Introduction to Extremal Graph Theory

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Slides of my Chorin Summer School Lectures, 2006

slightly polished:

smaller letters, slightly more explanation,

some extra topics included

More exercises

The slides will slightly be upgraded in October!

Part I: Classical Extremal Graph Theory

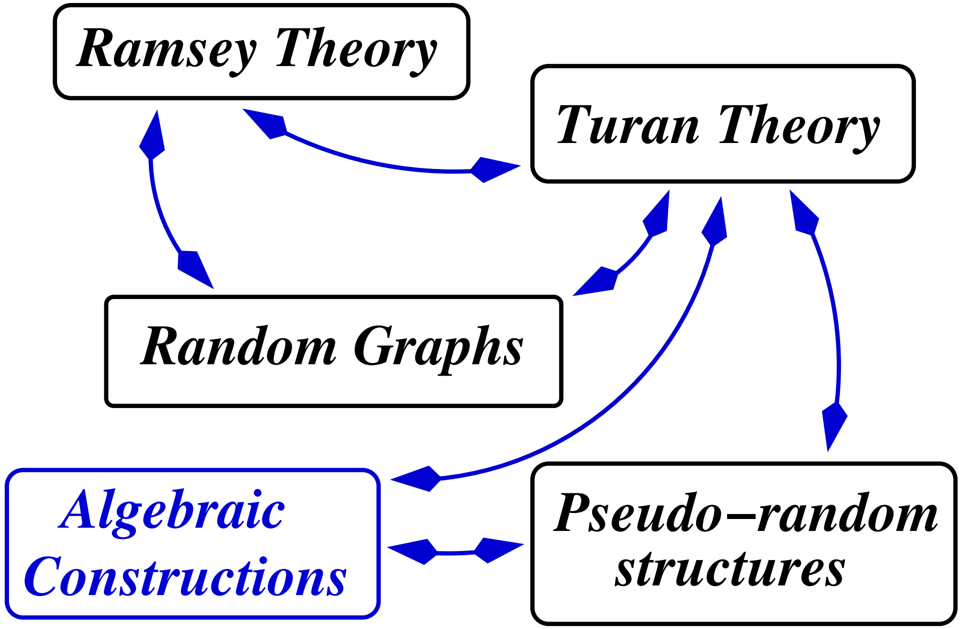
This series of slides/lectures covered a huge area. It contains some repetitions: otherwise it would be hopeless.

New parts (i.e. parts planned to be mentioned in my lecture for which I had not enough time:)

- Cube reduction theorem
- Erdős-Frankl-Rödl theory
- Some new exercises

Introduction

Extremal graph theory and Ramsey theory were among the early and fast developing branches of 20th century graph theory. We shall survey the early development of Extremal Graph Theory, including some sharp theorems.



Strong interactions
between these fields:
Here everything influenced everything

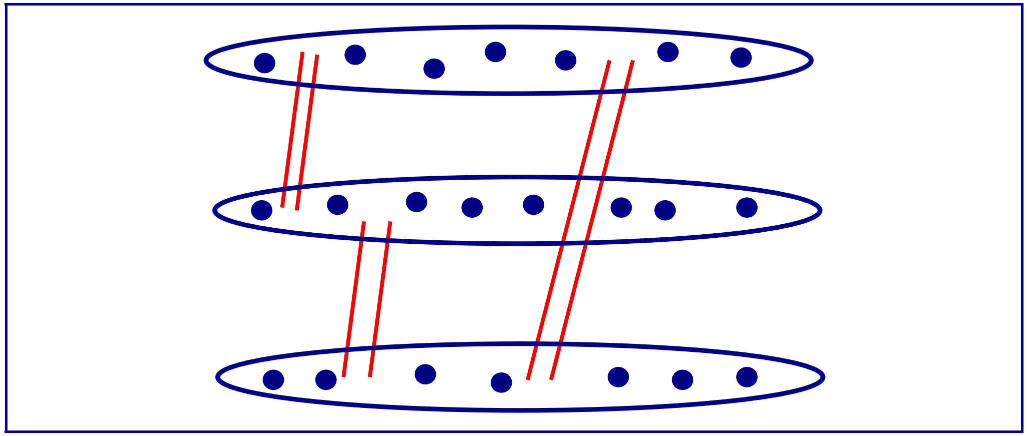
General Notation

- G_n, $Z_{n,k}$, $T_{n,p}$, H_ν... the (first) subscript n will almost always denote the number of vertices.
- K_p = complete graph on p vertices,
- P_k / C_k = path / cycle on k vertices.
- $\delta(x)$ is the degree of the vertex x.
- $v(G)$ / $e(G)$ = # of vertices / edges,
- $\delta(G)$ = mindeg, $\Delta(G)$ = maxdeg
- $\chi(G)$ = the chromatic number of G.
- $N(x)$ = set of neighbours of the vertex x, and
- $G[X]$= the subgraph of G induced by X.
- $e(X,Y)$ = # of edges between X and Y.

Special notation

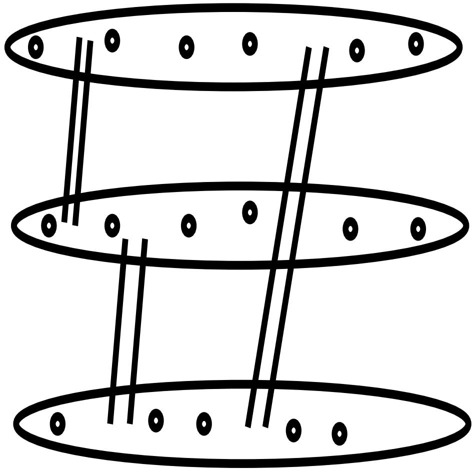
Turán type extremal problems for simple (?) graphs

- Sample graph L, $\mathcal{L}$
- $\mathbf{ex}(n,\mathcal{L}) =$ extremal number $= \max\limits_{L \not\subseteq \mathcal{L}} e(G_n)$.
  if $\quad {}_{L\in\mathcal{L}}$
- $\mathbf{EX}(n,\mathcal{L})$ = **extremal graphs**.
- $T_{n,p}$ = **Turán graph**, p-chromatic having most edges.



THE TURÁN GRAPH

The Turán graph

$T_{n,p}$ = 

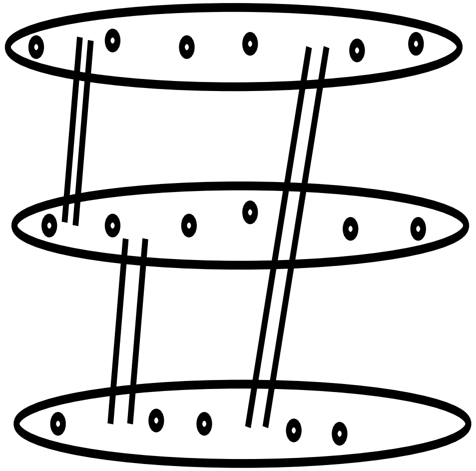
If $n \equiv r \pmod p$, $0 \leq r < p$, then

$$e(T_{n,p}) = \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(n^2 - r^2\right) + \binom{r}{2}.$$

So we can calculate $e(T_{n,p})$ but mostly we do not care for the formula!

Turán type graph problems

MANTEL 1903 (?) K_3
Erdős: C_4: Application in combinatorial number theory.
The first finite geometrical construction (Eszter Klein)



Turán theorem. *(1940)*

$$e(G_n) > e(T_{n,p}) \quad \Longrightarrow \quad K_{p+1} \subseteq G_n.$$

Unique extremal graph $T_{n,p}$.

GENERAL QUESTION: Given a family $\mathcal{L}$ of forbidden graphs, what is the maximum of $e(G_n)$ if G_n does not contain subgraphs $L \in \mathcal{L}$?

Main Line:

Many central theorems assert that for ordinary graphs the general situation is almost the same as for K_{p+1}.

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

- The extremal graphs S_n are very similar to $T_{n,p}$.
- the almost extremal graphs are also very similar to $T_{n,p}$.

The meaning of "VERY SIMILAR":

- One can delete and add $o(n^2)$ edges of an extremal graph S_n to get a $T_{n,p}$.
- One can delete $o(n^2)$ edges of an extremal graph to get a p-chromatic graph.

Stability of the class sizes

Exercise 1. Among all the n-vertex p-chromatic graphs $T_{n,p}$ is the (only) graph maximizing $e(T_{n,p})$.

Exercise 2. (Stability) If $\chi(G_n) = p$ and

$$e(G_n) = e(T_{n,p}) - s$$

then in a p-colouring of G_n, the size of the i^{th} colour-class,

$$\left| n_i - \frac{n}{p} \right| < c\sqrt{s+1}.$$

Exercise 3. Prove that if n_i is the size of the i^{th} class of $T_{n,p}$ and G_n is p-chromatic with class sizes $m_1, \ldots, m_p$, and if $s_i := |n_i - m_i|$, then

$$e(G_n) \leq e(T_{n,p}) - \sum \binom{s_i}{2}.$$

Extremal graphs

The "metric" $\rho(G_n, H_n)$ is the minimum number of edges to change to get from G_n a graph isomorphic to H_n.

Notation. $\mathbf{EX}(n, \mathcal{L})$: set of extremal graphs for $\mathcal{L}$.

Theorem 1 (Erdős-Simonovits, 1966). *Put*

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

If $S_n \in \mathbf{EX}(n, \mathcal{L})$, *then*

$$\rho(T_{n,p}, S_n) = o(n^2).$$

Product conjecture

Theorem 1 separates the cases $p = 1$ and $p > 1$:

$$\boxed{\mathbf{ex}(n, \mathcal{L}) = o(n^2) \quad \Longleftrightarrow \quad p = p(\mathcal{L}) = 1}$$

$$\boxed{p = 1\text{: degenerate extremal graph problems}}$$

Conjecture 1 (Sim.). *If*

$$\mathbf{ex}(n, \mathcal{L}) > e(T_{n,p}) + n \log n$$

and $S_n \in \mathbf{EX}(n, \mathcal{L})$, then S_n can be obtained from a $K_p(n_1, \ldots, n_p)$ only by adding edges.

This would reduce the general case to degenerate extremal graph problems:

The product conjecture, Reduction

Definition 1. *Given the vertex-disjoint graphs $H_1, \dots, H_p$, their product $\prod_{i=1}^{p} H_{n_i}$ is the graph H_n obtained by joining all the vertices of H_{n_i} to all vertices of H_{n_j}, for all $1 \le i < j \le p$.*

Exercise 4. Prove that if $\prod_{i=1}^{p} H_{n_i}$ is extremal for $\mathcal{L}$ then H_{n_1} is extremal for some $\mathcal{M}_1$. (Hint: Prove this first for $p = 1$.)

Definition 2 (Decomposition). $\mathcal{M} = \mathcal{M}(\mathcal{L})$ is the family of decomposition graphs of $\mathcal{L}$, where M is a decomposition graph for $\mathcal{L}$ if some $L \in \mathcal{L}$ can be $p+1$-colored so that the first two colors span an M^* containing M.

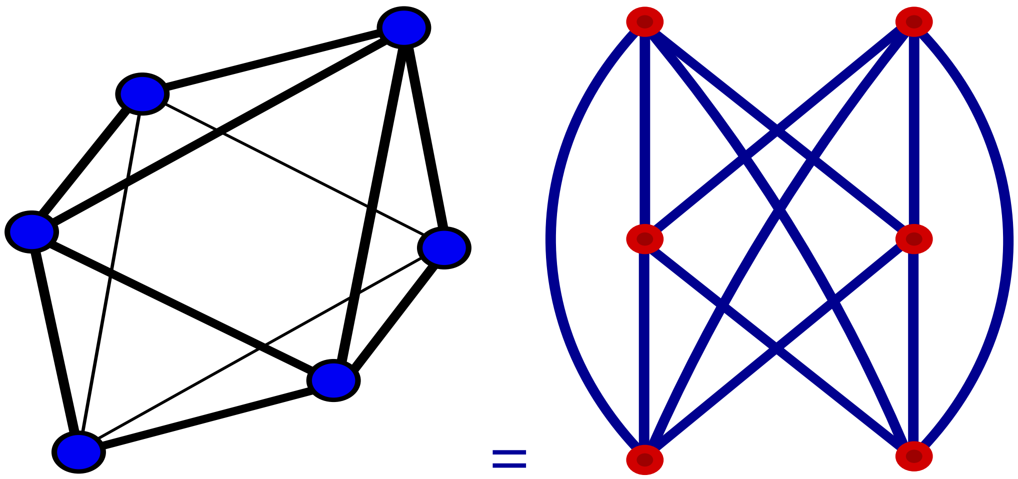
Exercise 5. Prove that if $\prod_{i=1}^{p} H_{n_i}$ is extremal for $\mathcal{L}$ then H_{n_i} is extremal for some $\mathcal{M}_i \subseteq \mathcal{M}$ and $p(\mathcal{M}) = 1$: the problem of $\mathcal{M}$ is degenerate.

Example: Octahedron Theorem

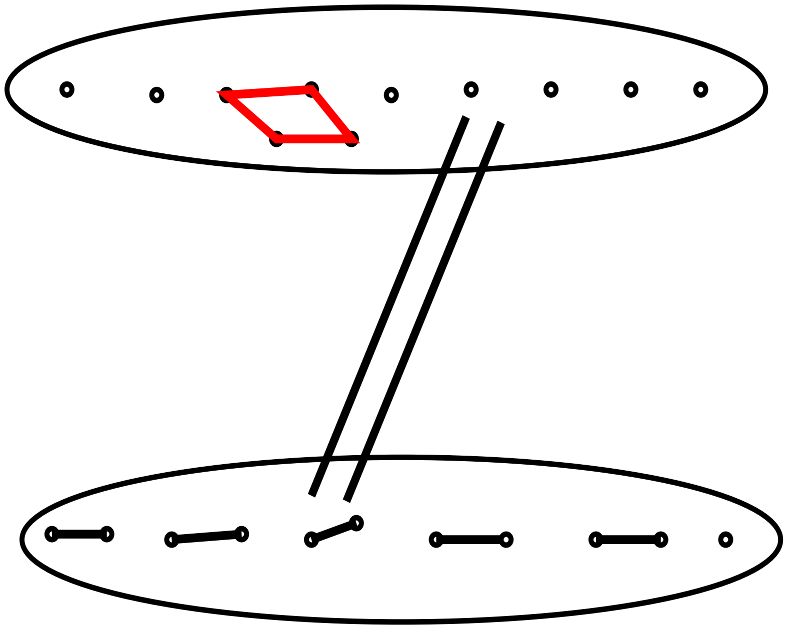
Erdős-Sim. theorem. For O_6, the extremal graphs S_n are "products":

$U_m \otimes W_{n-m}$ where U_m is extremal for C_4 and W_{n-m} is extremal for P_3. for $n > n_0$.

