THE COLORING NUMBERS OF THE DIRECT PRODUCT OF TWO HYPERGRAPHS

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1. Definitions. In the following, $H = (X, \mathscr{E})$ will denote a hypergraph with vertex set $X = \{x_1, x_2, \ldots, x_n\}$, and edge family $\mathscr{E} = (E_i / i \in I)$. $n(H) = n$ is the order of H, $m(H) = |I|$ is the number of edges, and $r(H) = \max|E_i|$ is the rank of H. A set $S \subseteq X$ is said to be stable if it contains no edge; the maximum cardinal of a stable set is denoted by $\beta(H)$ and is called the stability number of H.

A set $T \subset X$ is said to be a transversal if it meets each edge; the minimum cardinal of a transversal of H is denoted by $\tau(H)$ and is called the transversal number of H. Other numbers can be associated with hypergraph H; for instance, $\nu(H)$ denotes the maximum number of pairwise disjoint edges; $\rho(H)$ denotes the minimum number of edges which together cover X; $\delta(H)$, the maximum degree, is the maximum number of edges which meet at the same vertex. $\chi(H)$, the chromatic number, is the least integer k for which there exists a partition of X into k stable sets.

It is well known that the following inequalities hold:

(1) $\chi(H)\ \beta(H) \geq n(H)$

(2) $\chi(H) + \beta(H) \leq n(H) + 1$

(3) $\beta(H) = n(H) - \tau(H)$

(4) $\tau(H) \geq \nu(H)$

(5) $\tau(H) \leq r(H)\ \nu(H)$

(For a proof see [1]).

Given two hypergraphs $H = (X, \mathscr{E})$ and $H' = (Y, \mathscr{F})$, with $\mathscr{E} = (E_i / i \in I)$, $\mathscr{F} = (F_j / j \in J)$, their direct product is a hypergraph $H \times H'$ with vertex set $X \times Y$ and with edges $E_i \times F_j$ for $(i, j) \in I \times J$.

The aim of this paper is to find upper bounds and lower bounds for the numbers associated with hypergraph $H \times H'$. These results can easily be extended to the direct product of more than two hypergraphs.

First, it should be noticed that we have:

$$(6) \quad r(H \times H') = r(H)\, r(H')$$

Moreover, some of the associated numbers of $H \times H'$ can be obtained from other coefficients by the duality principle, using the following result:

<u>Proposition 1</u>. $(H \times H')^* = H^* \times H'^*$

By definition of the dual, $(H \times H')^*$ has vertex set $\{(e_i, f_j) / i \in I, j \in J\}$; the edge corresponding to a vertex (x_p, y_q) of $H \times H'$ must contain all the (e_i, f_j) such that $E_i \ni x_p$ and $F_j \ni y_q$, and therefore is the set $X_p \times Y_q$. Here X_p is the set of e_i's such that $E_i \ni x_p$, x_q is similarly defined. Hence the edge family of $H \times H'$ is

$$\{(X_p, X_q) / 1 \leq p \leq m, 1 \leq q \leq h\}.$$

The proposition follows.

2. <u>The Transversal Number</u>. Let H and H' be two hypergraphs of order m and n, respectively. From (3) we have

$$\beta(H \times H') = mn - \tau(H \times H').$$

So, the problem of finding a lower bound for β is the same as the problem of finding an upper bound for τ. This problem often occurs in Combinatorics.

<u>Example 1</u>. What is the least number of points in a $m \times n$ rectangular unit lattice (integer points of the plane), such that each square of side r has at least one of these points as a corner? The answer is $\tau(D_m^r \times D_n^r)$, where D_n^r is a simple graph with vertices $1, 2, \dots, n$, two vertices x, y being joined if $|x - y| = r$.

One can easily show that if $r = 1$ and mn is even, we have

$$\tau(D_m^1 \times D_n^1) = [m/2]^* \, [n/2]^*,$$

where $[x]^*$ denotes the smallest integer $\geq x$.

