i > \frac{1}{2}d_i + c'\sqrt{n}$. These events are independent for $i = 2, \dots, n$. Hence, apart from an exponentially unlikely event, we get

$$e_R(A_n, B_n) = \sum \delta_i > \sum \frac{1}{2}\left(d_i + \frac{1}{2}p_1 \cdot c'\sqrt{i}\right) > \frac{1}{4}pn^2 + c_8 n^{3/2}.$$

(ii) We show that for every x the degree

$$d_F(x) \geq \frac{1}{2}pn - o(n). \tag{16}$$

Delete the $O(1)$ horizontal edges in $F^n - x$, and put x into $A - x$ or $B - x$, according to whether $e_R(x, B - x)$ or $e_R(x, A - x)$ is the larger. Thus we deleted $\leq d_F(x)$ edges but added at least $\frac{1}{2}pn - o(n)$ edges, getting a bipartite Z^n with $0 \leq e(F^n) - e(Z^n) \leq d_F(x) - \frac{1}{2}pn + o(n)$.

(iii) We show that if (x, y) is a horizontal edge in A, then all but $O_p(1)$ of the vertices in $N_R(x) \cap N_R(y)$ belong to A. Indeed, if v of them, $\{z_1, \dots, z_v\}$ belonged to $B - X_B$ and none of the edges $(xz_i), (yz_i)$ ($i = 1, \dots, v$) were among the $O_p(1)$ missing edges (i.e., all they belonged to F^n), then we could find $w_1, \dots, w_{v-2} \in A - X_A$ completely joined to $\{z_1, \dots, z_v\}$. Thus $x, y, w_1, \dots, w_{v-2}$ and $\{z_1, \dots, z_v\}$ would span a $T^{2v,2,1} \subseteq F^n$, a contradiction. Thus all but $O_p(1)$ vertices of $N_R(x) \cap N_R(y)$ belong to A.

At this point for $p > \frac{1}{2}$ it is trivial that F^n is bipartite: if (x, y) were an edge in A, then

$$\begin{aligned} d_F(x) &\leq |N_R(x) \cap B| + O_p(1) \leq d_R(x) - |N_R(x) \cap A| + O_p(1) \\ &\leq d_R(x) - |N_R(x) \cap N_R(y)| + O_p(1) \leq (p - p^2)n + o(n), \end{aligned}$$

and $p - p^2 < \frac{1}{2}p$, contradicting (16).

In the other case we need a more involved argument.

(iv) Let now $\frac{2}{5} < p < \frac{1}{2}$. We shall prove that if there is a horizontal edge (x, y) in the optimal partition $[A/B]$,

$$x \in A, \qquad y \in A, \qquad (x, y) \in F^n, \tag{17}$$

then

$$e_F(A, B) < \frac{p}{4}n^2 + c_\delta n^{3/2},$$

where $c_\delta \to 0$ as $\delta \to 0$. Obviously, this will prove that $e(A) = e(B) = 0$. Our aim is to prove that (x, y) determines the partition $[A/B]$ up to δn vertices: for every $\delta > 0$, if $p > p_0 = \frac{1}{2} - \frac{\delta}{10}$, then (15) holds. Since $N_R(x) \cap N_R(y)$ is almost completely in A, and

$$B \cap (N_R(x) - N_R(y)) = (B \cap N_R(x)) - (N_R(x) \cap N_R(y)),$$

therefore

$$|B \cap (N_R(x) - N_R(y))| \geq p\frac{n}{2} - o(n) \geq \frac{n}{4} - \frac{\delta n}{8}.$$

Similarly,

$$|B \cap (N_R(y) - N_R(x))| \geq \frac{n}{4} - \frac{\delta n}{8}.$$

Thus

$$|B \cap \Delta(x, y)| \geq \frac{n}{2} - \frac{\delta n}{4}. \tag{18}$$

By (13),

$$\frac{n}{2} + o(n) > |\Delta(x, y)| = 2(p - p^2)n - o(n) > \frac{n}{2} - \frac{\delta n}{25}.$$

This, (18), and $|B| = \frac{n}{2} + o(n)$, imply (15).

(v) Now we are going to estimate the probability that for a random graph R^n there exists an edge (x, y) and a partition $[A/B]$ satisfying both (14) and (15).

For any fixed pair (x, y) we may regard R^n in the following way: first we select the edges incident with x or y, with probability p, independently, then we select the other edges. If we have already the neighbors of x and y, then we have also $\Delta(x,y)$. B differs from $\Delta(x,y)$ only in $\leq \delta n$ vertices; thus we can fix the symmetric difference D of $\Delta(x,y)$ and B in at most $\binom{n}{\delta n}$ ways. For each D we can choose $[A/B]$ in at most $2^{\delta n}$ ways. Hence, given the neighbors of x, y in R^n, we can choose the optimal partition $[A/B]$ in fewer than $4^{\delta n}$ ways. Now we select the edges of $R^n - \{x, y\}$. For fixed $[A/B]$, the probability of

$$e_R(A,B) > \frac{p}{4}n^2 + c_8\frac{n^{3/2}}{2}$$

is exponentially small, by the Chernoff inequality. More precisely, there exists a $\lambda = \lambda_p > 0$ independent of δ (and n), such that this probability is at most $(1 - \lambda)^n$.

The edge in (17) can be chosen in at most $\binom{n}{2}$ ways. Thus the probability that there is an edge (x,y) joining vertices of the same class in an optimal partition is

$$< n^2(1 - \lambda)^n 4^{\delta n} \to 0,$$

if δ is sufficiently small. This completes the proof. (Observe that (iv)–(v) also covered the case $p = \frac{1}{2}$.)

OPEN PROBLEMS

1. We have already mentioned that perhaps the most intriguing problem we could not settle here was the problem of C_4:

What is the maximum of $e(F^n)$ if we exclude C_4 instead of a 3-chromatic L?

2. What happens if $p \to 0$ as $n \to \infty$? Obviously, if $p \to 0$ very slowly, then our theorems will still hold. However, if, e.g., R^n is a random graph with roughly n^{2-c} edges for some small $c > 0$, the proofs completely break down and the theorems still may hold.
3. Erdös and the authors asked the following problem. Assume that G^n contains no K_3 and for any two nonadjacent vertices x and y there exist $\frac{n}{4}$ common neighbors. What is the maximum number of edges this graph can have? This problem was settled by P. Frankl and J. Pach in a more general form [FP].

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