→ ErdSimOcta



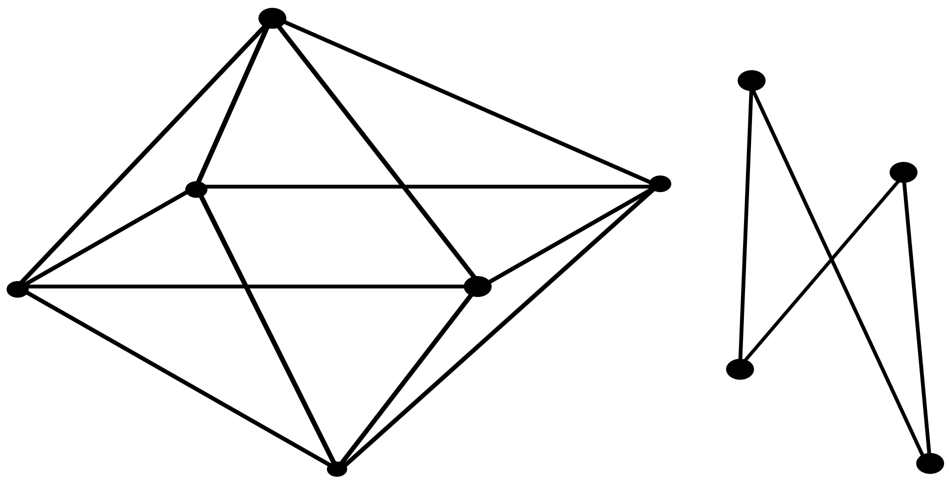
EXCLUDED: OCTAHEDRON

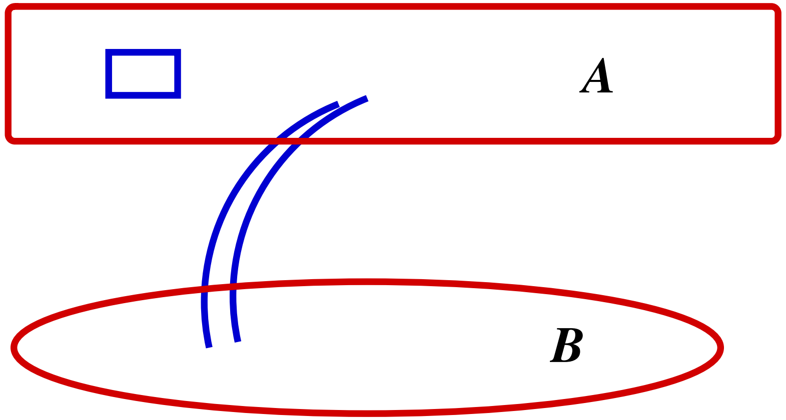


EXTREMAL = PRODUCT

Decomposition decides the error terms

Definition (Decomposition, alternative def.). For a given $\mathcal{L}$, $\mathcal{M} := \mathcal{M}(\mathcal{L})$, $\mathcal{M}$ is the family of all those graphs M for which there is an $L \in \mathcal{L}$ and a $t = t(L)$ such that $L \subseteq M \otimes K_{p-1}(t, \dots, t)$.
We call $\mathcal{M}$ the **decomposition family** of $\mathcal{L}$.





If B contains a C_4 then G_n contains an octahedron: K(3,3,3).

The product conjecture, II.

Conjecture (Product). *If no p-chromatic $L \in \mathcal{L}$ can be $p+1$-colored so that the first two color classes span a tree (or a forest) then all (or at least one of) the extremal graphs are products of p subgraphs of size $\approx \frac{n}{p}$.*

Structural stability

Erdős-Sim. Theorem.

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $L \not\subseteq G_n$ for any $L \in \mathcal{L}$ and

$$e(G_n) \geq \left(1 - \frac{1}{p}\right) \binom{n}{2} - \delta n^2,$$

then

$$\rho(G_n, T_{n,p}) \leq \varepsilon n^2$$

Structural stability: $o(.)$ form

Erdős-Sim. Theorem.
Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

If G_n is almost extremal:

- It is $\mathcal{L}$-free, and
- $$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - o(n^2),$$

then

$$\rho(G_n, T_{n,p}) = o(n^2).$$

Corollary: The almost extremal graphs are almost-p-colorable

Improved error terms, depending on $\mathcal{M}$.

Erdős-Sim. Theorem.
Put

$$p := \min_{L\in\mathcal{L}} \chi(L) - 1.$$

Let $\mathcal{M} = \mathcal{M}(\mathcal{L})$ be the decomposition family. Let $\mathbf{ex}(n, \mathcal{M}) = O(n^{2-\gamma})$. Then, if G_n is almost extremal:

- It is $\mathcal{L}$-free, and
- $$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - O(n^{2-\gamma}),$$

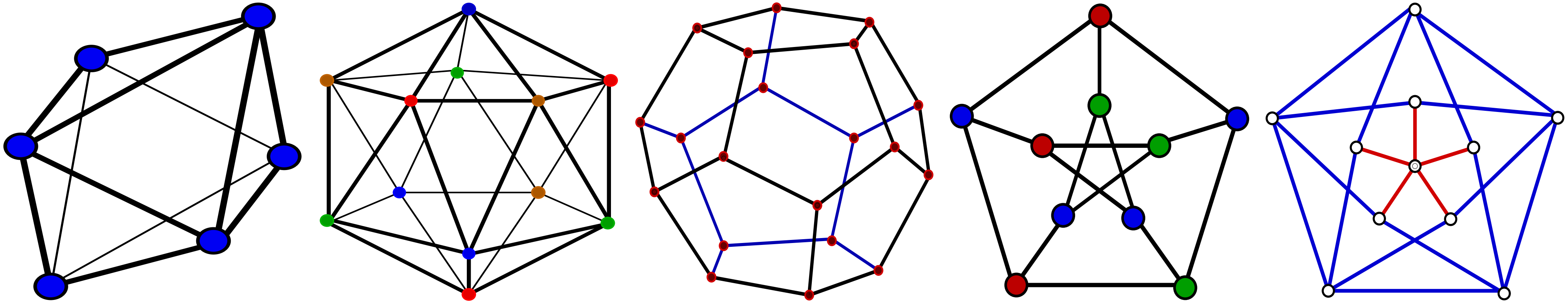
then we can delete $O(n^{2-\gamma})$ edges from G_n to get a p-chromatic graph.

Remark: For extremal graphs $\rho(S_n, T_{n,p}) = O(n^{2-\gamma})$.

Applicable and gives also exact results

Examples:

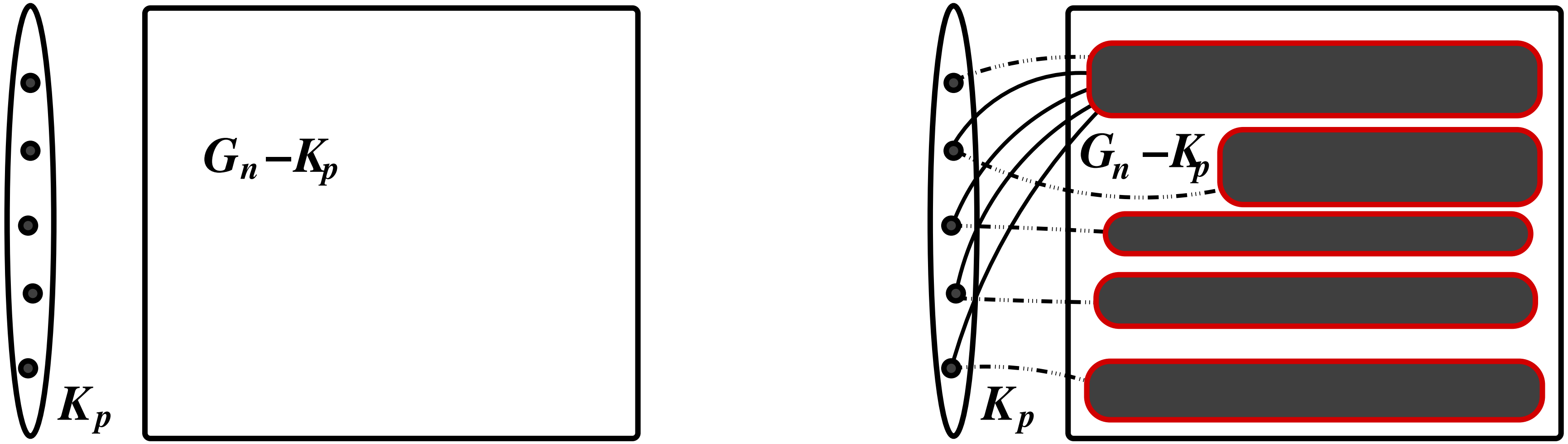
Octahedron, Icosahedron, Dodecahedron, Petersen graph, Grötzsch



In all these cases the stability theorem yields exact structure for $n > n_0$.

Original proof of Turán's thm

- We may assume that $K_p \subseteq G_n$.
- We cut off K_p.
- We use induction on n (from $n - p$).

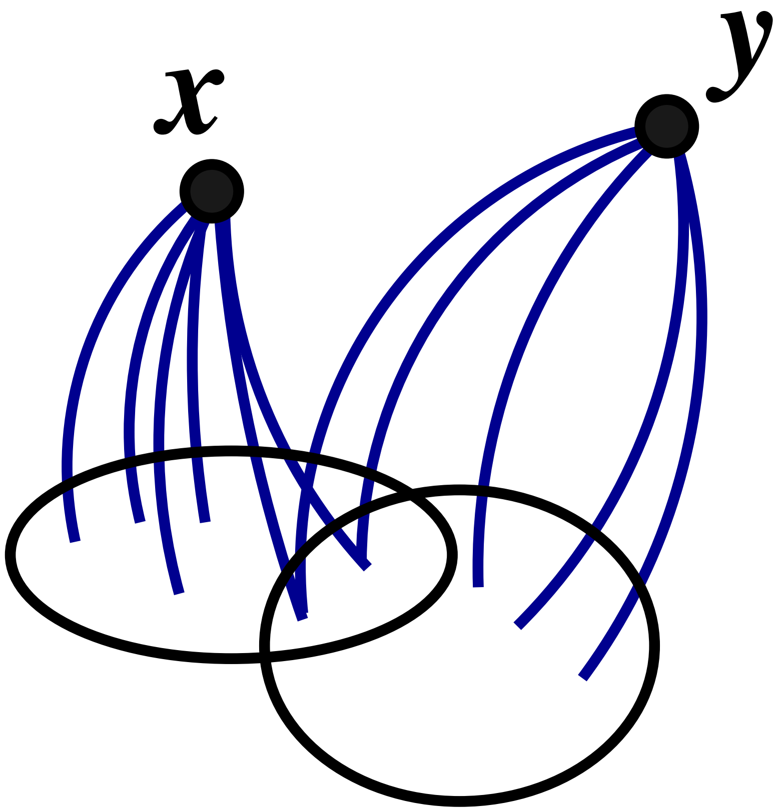


- We show the uniqueness

This "splitting off" method can be used to prove the structural stability and many other results. However, there we split of, say a large but fixed $K_p(M, \ldots, M)$.

Zykov's proof, 1949

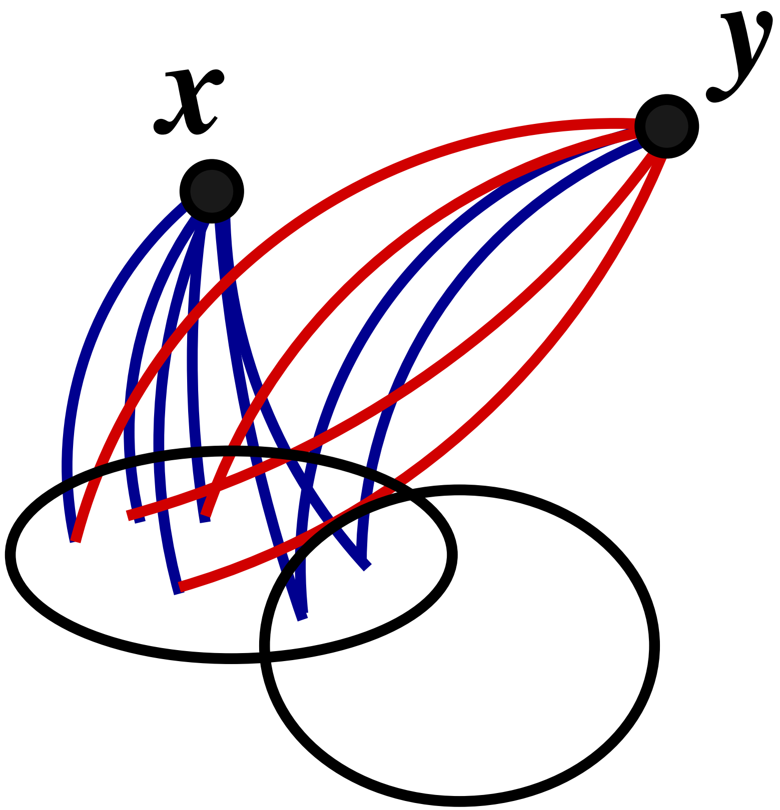
…and why do we like it?



Assume $deg(x) \geq deg(y)$.

Zykov's proof, 1949.

…and why do we like it?