Example 2. The Zarankiewicz problem. Let $1 \leq r \leq m, 1 \leq s \leq n$. Zarankiewicz has asked for the least integer $k_{rs}(m, n)$ such that every subset of $k_{rs}(m, n)$ points of an $m \times n$ rectangular unit lattice should contain rs points situated in r columns and s rows. If K_m^r denotes the complete r-uniform hypergraph on m points, we have

$$\beta(K_m^r \times K_n^s) = k_{rs}(m, n) - 1$$

An extensive literature exists on this problem (see Guy, [7], [8]). For the sake of simplicity, consider first the case $m = n$. It is known [9] that if $r \leq s$, then

(i) $$\beta(K_n^r \times K_n^s) \leq c_{r,s} n^{2-1/r}$$

where $c_{r,s}$ is a constant. Furthermore, if $r = s = 2$, (i) is sharp, that is if $n \to \infty$, we have

$$\frac{\beta(K_n^2 \times K_n^2)}{n^{3/2}} \to 1$$

It follows easily from [2] that if $s \geq 3$, then

(ii) $$c'_s n^{5/3} \leq \beta(K_n^3 \times K_n^s) \leq c''_s n^{5/3}$$

Unfortunately, the lower bounds for the general case are far from the upper bound given in (i).

Another simple case is when n is much greater than m . Thus if $n \geq (s-1)\binom{m}{r}$, Čulik [5] has determined the exact value:

$$\beta(K^r \times K_n^s) = (r-1)n + (s-1)\binom{m}{r} .$$

For example, $\beta(K_4^2 \times K_6^2) = 6 + 6 = 12$, and a maximum stable set with 12 vertices is given by the ones in the following array:

$$n = 4 \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right. \quad \underbrace{\qquad\qquad}_{m = 6}$$

<u>Proposition 2</u>. <u>Let</u> H <u>and</u> H' <u>be two hypergraphs</u>. <u>Then</u>

$$\tau(H \times H') \leq \tau(H)\,\tau(H') \ .$$

Let $T \subset X$ and $T' \subset Y$ be two minimum transversals respectively for H and H'. Since $T \times T'$ is a transversal for $H \times H'$, we have

$$\tau(H \times H') \leq |T \times T'| = \tau(H)\,\tau(H') \ .$$

Q.E.D.

Instead of showing that the inequality of Proposition 1 is the best possible, we shall show that a very large class of hypergraphs H satisfy

$$\tau(H \times H') = \tau(H)\,\tau(H') \qquad \text{for all } H' .$$

First, we shall prove two lemmas. In fact, these lemmas have been proved independently by L. Lovász and the authors and can be used for a different purpose (see [10]). Let s be a positive integer. Let $\varphi(x)$ be an integer function on X ; for $A \subset X$, let

$$\varphi(A) = \sum_{x \in A} \varphi(x) \ .$$

If $\varphi(E_i) \geq s$ for all $i \in I$, the function φ is said to be an <u>s-covering</u> for H. The minimum of $\varphi(X)$ over all s-coverings φ will be denoted by $\tau_s(H)$. Clearly, $\tau(H) = \tau_1(H)$.

Now, let H be a hypergraph with vertices $x_1, x_2, \ldots, x_n$, with $m(H)$ edges, and with maximum degree $\delta(H)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n non-negative real numbers. Let

$$\tau^*(H) = \min\{\sum_{i=1}^{n} \alpha_i \ / \sum_{x_i \in E_j} \alpha_i \geq 1 \quad \text{for all } j\}$$

<u>Lemma 1</u>. <u>Let</u> H <u>be a hypergraph</u>. <u>Then</u>

$$\max\{\nu(H) \ , \frac{m(H)}{\delta(H)}\} \leq \tau^*(H) \leq \frac{\tau_s(H)}{s} \leq \tau(H)$$

We have $\tau_s(H) \leq s\,\tau(H)$, because if T is a minimum transversal set and if $\varphi_T(X)$ is its characteristic function, then $s\varphi_T$ is an s-covering, and, consequently,

$$\tau_s(H) \leq s\varphi_T(X) = s\,\tau(H) .$$

We have $\frac{1}{s}\tau_s(H) \geq \tau^*(H)$, because if φ is a minimum s-covering, then by putting $\alpha_i = \frac{1}{s}\varphi(x_i)$, we obtain

$$\tau^*(H) \leq \sum_{i=1}^{n} \alpha_i = \frac{1}{s}\tau_s(H) .$$

Now, we shall show that $\tau^*(H) \geq \frac{m(H)}{\delta(H)}$. Consider n real numbers α_i such that $\sum_{x_i \in E_j} \alpha_i \geq 1$ for all j and such that $\Sigma \alpha_i = \tau^*(H)$. Denote by $\delta_x(H)$ the degree of vertex x . We have

$$m(H) \leq \sum_{j=1}^{m} \sum_{x_i \in E_j} \alpha_i \leq \sum_{i=1}^{n} \alpha_i \delta_{x_i}(H) \leq \delta(H) \sum_{i=1}^{n} \alpha_i = \delta(H)\,\tau^*(H) .$$

Also, if $(E'_1, E'_2, \ldots, E'_\nu)$ is a maximum matching of H , then

$$\tau^*(H) = \sum^{n} \alpha_i \geq \sum_{k=1}^{\nu} \sum_{x_i \in E'_k} \alpha_i \geq \nu(H)$$

The first inequality follows.

<u>Lemma 2</u>. $s^{-1}\tau_s(H)$ <u>tends to a limit, and</u>

$$\lim_{s \to \infty} \frac{\tau_s(H)}{s} = \tau^*(H)$$

A well known theorem of Fekete states that if a sequence (a_n) of positive numbers is such that $a_{m+n} \leq a_m + a_n$, then the sequence $(\frac{a_n}{n})$ tends to a limit. Let φ be a minimum p-covering and φ' be a minimum q-covering. Then $\varphi + \varphi'$ is a $(p+q)$ - covering, and therefore

$$\tau_{p+q}(H) \leq \varphi(X) + \varphi'(X) = \tau_p(H) + \tau_q(H) .$$

Hence, by Fekete's theorem, there exists a number ξ such that $\frac{\tau_s(H)}{s} \to \xi$.

By Lemma 1, $\xi \geq \tau^*(H)$.

Furthermore, the α_i's whose sum is $\tau^*(H)$ are defined by a linear programming problem with integral coefficients, and therefore, the α_i's are rational, and we can write

$$\alpha_i = \frac{\alpha'_i}{s} , \ \alpha'_i \text{ and } s \text{ integers.}$$

Hence

$$\frac{\tau_s(H)}{s} \leq \frac{1}{s}\,\Sigma\alpha_i' = \Sigma\alpha_i = \tau^*(H)$$

This shows that $\xi = \tau^*(H)$.

Q.E.D.

Theorem 1. A necessary and sufficient condition for a hypergraph H to satisfy $\tau(H \times H') = \tau(H)\,\tau(H')$ for all H' is that $\tau(H) = \tau^*(H)$.

Necessity. Assume that $\tau(H) \neq \tau^*(H)$. Then, by Lemma 1, $\tau(H) > \tau^*(H)$ and by Lemma 2, there exists an integer $s > 2$ such that $\frac{\tau_s(H)}{s} < \tau(H)$. We shall show that there exists a hypergraph H' such that $\tau(H \times H') < \tau(H)\,\tau(H')$.

Let $\varphi(x)$ be a minimal s-covering for H. Put $\varphi(X) = \tau_s(H) = t$, $Y = \{1, 2, \ldots, t\}$.

It is always possible to associate with each $x \in X$ a set $A(x) \subset Y$ so that:

(1) $|A(x)| = \varphi(x)$ for all $x \in X$,

(2) $x \neq x'$ implies $A(x) \cap A(x') = \emptyset$.

Let $H' = K_t^{t-s+1} = (Y, (F_j))$. We shall show that the direct product $H \times H'$ admits

$$T_o = \{(x, y) / x \in X,\ y \in A(x)\}$$

as a transversal.

Clearly, $E_i \times Y$ contains at least s different elements of T_o. Since no two of them have the same projection on Y, $E_i \times F_j$ contains at least one element of T_o, for all i, j. Thus, T_o is a transversal of $H \times H'$. Moreover, $\tau(H') = s$. Hence

$$\tau(H \times H') \leq |T_o| = \tau_s(H) < s\,\tau(H) = \tau(H)\,\tau(H').$$

Q.E.D.

Sufficiency. Let H be a hypergraph such that $\tau(H) = \tau^*(H)$. Then by Lemma 1, $\tau_s(H) = s\,\tau(H)$ for every integer s. Let $T_o \subset X \times Y$ be a minimum transversal of $H \times H'$. Let

$$\varphi_o(x) = |\{y / (x, y) \in T_o,\ y \in Y\}|.$$

Since the projection on Y of $(E_i \times Y) \cap T_o$ is a transversal of H',

$$\varphi_o(E_i) = |(E_i \times Y) \cap T_o| \geq \tau(H').$$

Thus, φ_o is an s-covering for $s = \tau(H')$. Hence,

$$\tau(H\times H') = |T_o| = \varphi_o(X) \geq \tau_s(H) = s\,\tau(H) = \tau(H')\,\tau(H) \ .$$

Therefore, the equality holds.