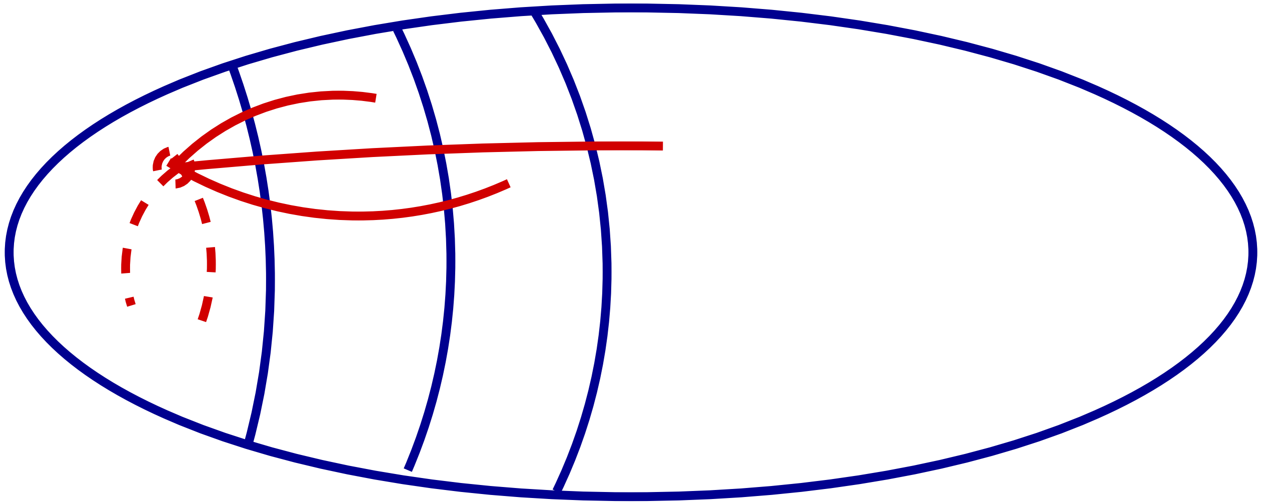
We replace $N(x)$ by $N(y)$.

Lemma. If $G_n \not\supseteq K_\ell$ and we symmetrize, the resulting graph will neither contain a K_ℓ.

How to use this?

We can use a parallel symmetrization.

● = max degree



Uniqueness?

- FÜREDI proved the stability for K_{p+1}, analyzing this proof: If there are many edges among the nonneighbours of the base x_i then we gain a lot.

Other directions

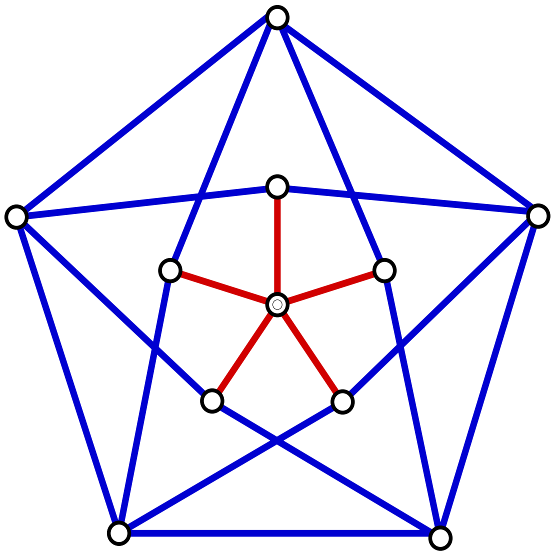
- Prove exact results for special cases
- Prove good estimates for the bipartite case: $p = 1$
- Clarify the situation for digraphs
- Prove reasonable results for hypergraphs

- Investigate fields where the problems have other forms, yet they are strongly related to these results.

Examples: 1. Critical edge

Critical edge theorem. *If $\chi(L) = p+1$ and L contains a color-critical edge, then $T_{n,p}$ is the (only) extremal for L, for $n > n_1$. [If and only if]*

SIM., (Erdős)

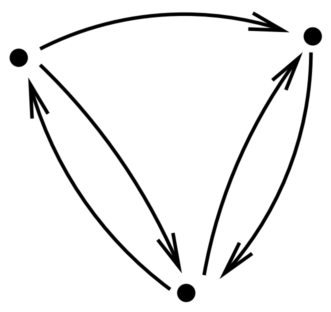


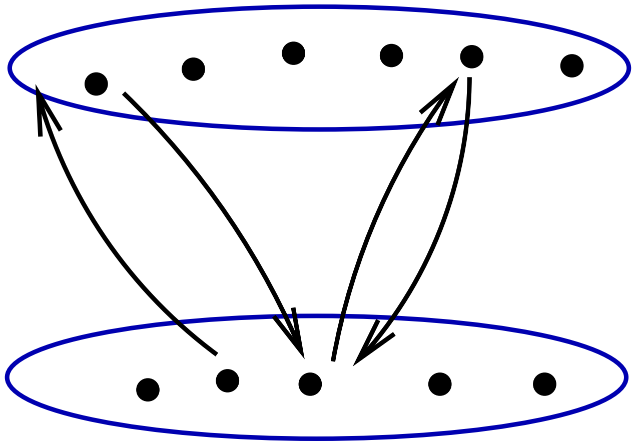
GRÖTZSCH GRAPH

Complete graphs
Odd cycles

Examples: 2. A digraph theorem

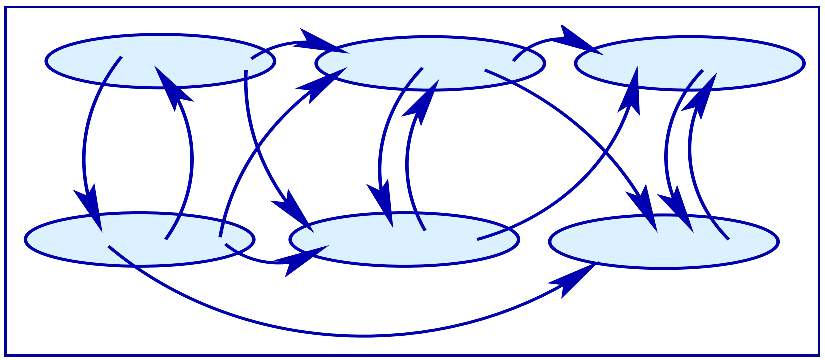
We have to assume an upper bound s on the multiplicity. (Otherwise we may have arbitrary many edges without having a K_3.) Let $s = 1$.

L: 



$$\mathbf{ex}(n, L) = 2\mathbf{ex}(n, K_3) \qquad (n > n_0?)$$

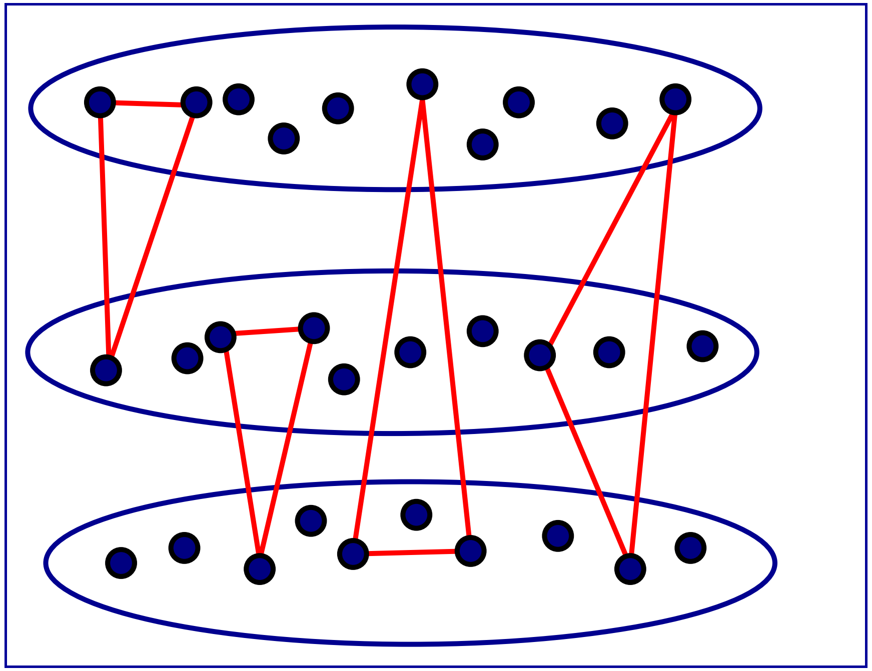
Many extremal graphs: We can combine arbitrary many oriented double Turán graph by joining them by single arcs.

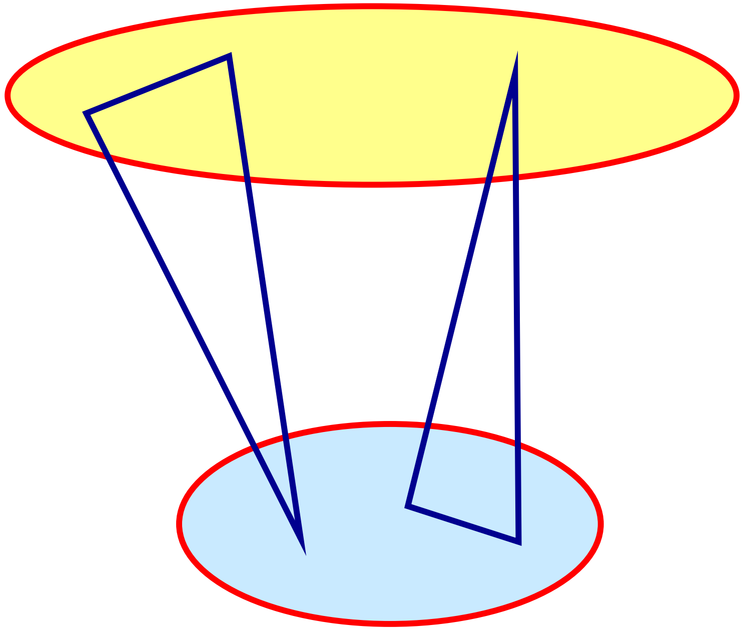


Example 3. The famous Turán conjecture

Consider 3-uniform hypergraphs.

Conjecture 2 (Turán). *The following structure (on the left) is the (? asymptotically) extremal structure for* $K_4^{(3)}$*:*





For $K_5^{(3)}$ one conjectured extremal graph is just the above "complete bipartite" one (on the right)!

Examples: Degree Majorization

ERDŐS

For every K_{p+1}-free G_n there is a p-chromatic H_n with

$$d_H(v_i) \geq d_G(v_i).$$

(I.e the degrees in the new graph are at least as large as originally.)

BOLLOBÁS-THOMASON, ERDŐS-T. SÓS

If $e(G_n) > e(T_{n,p})$ edges, then G_n has a vertex v with

$$e(G[N(v)]) \geq \mathbf{ex}(d(v), K_p).$$

(I.e the neighbourhood has enough edges to ensure a K_p.)

Application of symmetrization

Exercise 6. Prove that symetrization does not produce new complete graphs: if the original graph did not contain K_ℓ, the new one will neither.

Exercise 7. Prove the degree-majorization theorem, using symmetrization.

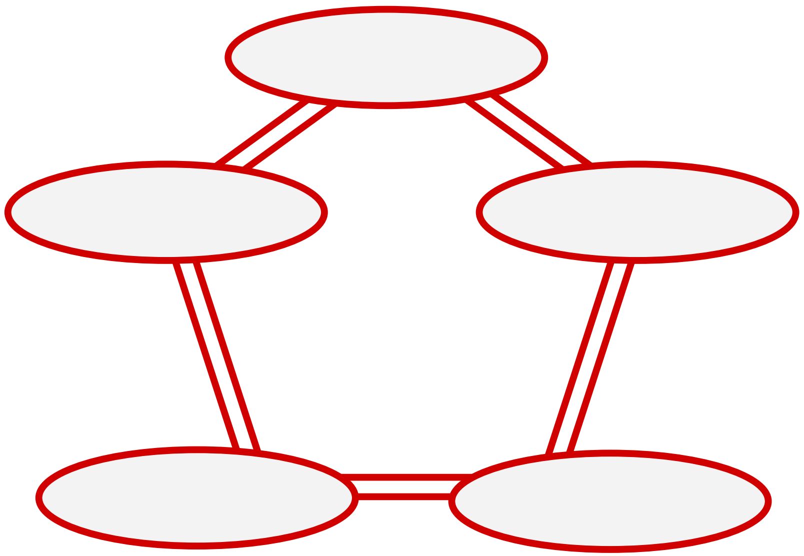
Exercise 8. (BONDY) Prove the Bollobás-Thomason- Erdős-T. Sós theorem, using symmetrization.

Exercise 9. Is it true that if a graph does not contain C_4 and you symmetrize, the new graph will neither contain a C_4?

Examples:

ERDŐS:

Prove that each triangle-free graph can be turned into a bipartite one deleting at most $n^2/25$ edges.



The construction shows that this is sharp if true.

Partial results: ERDŐS-FAUDREE-PACH-SPENCER

ERDŐS-GYŐRI-SIM.

GYŐRI

FÜREDI

Introduction, history, some central theorems

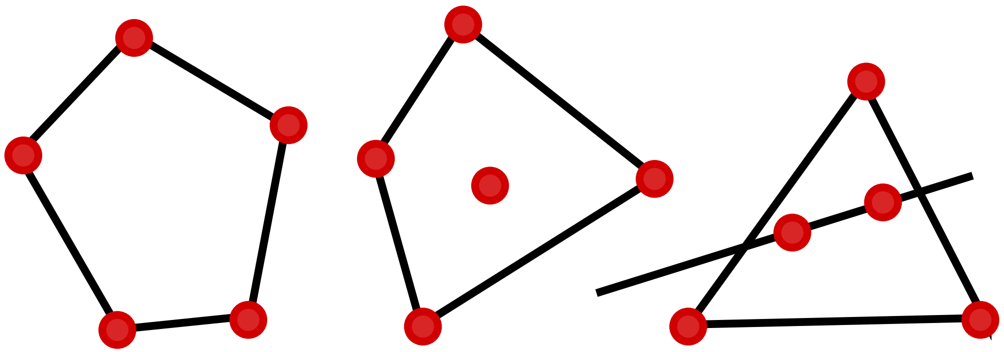
- Ramsey theory (ERDŐS-SZEKERES)
  – The ESZTER KLEIN problem
  – The Lemma: Either G_n or $\overline{G_n}$ contains a large K_ℓ.
- Extremal graph theory (TURÁN/ERDŐS)

The Eszter Klein problem:

Exercise 10. Prove that among 5 points there are always 4 in a convex position.

Problem 1. *Let $f(k)$ denote the smallest integer for which among any $f(k)$ points in the plane there are k in convex position.*

- *Is there such an integer at all?*
- *If YES, estimate it.*



Proof of statement of Ex 1.