Q.E.D.

Corollary 1. If H satisfies $\nu(H) = \tau(H)$ (and in particular if H is balanced) then $\tau(H\times H') = \tau(H)\tau(H')$ for every H'.

This follows immediately from Lemma 1.

In particular, if H is balanced, i.e. if each odd cycle of H possesses an edge containing three vertices of the cycle, it is known ([1]) that $\nu(H) = \tau(H)$, and consequently, the required equality holds.

Corollary 2. Let G be a graph. Then

$$\tau(G\times H') = \tau(G)\tau(H')$$

for every hypergraph H' if and only if

$$\tau(G) = \nu(G) \ .$$

By a theorem of Lovász [10], $\tau(G) = \tau^*(G)$ if and only if $\tau(G) = \nu(G)$. The proof follows.

Q. E. D.

Corollary 3. Let H be a hypergraph such that $m(H) = \tau(H)\,\delta(H)$. Then $\tau(H\times H') = \tau(H)\tau(H')$ for every hypergraph H'.

This follows immediately from Lemma 1.

Corollary 4. Let H and H' be two hypergraphs. Then

$$\rho(H\times H') \leq \rho(H)\rho(H') \ .$$

Furthermore, if H is balanced, then $\rho(H\times H') = \rho(H)\rho(H')$ for every H'.

Clearly, if H^* is the dual of H , then $\rho(H) = \tau(H^*)$. If H is balanced, then H^* is also balanced.

Thus, the result follows immediately from Proposition 1, Proposition 2 and Corollary 1.

<u>Corollary 5</u>. <u>Let</u> H <u>and</u> H' <u>be two hypergraphs</u>. <u>Then</u>

$$\beta(H \times H') \geq \beta(H)\, n(H') + \beta(H')\, n(H) - \beta(H)\, \beta(H')$$

<u>Equality holds for every</u> H' <u>if and only if</u> $\tau(H) = \tau^*(H)$.

We have

$$\beta(H \times H') = n(H \times H') - \tau(H \times H') \geq n(H)\, n(H') - \tau(H)\, \tau(H') = n(H)\, n(H')$$

$$- (n(H) - \beta(H))\, (n(H') - \beta(H')) = \beta(H)\, n(H') + \beta(H')\, n(H) - \beta(H)\, \beta(H')$$

The equality holds iff it holds in Theorem 1.

<u>Theorem 2</u>. <u>Let</u> H <u>and</u> H' <u>be two hypergraphs</u>. <u>Then</u>

$$\tau(H \times H') \geq \tau(H) + \tau(H') - 1$$

<u>A hypergraph</u> $H = (E_i \,/\, i \in I)$ <u>satisfies</u> $\tau(H \times H') = \tau(H) + \tau(H') - 1$ <u>for every</u> H' <u>if and only if</u> $\bigcap_{i \in I} E_i \neq \emptyset$.

1. Let H and H' be two hypergraphs on X and Y respectively. Let T_o be a minimum transversal of $H \times H'$, and for $x \in X$, let

$$\varphi_o(x) = |\{y \,/\, (x\,,y) \in T_o\,, y \in Y\}|$$

Clearly, φ_o is an s-covering of H for $s = \tau(H')$. Let $T_1 \subset X \times Y$ be obtained from T_o by removing exactly $s - 1$ vertices, and let

$$\varphi_1(x) = |\{y \,/\, (x\,,y) \in T_1\,, y \in Y\}|$$

We have, for all edges E_i of H,

$$\varphi_1(E_i) \geq \varphi_o(E_i) - (s-1) \geq 1$$

Hence, φ_1 is a 1-covering of H , and therefore $\varphi_1(X) \geq \tau(H)$. Hence

(1) $$\tau(H \times H') = \varphi_o(X) = \varphi_1(X) + (s-1) \geq \tau(H) + \tau(H') - 1 .$$

2. Now, consider a hypergraph $H = (E_i \,/\, i \in I)$ such that $\bigcap E_i \neq \emptyset$. Then $\tau(H) = 1$.

Let $x_o \in \bigcap E_i$. Clearly, $H \times H'$ has a transversal $T_o \subset \{x_o\} \times Y$ such that

$$|T_o| = \tau(H') = \tau(H) + \tau(H') - 1$$

Hence, by part 1 of the theorem, T_o is a minimum transversal of $H \times H'$, and

$$\tau(H \times H') = |T_o| = \tau(H) + \tau(H') - 1$$

Since this equality holds for every H', the second part of the theorem is proved.