Erdős-Szekeres conjecture

$$f(k) \leq 2^{k-2} + 1.$$

Two proofs of a weaker result:
Reinventing RAMSEY theorem:

Theorem 2 (Erdős-Szekeres).
If $R(k,\ell)$ is the RAMSEY threshold, then

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}.$$

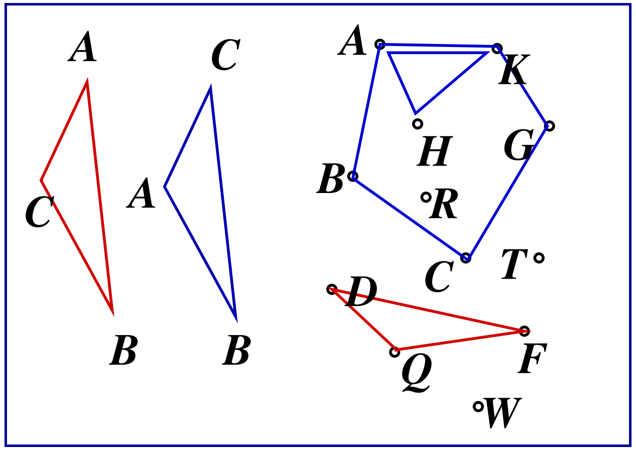
Exercise 11. Prove this, using induction on $k+\ell$.

Exercise 12. Can you prove something similar for 3 colours?

Proof of E-Sz thm, using Ramsey

- The hypergraph RAMSEY is needed.
- color the convex 4-gons RED, the others BLUE
- Show that if all the 4-tuples are RED for $P_1, \dots, P_k$, then they are in convex position.

Remark 1. One can improve this proof: apply RAMSEY to 3-uniform hypergraphs: Color the triangles in RED-BLUE: Clockwise $\Delta P_a P_b P_c$ RED, the others BLUE $\qquad (a < b < c)$



- Show that if all the 3-tuples are RED for $P_1, \dots, P_k$, then they are in convex position.

Turán's approach

In which other way can we ensure a large $K_k \subseteq G_n$?

E.g., if $e(G_n)$ is large?

Later TURÁN used to say: RAMSEY and his theorems are applicable because they are generalizations of the Pigeon Hole Principle.

Turán asked for several other sample graphs L to determine $\mathbf{ex}(n, L)$:

- Platonic graphs: Icosahedron, cube, octahedron, dodecahedron.
- path P_k

Exercises:

Exercise 13. Let $\mathcal{C}$ denote the family of all cycles. Determine $\mathbf{ex}(n, \mathcal{C})$.

Exercise 14. Determine $\mathbf{ex}(n, P_4)$.

Exercise 15. Prove that if $d_{\min}(G_n) \geq k-1$ then G_n contains all trees on k vertices.

Exercise 16. Prove that for any tree T_k,

$$\mathbf{ex}(n, T_k) \leq (k-1)n.$$

Erdős-Sós conjecture

$$\mathbf{ex}(n, T_k) \leq \frac{1}{2}(k-1)n.$$

AJTAI-KOMLÓS-SIM.-SZEMERÉDI: True if $k > k_0$.

Classification of extremal graph problems and lower bound constructions

- The asymptotic structure of extremal graphs

- Degenerate extremal graph problems:
  – $\mathcal{L}$ contains a bipartite L:
  – $\mathbf{ex}(n, \mathcal{L}) = o(n^2)$.

- Lower bounds using random graphs and finite geometries:
  – Here random methods are weak
  – Finite geometry sometimes gives sharp results.

The Erdős-Stone theorem (1946)

$$\boxed{\mathbf{ex}(n, K_{p+1}(t,\ldots,t)) = \mathbf{ex}(n, K_{p+1}) + o(n^2)}$$

Motivation from topology

General asymptotics

Erdős-Stone-Sim. → ErdSimLim

If

$$\min_{L\in\mathcal{L}} \chi(L) = p+1$$

then

$$\mathbf{ex}(n,\mathcal{L}) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

So the asymptotics depends only on the minimum chromatic number

Erdős-Stone-Sim. thm

$$\mathbf{ex}(n, \mathcal{L}) = \mathbf{ex}(n, K_{p+1}) + o(n^2).$$

How to prove this from ERDŐS-STONE?

– pick $L \in \mathcal{L}$ with $\chi(L) = p + 1$.

– pick t with $L \subseteq K_{p+1}(t, \ldots, t)$.

– apply ERDŐS-STONE:

$$\mathbf{ex}(n, \mathcal{L}) \geq e(T_{n,p})$$

but

$$\begin{aligned} \mathbf{ex}(n, \mathcal{L}) \leq \mathbf{ex}(n, L) &\leq \mathbf{ex}(n, K_{p+1}(t, \ldots, t)) \\ &\leq e(T_{n,p}) + \varepsilon n^2. \quad \square \end{aligned}$$

Classification of extremal problems

- nondegenerate: $p > 1$
- degenerate: $\mathcal{L}$ contains a bipartite L
- strongly degenerate: $T_\nu \in \mathcal{M}(\mathcal{L})$

where $\mathcal{M}$ is the decomposition family.

Importance of Decomposition

This determines the real error terms in our theorems. E.g., if $\mathcal{M}$ is the family of decomposition graphs.

$$e(T_{n,p}) + \mathbf{ex}(n/p, \mathcal{M}) \le \mathbf{ex}(n, \mathcal{L}) \le e(T_{n,p}) + c \cdot \mathbf{ex}(n/p, \mathcal{M})$$

for any $c > p$, and n large.

Exercise 17. What is the decomposition class of K_{p+1}?

Exercise 18. What is the decomposition class of the octahedron graph $K_3(2,2,2)$? More generally, of $K(p,q,r)$?

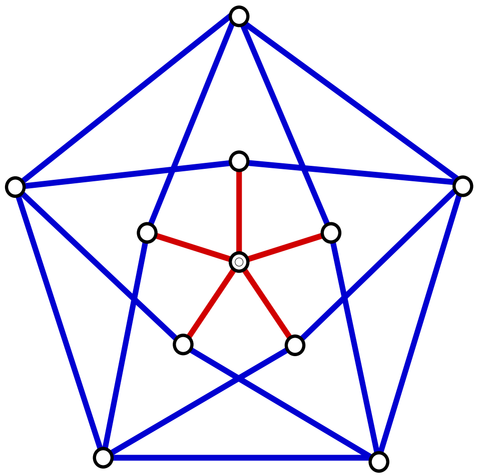
Exercise 19. What is the decomposition class of the Dodecahedron graph D_{20}? And of the icosahedron graph I_{12}?

The corresponding theorems

Def. e is color-critical edge if $\chi(L-e) < \chi(L)$.

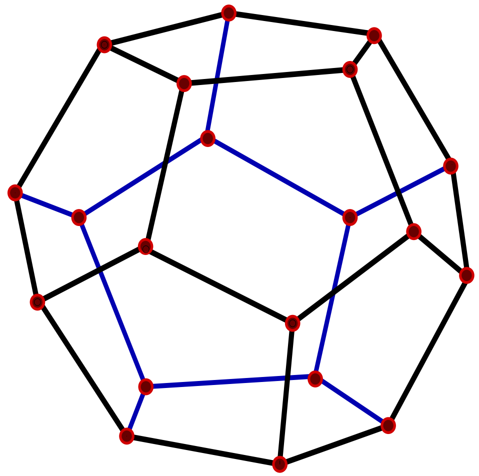
Critical edge, (Sim.) theorem. *If $\chi(L) = p+1$ and L contains a color-critical edge, then $T_{n,p}$ is the (only) extremal for L, for $n > n_1$.*

+ Erdős

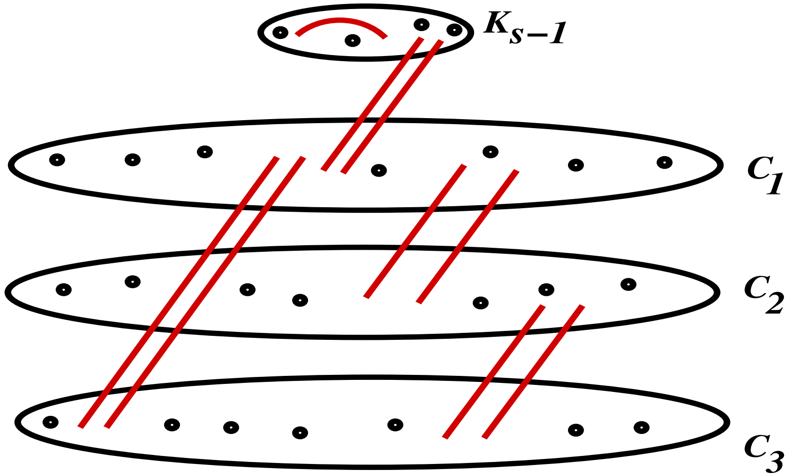


Complete graphs
Odd cycles

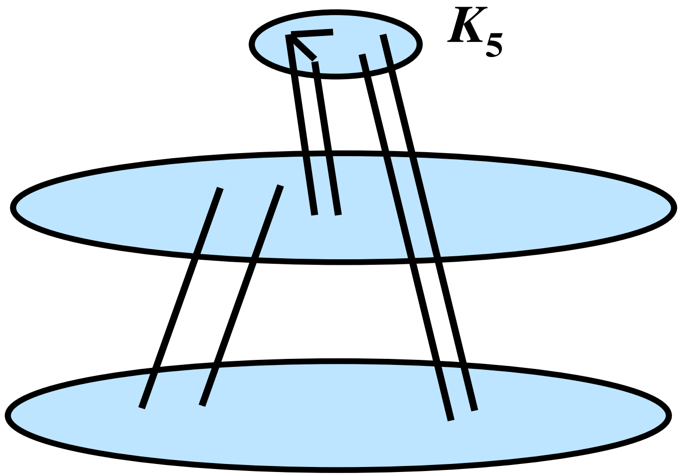
Dodecahedron Theorem (Sim.)



Dodecahedron: D_{20}



$H(n,d,s)$

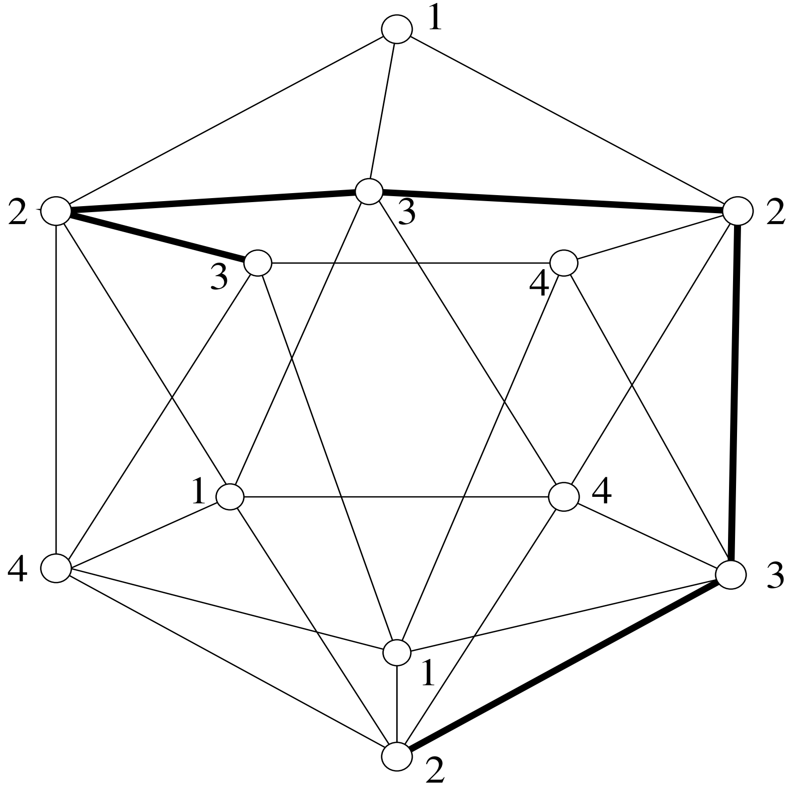


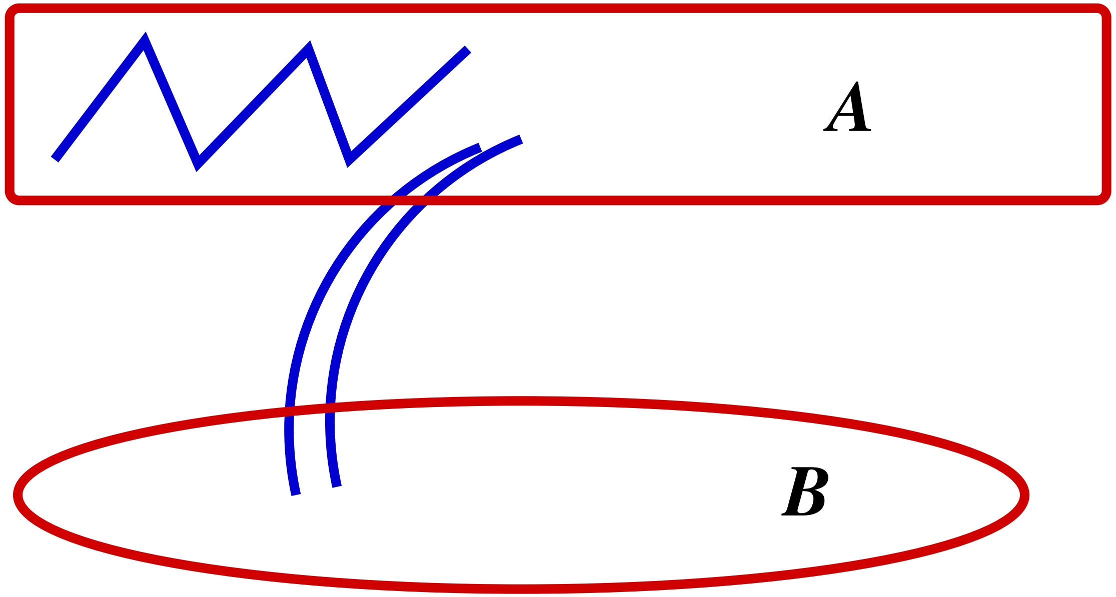
$H(n,2,6)$

For D_{20}, $H(n,2,6)$ is the (only) extremal graph for $n > n_0$.

($H(n,2,6)$ cannot contain a D_{20} since one can delete 5 points of $H(n,2,6)$ to get a bipartite graph but one cannot delete 5 points from D_{20} to make it bipartite.)

Example 2: the Icosahedron





If B contains a P_6, then G_n contains an icosahedron

The decomposition class is: P_6.

Application in combin. number theory

Erdős (1938): → ErdTomsk

Maximum how many integers $a_i \in [1, n]$ can be found under the condition: $a_i a_j \neq a_k a_\ell$, unless $\{i, j\} = \{k, \ell\}$?

This lead ERDŐS to prove:

$$\mathbf{ex}(n, C_4) \leq cn\sqrt{n}.$$

The first finite geometric construction to prove the lower bound (ESZTER KLEIN)

Crooks tube

First "attack":

The primes between 1 and n satisfy Erdős' condition.

Can we conjecture
$$g(n) \approx \pi(n) \approx \frac{n}{\log n}?$$

YES!

Proof idea: If we can produce each non-prome $m \in [1, n]$ as a product:

$$m = xy, \quad \text{where} \quad x \in X, \ y \in Y,$$

then

$$g(n) \leq \pi(n) + \mathbf{ex}_B(X, Y; C_4).$$

where $\mathbf{ex}_B(U, V; L)$ denotes the maximum number of edges in a subgraph of $G(U, V)$ without containing an L.

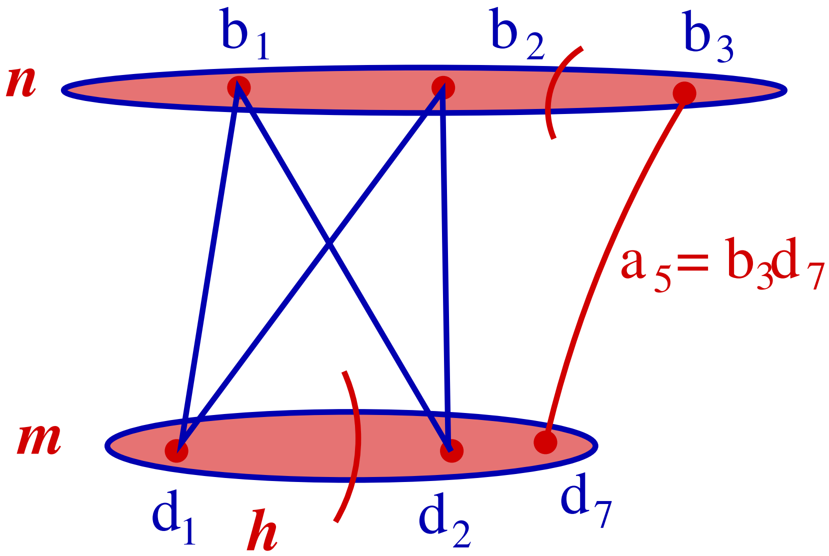
The number theoretical Lemma:

Consider only integers. Let $\mathcal{P}$ = primes,

$$\mathcal{B} := [1, n^{2/3}] \bigcup [n^{2/3}, n] \cap \mathcal{P} \quad \text{and} \quad \mathcal{D} := [1, n^{2/3}].$$

Lemma 1 (Erdős, 1938). $[1, n] \subseteq \mathcal{B} \cdot \mathcal{D} = (\mathcal{B}_1 \cdot \mathcal{D}) \cup (\mathcal{B}_2 \cdot D)$.

Lemma 2 (Erdős, 1938). *Representing each $a_i = b_i d_i$, the obtained bipartite graph has no C_4.*



$e(G(\mathcal{B}_1, \mathcal{D})) \leq 3m\sqrt{m} = 3n.$
$\mathcal{B}_2$ is joined only to $[1, n^{1/3}]$:
$e(G(\mathcal{B}_2, \mathcal{D})) \leq \pi(n) + h^2$
$= \pi(n) + n^{2/3}.$

Kővári-T. Sós-Turán theorem

One of the important extremal graph theorems is that of
KŐVÁRI, T. SÓS AND TURÁN, → KovSosTur

solving the extremal graph problem of $K_2(p, q)$.

Theorem (Kővári–T. Sós–Turán). *Let* $2 \leq p \leq q$ *be fixed integers. Then*

$$\mathbf{ex}(n, K(p,q)) \leq \frac{1}{2}\sqrt[p]{q-1}\, n^{2-1/p} + \frac{1}{2}pn.$$

Is the exponent $2-(1/p)$ sharp?

Conjecture (KST Sharp). *For every integers p, q,*

$$\mathbf{ex}(n, K(p,q)) > c_{p,q}\, n^{2-1/p}.$$

Known for $p = 2$ and $p = 3$: Finite geometric constructions

ERDŐS, RÉNYI, V. T. SÓS, $\rightarrow$ $\boxed{\text{ErdRenyiSos}}$

W. G. BROWN $\rightarrow$ $\boxed{\text{BrownThom}}$

Random methods: $\rightarrow$ $\boxed{\text{ErdRenyiEvol}}$

$$\mathbf{ex}(n, K(p,q)) > c_p n^{2-\frac{1}{p}-\frac{1}{q}}.$$

Füredi on $K_2(3,3)$: The Brown construction is sharp.

$\boxed{\text{Kollár-Rónyai-Szabó: } q > p!\ .}$ Commutative Algebra constr.

$\boxed{\text{Alon-Rónyai-Szabó: } q > (p-1)!\ .}$

Unknown:

- Missing lower bounds: Constructions needed

- "Random constructions" do not seem to give the right order of magnitude when $\mathcal{L}$ is finite

We do not even know if

$$\frac{\mathbf{ex}(n, K(4,4))}{n^{5/3}} \to \infty.$$

- Partial reason for the bad behaviour: Lenz Construction

Lenz Construction

Exercise 20. Prove that $\mathbb{E}^4$ contains two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ so that each $x \in \mathcal{C}_1$ is at distance 1 from each point of $\mathcal{C}_2$.

Exercise 21. Is there any bipartite L which can be excluded from the unit distance graph in $\mathbb{E}^4$?

Exercise 22. Find an L of chromatic number 3 which can be excluded from the unit distance graph in $\mathbb{E}^4$.

Sketch of the proof of KST Thm

Lemma (Convexity). *Extending $\binom{n}{p}$ to all $x > 0$ by*

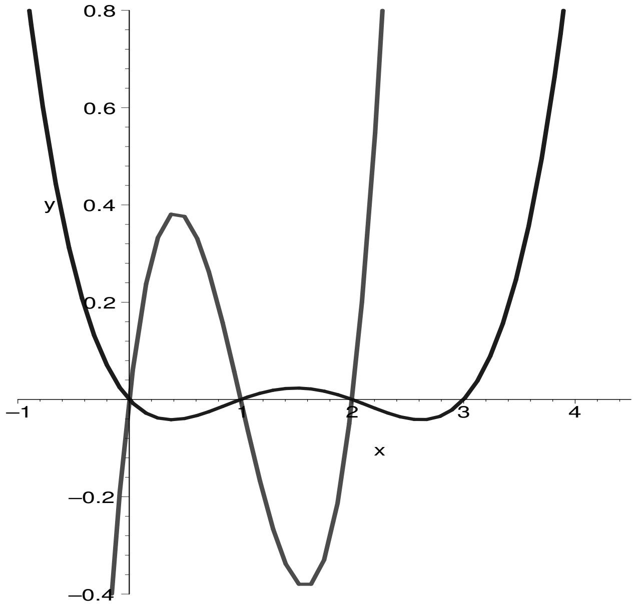
$$\binom{x}{p} := \begin{cases} \frac{x(x-1)\dots(x-p+1)}{p!} & \textit{for } x \geq p-1, \\ 0 & \textit{otherwise} \end{cases}$$

we get a convex function.

Proof of the Lemma. Rolle theorem

$$\begin{cases} \frac{x(x-1)\dots(x-p+1)}{p!} & \text{for } x \geq p-1, \\ 0 & \text{otherwise} \end{cases}$$

is also convex in $(-\infty, \infty)$. ∎



Needed:

Exercise 23.

$$\frac{(n-p+1)^p}{p!} \le \binom{n}{p} \le \frac{n^p}{p!}$$

Exercise 24. When is

$$\frac{(n-p+1)^p}{p!} \le \binom{n}{p} \le \frac{n^p}{p!}$$

almost sharp?

Exercise 25. For $a \gg c \gg b$, estimate

$$\frac{\binom{a-b}{c-b}}{\binom{a}{c}}.$$

Proof of KST Thm

Count $\mathcal{P} = \#K_{p,1} \subseteq G_n$

(a) $\mathcal{P} \leq (q-1)\binom{n}{p}$.

(b) If $d_1, \ldots, d_n$ are the degrees in G_n, then $\mathcal{P} = \sum \binom{d_i}{p}$.

Jensen's Inequality: if $E := e(G_n)$, then

$$n\binom{2E/n}{p} \leq \sum \binom{d_i}{p} \leq (q-1)\binom{n}{p} \quad \Longrightarrow \quad n\left(\frac{2E}{n} - p\right)^p \leq (q-1)n^p,$$

$$\sqrt[p]{n}\left(\frac{2E}{n} - p\right) \leq \sqrt[p]{q-1} \cdot n.$$

Rearranging:

$$\frac{2E}{n} - p \leq \sqrt[p]{q-1} \cdot n^{1-\frac{1}{p}}. \quad \Longrightarrow \quad E \leq \frac{1}{2}\sqrt[p]{q-1} \cdot n^{2-\frac{1}{p}} + \frac{1}{2}pn.$$

Degenerate problems

Given a family $\mathcal{L}$ of forbidden graphs,

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2).$$

if and only if there is a bipartite graph in $\mathcal{L}$.
Moreover, if $L_0 \in \mathcal{L}$ is bipartite, then

$$\mathbf{ex}(n, \mathcal{L}) = O(n^{2-2/v(L_0)}).$$

Proof. Indeed, if a graph G_n contains no $L \in \mathcal{L}$, then it contains no L_0 and therefore it contains no $K_2(p, v(L_0) - p)$, yielding an $L \subseteq G_n$. . ∎

Strongly degenerate problems

Given a finite family $\mathcal{L}$ of forbidden graphs,

$$\mathbf{ex}(n, \mathcal{L}) = O(n).$$

if and only if there is a tree (or a forest) graph in $\mathcal{L}$.

Proof.

– if there is a tree in $\mathcal{L}$ then $\mathbf{ex}(n, \mathcal{L}) = O(n)$.

– By Erdős's lower bound, if there is no tree in $\mathcal{L}$ and the largert L has v vertices, we may take a G_n with girth $> v$ and

$$e(G_n) > n^{1+c_{\mathcal{L}}}.$$

∎

Exercise 26. Show that the finiteness cannot be omitted.

Mader type theorems:

Let $\top(L)$ be the family of topological L's. Then
$\mathbf{ex}(n, \top(L)) = O(n)$.

Mader theorem. *(1967)* There exists a constant $c_p > 0$ for which, if

$$e(G_n) > c_p n$$

then G_n contains a topological K_p.

$\longrightarrow$ $\boxed{\text{Mader67}}$

Further Mader theorems (1998, 2005)

Conjecture (G. Dirac). *Every* G_n $(n \geq 3)$ *with* $e(G_n) \geq 3n - 5$ *contains a subdivision of* K_5.

WOLGANG MADER: YES.

Conjecture (C. Thomassen). *Every non-planar 4-connected graph with at least* $3n - 6$ *edges contains a subdivision of* K_5.

MADER also proved this, by characterizing

graphs with $3n - 6$ edges not containing a subdivision of K_5.

$\rightarrow$ Mader05

Supersaturated Graphs: Degenerate

Prove that if

$$E = e(G_n) > c_0 n^{2-(1/p)},$$

then the number of $K_{p,q}$'s in G_n

$$\#K(p,q) \geq c_{p,q} \frac{E^{pq}}{n^2}$$

The meaning of this is that an arbitrary G_n having more edges than the (conjectured) extremal number, must have – up to a multiplicative constant, – at least as many $K_{p,q}$ as the corresponding random graph,

see conjectures Erdős and Sim. and of Sidorenko

Supersaturated, Non-Degenerate

If

$$e(G_n) > \mathbf{ex}(n, L) + cn^2,$$

then G_n contains $\geq c_L n^{v(L)}$ copies of L

This extends to multigraphs, hypergraphs, directed multihypergraphs.

BROWN-SIMONOVITS $\rightarrow$ BROWNSIMDM

Bondy-Simonovits

Theorem (Even Cycle: C_{2k}). $\mathrm{ex}(n, C_{2k}) = O(n^{1+(1/k)})$.

More explicitly:

Theorem (Even Cycle: C_{2k}). $\mathrm{ex}(n, C_{2k}) \leq c_1 k n^{1+(1/k)}$.

Conjecture (Sharpness). *Is this sharp, at least in the exponent? The simplest unknown case is C_8,*

It is sharp for C_4,C_6, C_{10}

Could you reduce k in $c_1 k n^{1+(1/k)}$?

Sketch of the proof:

Lemma 3. *If D is the average degree in G_n, then G_n contains a subgraph G_m with*

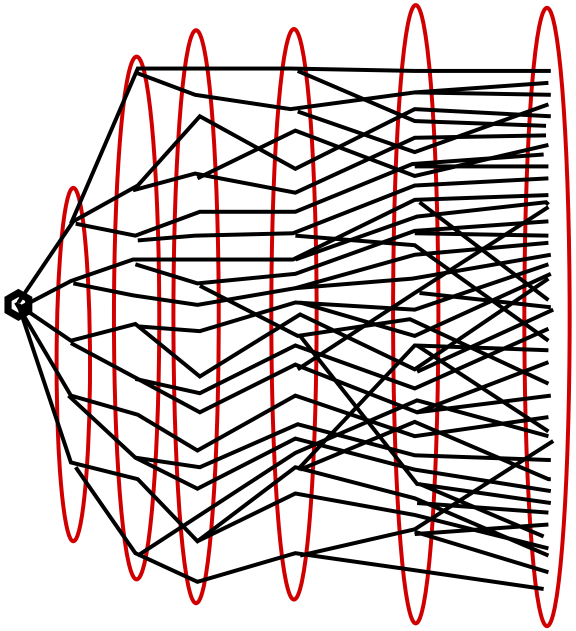
$$d_{\min}(G_m) \geq \frac{1}{2}D \quad \text{and} \quad m \geq \frac{1}{2}D.$$

Exercise 27. Can you improve this lemma?