3. It remains to show that if $\tau(H) > 1$, there exists a hypergraph H' such that $\tau(H \times H') > \tau(H) + \tau(H') - 1$. Take any balanced hypergraph H' with $\tau(H') = s \geq 2$. By Corollary 1 to Theorem 1, we have

$$\tau(H \times H') = s\,\tau(H) > \tau(H) + (s-1) = \tau(H) + \tau(H') - 1$$

The required inequality follows.

Q.E.D.

Remark. Proposition 2 shows that, for all p, q,

$$\max\{\tau(H \times H') \,/\, \tau(H) = p\,,\ \tau(H') = q\} = pq \tag{1}$$

However, Theorem 2 shows only that

$$\min\{\tau(H \times H') \,/\, \tau(H) = p\,,\ \tau(H') = q\} = p + q - 1 \tag{2}$$

holds for $p = 1$ (or $q = 1$). However, it is easy to show that (2) holds for all pq.

Put $H = K^q_{p+q-1}$, $H' = K^p_{p+q-1}$ (the complete hypergraphs on $p+q-1$ vertices with ranks respectively q and p). Clearly, $\tau(H) = p$, $\tau(H') = q$. If the vertex set of H is $\{x_1, \ldots, x_{p+q-1}\}$ and the vertex set of H' is $\{y_1, \ldots, y_{p+q-1}\}$, then $T_o = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{p+q-1}, y_{p+q-1})\}$ is a transversal of $H \times H'$ because otherwise there exists an edge E_i of H and an edge F_j of H' such that $(E_i \times F_j) \cap T_o = \emptyset$, which contradicts that $|E_i| + |F_j| = p + q$. Thus, (2) follows from Theorem 2.

In fact, we can have a better inequality by using the number τ^*. We have

<u>Theorem 3</u>. <u>Let</u> H <u>and</u> H' <u>be two hypergraphs</u>. <u>Then</u>

$$\tau(H \times H') \geq \max\{\tau^*(H)\,\tau(H')\,,\ \tau(H)\,\tau^*(H')\}.$$

Let T_o be a minimum transversal of $H \times H'$, and let

$$\varphi_o(x) = |\{y \,/\, (x, y) \in T_o\,,\ y \in Y\}|$$

φ_o is an s-covering of H for $s = \tau(H')$. Hence, by Lemma 1,

$$\tau(H\times H') = |T_o| = \varphi_o(X) \geq \tau_s(H) \geq s\,\tau^*(H) = \tau(H')\,\tau^*(H) \;.$$

The required inequality follows.

<u>Corollary 1</u>. $\tau(H\times H') \geq \max\left\{\frac{m(H)}{\delta(H)}\,\tau(H'),\ \frac{m(H')}{\delta(H')}\,\tau(H)\right\}$

This follows immediately from Lemma 1.

<u>Corollary 2</u>. $\tau(H\times H') \geq \max\{\nu(H)\,\tau(H')\ ,\ \nu(H')\,\tau(H)\}$

This follows immediately from Lemma 1.

3. <u>The Chromatic Number</u>. We shall now consider the chromatic number of the direct product $H\times H'$.

<u>Example</u>. (Polarized partition relations among cardinal numbers, [6], [4]). What is the least number of colors required to color the points of an $m\times n$ rectangle unit lattice so that rs points situated in r columns and s rows cannot have the same color? Clearly, this number is $\chi(K_m^r\times K_n^s)$.

For instance, $\chi(K_5^2\times K_4^2) = 2$, and a bicoloring of the 6×4 rectangle unit lattice is shown in Example 2, Section 2.

Also, we have

$$\chi(K_5^2\times K_5^2) = 3$$

Otherwise, there exists a bicoloring of the 5×5 matrix $((a_j^i))$ where the 0's denote the points of the first color and the 1's the points of the second color. Since the first column $(a_1^1,a_1^2,a_1^3,a_1^4,a_1^5)$ necessarily has three entries of equal value, suppose $a_1^1 = a_1^2 = a_1^3 = 0$.

The first two rows have, in each column, one of the combinations 00, 11, 01, 10, and there exist two columns with the same combination (because $2^2 < 5$).

Since this repeated combination cannot be 00 nor 11, we may assume

$$a_2^1 = a_3^1 = 0$$

$$a_2^2 = a_3^2 = 1.$$

None of a_2^3, a_3^3 can be zero; hence

$$a_2^3 = a_3^3 = 1 .$$

Since the submatrix

$$\left(\left(\begin{matrix} a_2^2 & a_3^2 \\ a_2^3 & a_3^3 \end{matrix}\right)\right)$$

has only ones , the 0's and 1's in $((a_j^i))$ do not define a bicoloring of $K_5^2 \times K_5^2$.

Q.E.D.

This argument has been extended by Chvátal [3], [4], who showed that

(A) $$c_1 n^{1/r} \leq \chi(K_n^r \times K_n^r) \leq c_2 n^{1/r}$$

In fact, the lower bound also follows from a result of Kövary, Sós, Turán [9], while the upper bound was obtained by so-called probabilistic methods. Moreover, replacing the probabilistic method by a finite geometrical construction, onē can show that

(B) $$\chi(K_n^2 \times K_n^2) / n^{1/2} \to 1$$

Finally, Sterboul [11] showed that in some cases, the same kind of arguments gives the exact value of $\chi(K_m^2 \times K_n^2)$.

The problem of finding a lower bound for $\chi(H \times H')$ was also considered by Chvátal [3], who gave the two following inequalities:

$$\chi(H \times H') \geq \min\{\chi(H) , \chi(H')^{1/n(H)}\} ,$$
$$\chi(H \times H') \geq \min\{\chi(H) , m(H)^{-1} \chi(H')\} .$$

An obvious result is:

<u>Proposition 3</u>. $\chi(H \times H') \leq \min\{\chi(H) , \chi(H')\}$

Assume that $\chi(H) \leq \chi(H')$, and let $g(x)$ be a coloring of H in $p = \chi(H)$ colors. Then $h(x , y) = g(x)$ is a coloring of $H \times H'$ in p colors. Hence $\chi(H \times H') \leq \chi(H)$.

Q.E.D.

Equality is obtained in some degenerate cases, for example when $\chi(H) = 2$. However, in general, Proposition 3 is far from being best possible. A better estimation for $\chi(H \times H')$, knowing $\chi(H) = p$ and $\chi(H') = q$, is:

<u>Theorem 4</u>. $\max\{\chi(H \times H') / \chi(H) = p , \chi(H') = q\} = \chi(K_p^2 \times K_q^2)$

We have only to show that if H and H' are two hypergraphs with $\chi(H) = p$, $\chi(H') = q$, then

$$\chi(H \times H') \leq \chi(K_p^2 \times K_q^2)$$

Consider a coloring $c(x)$ of H with p symbols $a_1, a_2, \ldots, a_p$, and a coloring $c'(y)$ of H' with q symbols $b_1, b_2, \ldots, b_q$. Consider a complete graph K_p^2 with vertex set $\{a_1, a_2, \ldots, a_p\}$ and a complete graph K_q^2 with vertex set $\{b_1, b_2, \ldots, b_q\}$. Let $g(a_i, b_j)$ be a coloring of $K_p^2 \times K_q^2$ in $t = \chi(K_p^2 \times K_q^2)$ colors. Now, put

$$h(x, y) = g(c(x), c'(y))$$

To show that $h(x, y)$ is a coloring of $H \times H'$, consider an edge $E \times F$ of $H \times H'$. E contains two vertices x_1 and x_2 with $c(x_1) \neq c'(x_2)$, and F contains two vertices y_1 and y_2 with $c'(y_1) \neq c'(y_2)$. Since $\{c(x_1), c(x_2)\} \times \{c'(y_1), c'(y_2)\}$ is an edge of $K_p^2 \times K_q^2$, it contains two points, say $(c(x_3), c'(y_3))$ and $(c(x_4), c'(y_4))$, with

$$g(c(x_3), c'(y_3)) \neq g(c(x_4), c'(y_4))$$

Hence, $E \times F$ contains two vertices (x_3, y_3) and (x_4, y_4) with $h(x_3, y_3) \neq h(x_4, y_4)$. This shows that $h(x, y)$ is a t-coloring of $H \times H'$. Hence $\chi(H \times H') \leq t = \chi(K_p^2 \times K_q^2)$.

Q.E.D.

The problem of finding a good estimate for

$$f(p, q) = \min\{\chi(H \times H') / \chi(H) = p , \chi(H') = q\}$$

seems to be difficult. In particular, we can ask if as p and q tend to infinity, $f(p, q)$ tends to infinity.

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