- So we may assume that G_n is bipartite and regular.

Assume also that it does not contain shorter cycles either.

Sketch of the proof: Expansion



Start with cheating: girth > $2k$:

- The ith level contains at least D^i different points.
- $D^i < n$, $i = 1, 2, \dots k$.

So $D < n^{1/k}$.

- $e(G_n) \le cDn \le \frac{1}{2} n^{1+1/k}$.

We still have the difficulty that the shorter cycles cannot be trivially eliminated. Two methods to overcome this:

- BONDY-SIMONOVITS and → BondySim
- FAUDREE-SIMONOVITS → FaudreeSim.

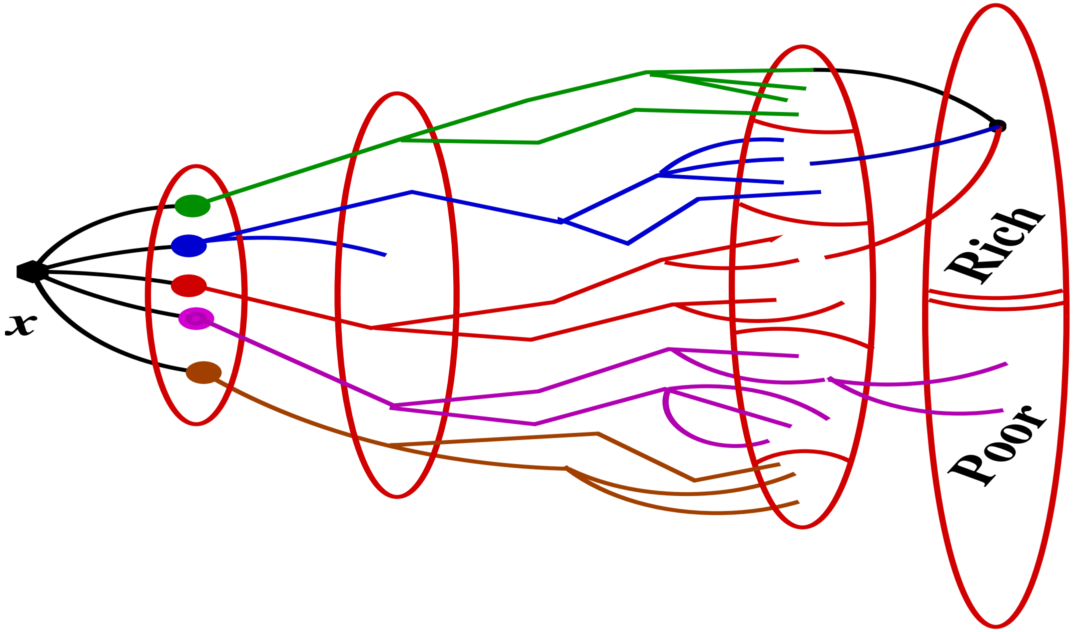
Both proofs use Expansion:

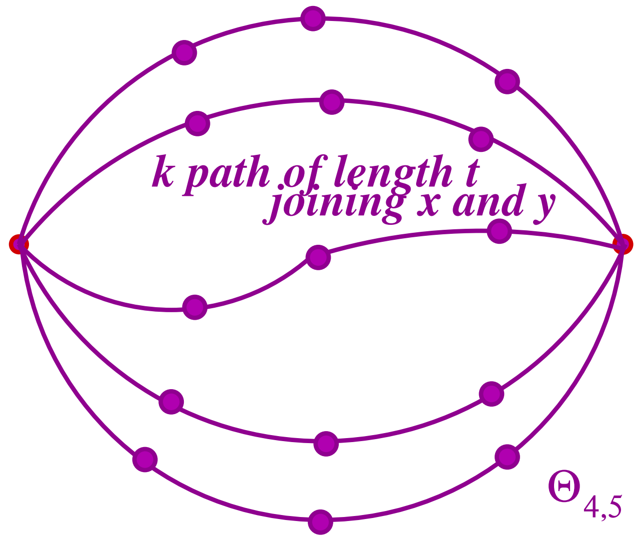
x is a fixed vertex, S_i is the i^{th} level, we need that

$$\frac{|S_{i+1}|}{|S_i|} > c_L \cdot d_{\min}(G_n) \quad \text{for} \quad i = 1, \dots, k.$$

Faudree-Simonovits method:

This gives more: $\mathbf{ex}(n, \Theta_{k,\ell}) = O(n^{1+(1/k)})$.





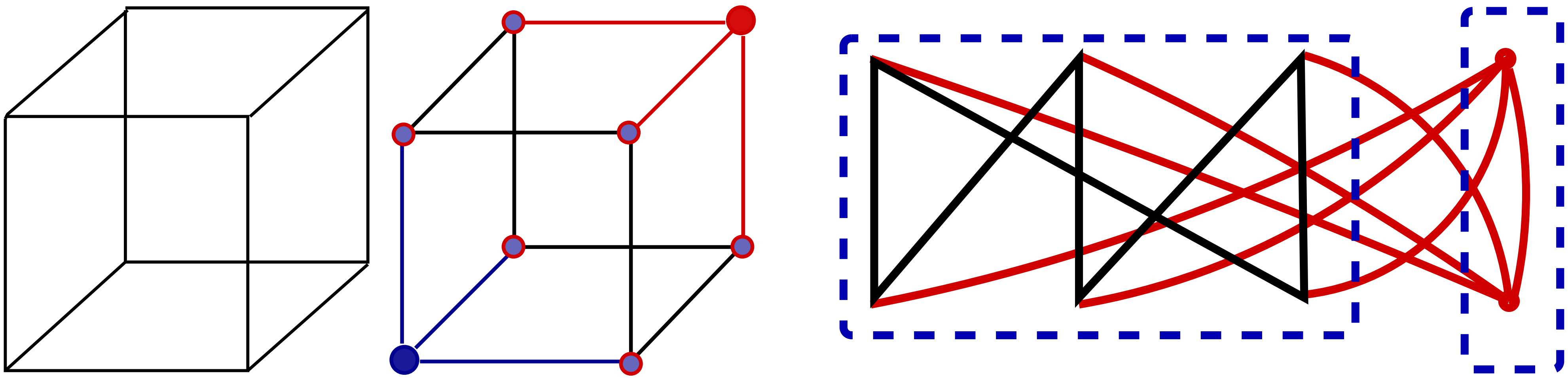
To prove the expansion we distinguish rich and poor vertices:

- Rich = connected to many different colours
- Poor: connected to few different colours.

Cube-reduction

Theorem (Cube, Erdős-Sim.). $\mathrm{ex}(n, Q_3) = O(n^{8/5})$.

New Proofs: PINCHASI-SHARIR, FÜREDI, ...

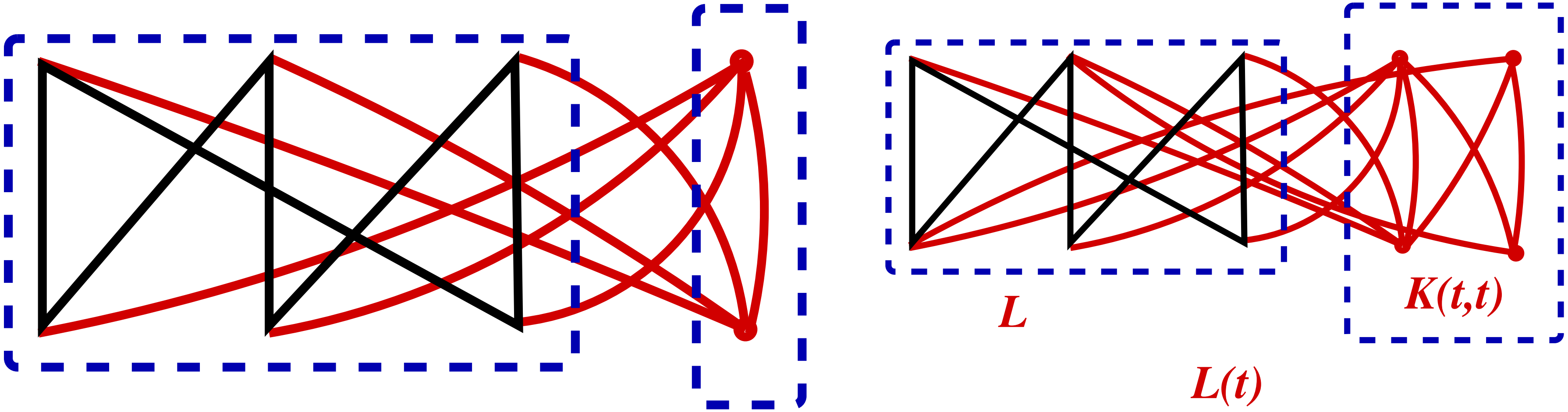


The cube is obtained from C_6 by adding two vertices, and joining two new vertices to this C_6 as above.

- We shall use a more general definition: $L(t)$.

General definition of $L(t)$:

- Take an arbitrary bipartite graph L and $K(t,t)$. 2-color them!
- join each vertex of $K(t,t)$ to each vertex of L of the opposite color



Theorem (Reduction, Erdős-Sim.). *Fix a bipartite L and an integer t. If $\mathbf{ex}(n,L)=n^{2-\alpha}$ and $L(t)$ is defined as above then $\mathbf{ex}(n,L(t))\le n^{2-\beta}$ for*

$$\frac{1}{\beta}-\frac{1}{\alpha}=t.$$

Examples

The ES reduction included many (most?) of the earlier upper bounds on bipartite L. Deleting an edge e of L, denote by $L - e$ the resulting graph.

Exercise 28. Deduce the KST theorem from the Reduction Theorem.

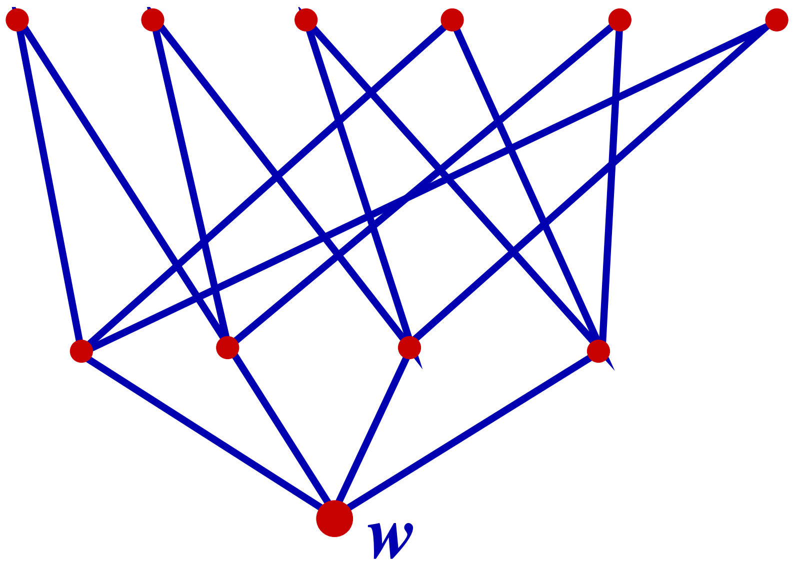
Exercise 29. Show that $\mathbf{ex}(n, Q_8 - e) = O(n^{3/2})$.

Exercise 30. Show that $\mathbf{ex}(n, K_2(p,p) - e) = O(n^{2-(1/p)})$.

Open Problem:
Find a lower bound for $\mathbf{ex}(n, Q_8)$, better than $cn^{3/2}$.
Conjectured: $\mathbf{ex}(n, Q_8) > cn^{8/5}$.

What is left out?

The graph F_{11} below is full of C_4's.



Erdős conjectured that **ex**$(n, F_{11}) = O(n^{3/2})$. The methods known tose days did not give this. Füredi proved the conjecture. $\rightarrow$ Fur11CCA

The general definition: In $F_{1+k+\binom{k}{\ell}}$ w is joined to k vertices $x_1, \dots, x_k$, and $\binom{k}{\ell}$ further vertices are joined to each ℓ-tuple $x_{i_1} \dots x_{i_\ell}$.

$F_{11} = F_{1+4+\binom{4}{2}}$.

An Erdős problem: Compactness?

We know that if G_n is bipartite, C_4-free, then

$$e(G_n) \le \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

We have seen that there are C_4-free graphs G_n with

$$e(G_n) \approx \frac{1}{2} n^{3/2} + o(n^{3/2}).$$

Is it true that if $K_3, C_4 \not\subseteq G_n$ then

$$e(G_n) \le \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2})\ ?$$

This does not hold for hypergraphs (BALOGH) or for geometric graphs (TARDOS)

Erdős-Sim., C_5-compactness:

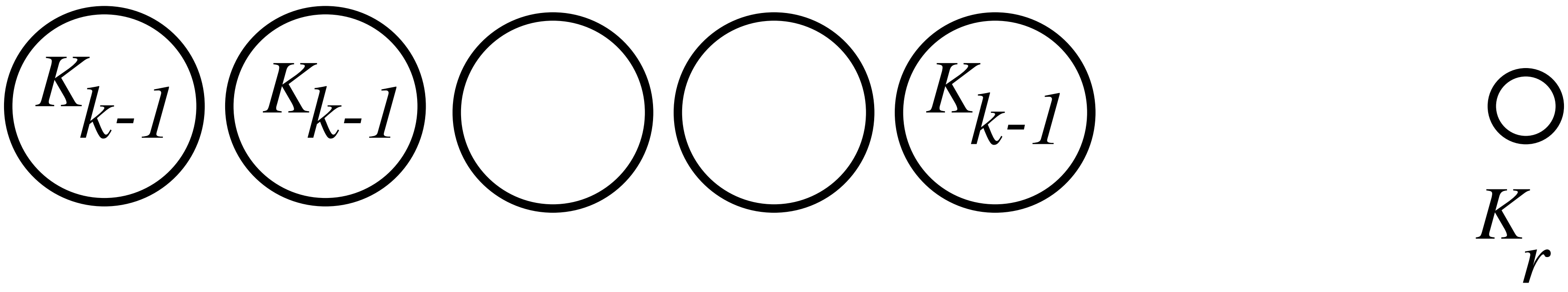
If $C_5, C_4 \not\subseteq G_n$ then $\rightarrow$ $\boxed{\text{ErdSimComp}}$

$$e(G_n) \le \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

Unfortunately, this is much weaker than the conjecture on C_3, C_4: excluding a C_5 is a much more restrictive condition.

Erdős-Gallai:

$$\mathbf{ex}(n, P_k) \leq \frac{1}{2}(k-2)n.$$



FAUDREE-SCHELP

KOPYLOV

Erdős-T. Sós:

Conjecture (Extremal number of the trees). *For any tree T_k,*
$\mathbf{ex}(n, T_k) \le \frac{1}{2}(k-2)n.$

- Motivation: True for the two extreme cases: path and star.
- fight for $\frac{1}{2}$
- Partial results

Theorem (Andrew McLennan). *The Erdős-Sós conjecture holds for trees of diameter 4,* *(2003)*

Theorem (Ajtai-Komlós-Sim.-Szemerédi). *If $k > k_0$ then true:*

$$\mathbf{ex}(n, T_k) \le \frac{1}{2}(k-2)n.$$

Lower bounds for degenerate cases

- Why is the random method weak?
- Why is the Lenz construction important?
- Finite geometries
- Commutative algebra method
  - Kollár-Rónyai-Szabó
  - Alon-Rónyai-Szabó
- Margulis-Lubotzky-Phillips-Sarnak method

→ Margu

→ LubPhilSar

- Lazebnik-Ustimenko-Woldar
  - Even cycle-extremal graphs

Why is the random method weak?

Let $\chi(L) = 2$, $v := v(L)$, $e = e(L)$.

- The simple Random method (threshold) gives an L-free graph G_n with $cn^{2-(v/e)}$ edges. For C_{2k} this is too weak.

- Improved method (first moment):

$$cn^{2-\frac{v-2}{e-1}}.$$

For C_{2k} this yields

$$cn^{2-\frac{2k-2}{2k-1}} = cn^{1+\frac{1}{2k-1}}.$$

Conjectured:

$$\mathbf{ex}(n, C_{2k}) > cn^{1+\frac{1}{k}}.$$

Random method, General Case:

Theorem (General Lower Bound). *If a finite $\mathcal{L}$ does not contain trees (or forests), then for some constants*
$c = c_{\mathcal{L}} > 0$, $\alpha = \alpha_{\mathcal{L}} > 0$

$$\mathbf{ex}(n, \mathcal{L}) > cn^{1+\alpha}.$$

Proof (Sketch).

- Discard the non-bipartite L's.
- Fix a large $k = k(\mathcal{L})$. (E.g., $k = \max v(L)$.)
- We know $\mathbf{ex}(n, \{C_4, \dots, C_{2k}\}) > cn^{2-\frac{v-2}{e-1}}$.
- Since each $L \in \mathcal{L}$ contains some $C_{2\ell}$ ($\ell \le k$),

$$\mathbf{ex}(n, \mathcal{L}) \ge \mathbf{ex}(n, C_4, \dots, C_{2k}) > cn^{1+\frac{1}{2k-1}}.$$

Constructions using finite geometries

$p \approx \sqrt{n}$ = prime ($n = p^2$)

Vertices of the graph F_n are pairs: $(a, b) \mod p.$

Edges: (a, b) is joined to (x, y) if $ac + bx = 1 \mod p.$

Geometry in the constructions: the neighbourhood is a straight line and two such nighbourhoods intersect in ≤ 1 vertex.

$\Longrightarrow$ No $C_4 \subseteq F_n$

loops to be deleted

most degrees are around $\sqrt{n}$:

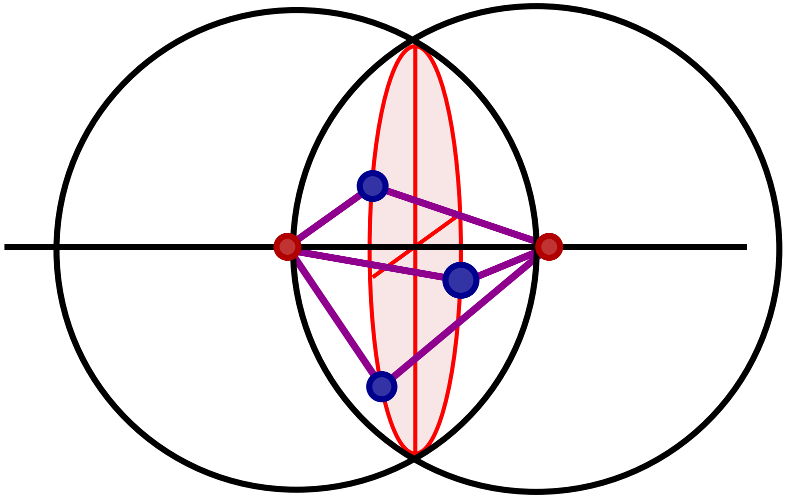
$$e(F_n) \approx \tfrac{1}{2} n \sqrt{n}$$

Finite geometries: Brown construction

Vertices: $(x, y, z) \bmod p$

Edges:

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = \alpha.$$



$$\mathbf{ex}(n, K(3,3)) > \frac{1}{2} n^{2-(1/3)} + o(\ldots).$$

$\longrightarrow$ BrownThom

The first missing case

The above methods do not work for $K(4,4)$.
We do not even know if

$$\frac{\mathbf{ex}(n, K_2(4,4))}{\mathbf{ex}(n, K_2(3,3))} \longrightarrow \infty.$$

$\boxed{\text{One reason for the difficulty: Lenz construction:}}$

$\mathbb{E}^4$ contains two circles in two orthogonal planes:

$$\mathcal{C}_1 = \{x^2 + y^2 = \frac{1}{2},\ z = 0,\ w = 0\} \quad \text{and} \quad \mathcal{C}_2 = \{z^2 + w^2 = \frac{1}{2},\ x = 0,\ y = 0\}$$

and each point of $\mathcal{C}_1$ has distance 1 from each point of $\mathcal{C}_2$: the unit distance graph contains a $K_2(\infty, \infty)$.

Other similar constructions

- E. Klein
- Reiman
- Hylten-Cavallius
- Mörs construction

- Singleton
- Benson construction

- Wenger construction

- Lazebnik-Ustimenko

Algebraic constructions

- Margulis

- Margulis II.
- Lubotzky-Phillips-Sarnak

- Lazebnik-Ustimenko

Margulis construction

Simplest case → Margu

Find a 4-regular graph with girth $c \log n$.

Take the Cayley graph generated by

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

They are independent: No (long) product of these matrices and their inverses is I unless it trivially simplifies to I.

$\Rightarrow$ Infinite Cayley = 4-regular tree. Take everything mod p.

G_n, $n \approx p^3$ vertices, $g(G_n) > c \log p$

General case, even degree: $\dots$

In Lubotzky-Phillips-Sarnak: quaternions: matrices with Gaussian integers

The Universe

Extremal problems can be asked (and are asked) for many other object types.

- Mostly simple graphs
- Digraphs $\longrightarrow$ $\boxed{\text{Brown-Harary, Brown, Erdős, Simonovits}}$
- Multigraphs $\longrightarrow$ $\boxed{\text{Brown-Harary, Brown, Erdős, Simonovits}}$
- Hypergraphs $\longrightarrow$ $\boxed{\text{Turán, \ldots}}$
- Geometric graph $\longrightarrow$ $\boxed{\text{Pach, Tóth, Tardos}}$
- Integers $\longrightarrow$ $\boxed{\text{Erdős,Sidon,Szemerédi,\ldots}}$
- groups
- other structures

The general problem

Given a **universe**, and a structure $\mathcal{A}$ with two (natural parameters) n and e on its objects G.
Given a property $\mathcal{P}$.

$$\mathbf{ex}(n, \mathcal{P}) = \max_{n(G)=n} e(G).$$

Determine $\mathbf{ex}(n, \mathcal{P})$ and

describe the **EXTREMAL STRUCTURES**

Examples: Hypergraphs, . . .

We return to this later.

Examples: Multigraphs, Digraphs, . . .

- BROWN-HARARY: bounded multiplicity: r
- BROWN-ERDŐS-SIM. $\rightarrow$ $\boxed{\text{BrownSimDM}}$
- $r = 2s$: digraph problems and multigraph problems seem to be equivalent:
  – each multigraph problem can easily be reduced to digraph problems
  – and we do not know digraph problems that are really more difficult than some corresponding multigraph problem

Examples: Numbers, ...

- Tomsk
- Sidon sequences
- Let $r_k(n)$ denote the maximum m such that there are m integers $a_1, \dots, a_m \in [1, n]$ without k-term arithmetic progression.

Szemerédi Theorem. *For any fixed k $r_k(n) = o(n)$ as $n \to \infty$.*

History (simplified):

- K. F. Roth: $r_3(n) = o(n)$
- Szemerédi
- Fürstenberg: Ergodic proof
- Fürstenberg-Katznelson: Higher dimension
- Polynomial extension, Hales-Jewett extension
- Gowers: much more effective

On the number of $\mathcal{L}$-free graphs

J. Balogh, B. Bollobás, and Simonovits

Erdős–Kleitman–Rothschild type results

Below $\mathcal{P}(n, \mathcal{L})$ is the family of n-vertex $\mathcal{L}$-free graphs.
Erdős, Kleitman and Rothschild $|\mathcal{P}(n, L)|$ = ? for $L = K_{p+1}$.

$$|\mathcal{P}(n, \mathcal{L})| \geq 2^{\mathbf{ex}(n,\mathcal{L})}. \tag{1}$$

Conjecture (Erdős).

$$|\mathcal{P}(n, L)| = 2^{(1+o(1))\mathbf{ex}(n,L)}.$$

Erdős-Kleitman-Rothschild

Let $\varphi(n) = o(n^{p-1})$ be a fixed function, and write $\mathcal{P}(n, K_{p+1}, \varphi)$ for the family of graphs containing at most $\varphi(n)$ copies of K_{p+1}. Then

$$|\mathcal{P}(n, K_{p+1}, \varphi)| \leq 2^{\left(1-\frac{1}{p}\right)\binom{n}{2}+o(n^2)}.$$

In particular,

$$|\mathcal{P}(n, K_{p+1})| \leq 2^{\left(1-\frac{1}{p}\right)\binom{n}{2}+o(n^2)}.$$

Theorem [EKR] is sharp.

Erdős-Frankl-Rödl Theorem

Let L be a graph with $\chi(L) \geq 3$. Then → $\boxed{\text{ErdFraRo}}$

$$|\mathcal{P}(n,L)| = 2^{(1+o(1))\mathbf{ex}(n,L)}$$

$$= 2^{\left(1-\frac{1}{\chi(L)-1}\right)\binom{n}{2}+o\left(n^2\right)}.$$

If L is a tree then Erdős' conjecture fails:

$$2^{c_1 n \log n} \leq |\mathcal{P}(n,L)| \leq 2^{c_2 n \log n},$$

for some positive constants c_1 and c_2.

If $L = C_4$: $\boxed{\text{difficult}}$

(see Kleitman and Winston) → $\boxed{\text{KleiWin}}$.

Starting point: Erdős-Frankl-Rödl Theorem

(2) $$|\mathcal{P}(n,\mathcal{L})| = 2^{(1+o(1))\mathbf{ex}(n,\mathcal{L})} = 2^{\left(1-\frac{1}{p}\right)\binom{n}{2}+o(n^2)},$$

where

(3) $$p+1 = \min_{L\in\mathcal{L}} \chi(L).$$

Improved

→ BalBollSim

Theorem (Balogh-Bollobás-Sim). *For every non-trivial family $\mathcal{L}$ of graphs there exists a constant positive $\gamma = \gamma_{\mathcal{L}}$ such that, for $p+1 = \min_{L\in\mathcal{L}} \chi(L)$,*

$$|\mathcal{P}(n,\mathcal{L})| \le 2^{\left(1-\frac{1}{p}\right)\binom{n}{2}+O\left(n^{2-\gamma}\right)}.$$

Sharper estimates?

Theorem (Balogh-Bollobás-Sim: "Sharp form"). *Assume that $\mathcal{L}$ is finite. Then for almost all $\mathcal{L}$-free graphs G_n we can delete $h = O_{\mathcal{L}}(1)$ vertices of G_n and partition the remaning vertices into p classes $U_1, \dots, U_p$ so that $G[U_i]$ are $\mathcal{M}$-free $(i = 1, \dots, p)$.*

There are even sharper results $\rightarrow$ $\boxed{\text{BalBollSimB}}$

Problems, Exercises

Exercise 31. Let the vertices of a graph be points in $\mathbb{E}^2$ and join two points by an edge if their distance is 1. Show that this graph contains no $K(2,3)$.

Exercise 32. Let the vertices of a graph be points in $\mathbb{E}^3$ and join two points by an edge if their distance is 1. Show that this graph contains no $K(3,3)$.

Exercise 33. If we take n points of general position in the d-dimensional Euclidean space (i.e., no d of them belong to a $d-1$-dimensional affine subspace) and join two of them if their distance is 1, then the resulting graph G_n can not contain K_{d+2}.

Exercise 34. If $a_1, \dots, a_p$ and $b_1, \dots, b_q$ are points in $\mathbb{E}^d$ and all the pairwise distances $\rho(a_i, b_j) = 1$, then the two affine subspaces defined by them are either orthogonal to each other or one of them reduces to one point.

Problems, Exercises, cont.

Exercise 35. Show that if we join two points in $\mathbb{E}^4$ when their distance is 1, then the resulting graph contains a $K(\infty, \infty)$.

Exercise 36. Let $v = v(L)$. Prove that if we put more than $n^{1-(1/v)}$ edges into some class of $T_{n,p}$ then the resulting graph contains L. Can you sharpen this statement?

Exercise 37. (Petty's theorem) If we have n points in $\mathbb{E}^d$ with an arbitrary metric $\rho(x, t)$ and its "unit distance graph" contains a K_p then $p \leq 2^d$. (Sharp for the d-dimensional cube and the ℓ_1-metric.)

Erdős on unit distances

Many of the problems in the area are connected with the following beautiful and famous conjecture, motivated by some grid constructions.

Conjecture (P. Erdős). *For every $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that if $n > n_0(\varepsilon)$ and G_n is the Unit Distance Graph of a set of n points in $\mathbb{E}^2$ then*

$$e(G_n) < n^{1+\varepsilon}.$$

Part II: Regularity Lemma for graphs

- Origins/connections to the existence of arithmetic progressions in dense sequences
- Connection to the quantitative Erdõs-Stone theorem
- First graph theoretic applications (Ruzsa-Szemerédi theorem, RAMSEY-TURN problems)
- Counting lemma, removal lemma, coloured regularity lemma

Regular pairs

Regular pairs are highly uniform bipartite graphs, namely ones in which the density of any reasonably sized subgraph is about the same as the overall density of the graph.

Definition (Regularity condition). Let $\varepsilon > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is ε-regular if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

The Regularity Lemma

The Regularity Lemma says that every dense graph can be partitioned into a small number of regular pairs and a few leftover edges. Since regular pairs behave as random bipartite graphs in many ways, the R.L. provides us with an approximation of an arbitrary dense graph with the union of a constant number of random-looking bipartite graphs.

Regularity Lemma

Theorem (Szemerédi, 1978). *For every $\varepsilon > 0$ and m there are $M(\varepsilon, m)$ and $N(\varepsilon, m)$ with the following property: for every graph G with $n \geq N(\varepsilon, m)$ vertices there is a partition of the vertex set into k classes*

$$V = V_1 + V_2 + \ldots + V_k$$

such that

- $m \leq k \leq M(\varepsilon, m)$,
- $||V_i| - |V_j|| < 1$, $(1 \leq i < j \leq k)$
- *all but at most εk^2, of the pairs (V_i, V_j) are ε-regular.*

See

→ SzemRegu, KomSim

The role of m

is to make the classes V_i sufficiently small, so that the number of edges inside those classes are negligible. Hence, the following is an alternative form of the R.L.

Theorem (Regularity Lemma – alternative form). *For every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that the vertex set of any n-graph G can be partitioned into k sets $V_1, \dots, V_k$, for some $k \le M(\varepsilon)$, so that*

- $|V_i| \le \lceil \varepsilon n \rceil$ *for every i,*
- $||V_i| - |V_j|| \le 1$ *for all i, j,*
- (V_i, V_j) *is ε-regular in G for all but at most εk^2 pairs (i, j).*

For $e(G_n) = o(n^2)$, the Regularity Lemma becomes trivial.

Clusters, Reduced Graph

The classes V_i will be called groups or clusters.

Given an arbitrary graph $G = (V, E)$, a partition P of the vertex-set V into $V_1, \ldots, V_k$, and two parameters ε, d, we define the Reduced Graph (or Cluster Graph) R as follows: its vertices are the clusters $V_1, \ldots, V_k$ and V_i is joined to V_j if (V_i, V_j) is ε-regular with density more than d.

Most applications of the Regularity Lemma use Reduced Graphs, and they depend upon the fact that many properties of R are inherited by G.

Defect form of the Cauchy-Schwarz

Lemma 1 (Improved Cauchy-Schwarz inequality). *If for the integers* $0 < m < n$,

$$\sum_{k=1}^{m} X_k = \frac{m}{n} \sum_{k=1}^{n} X_k + \delta,$$

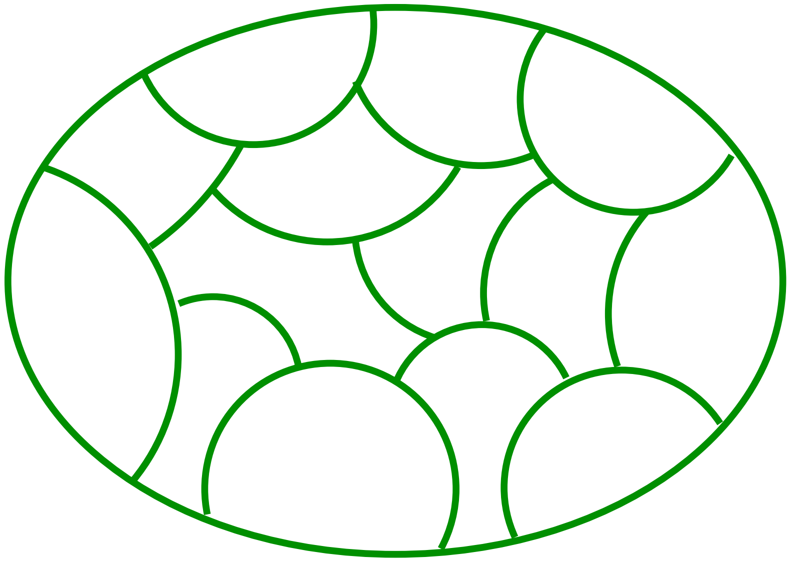
then

$$\sum_{k=1}^{n} X_k^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} X_k \right)^2 + \frac{\delta^2 n}{m(n-m)}.$$

How to prove Regularity Lemma?

- Use the Defect form of Cauchy-Schwarz.
- Index:

$$I(\mathcal{P}) = \frac{1}{k^2} \sum d(V_i, V_j)^2 < \frac{1}{2}.$$



- Improving the partition

Coloured Regularity Lemma

- If we have several colours, say, Black, Blue, Red, then we have a Szemerédi partition good for each colour simultaneously.

- How to apply this?

Inheritance

G_n inherits the properties of the cluster graph H_k.

- sometimes in an improved form!

Through a simplified example:

- If H_k contains a C_7 then G_n contains many: cn^7.

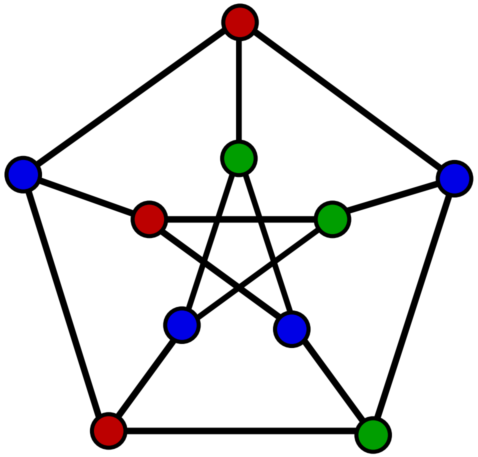
Counting Lemma

Through a simplified example:

Removal Lemma

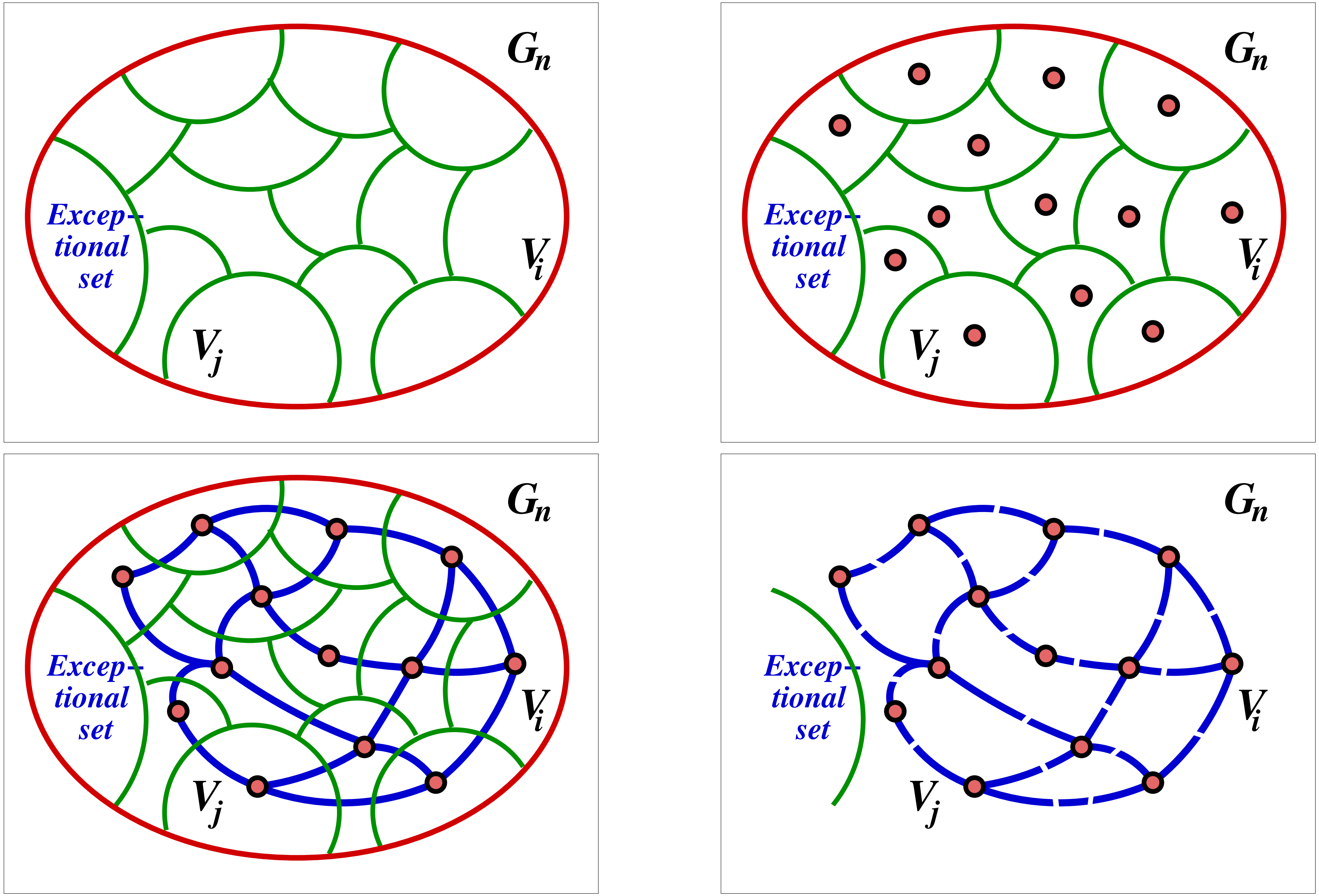
Through a simplified example:

For every $\varepsilon > 0$ there is a $\delta > 0$:
If a G_n does not contain δn^{10} copies of the Petersen graph, then we can delete εn^2 edges to destroy all the Petersen subgraphs.

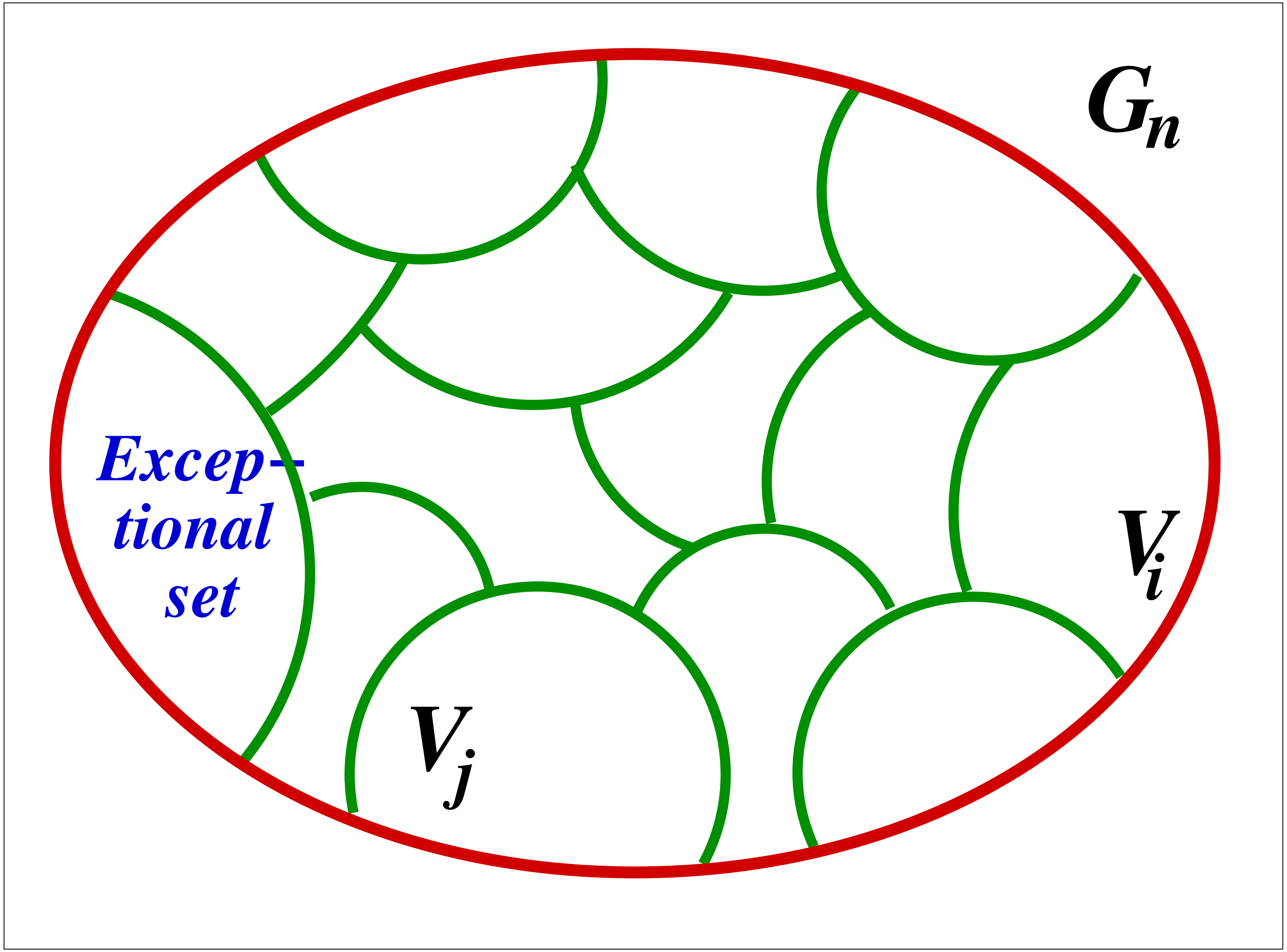


- something similar is applicable in PROPERTY TESTING.

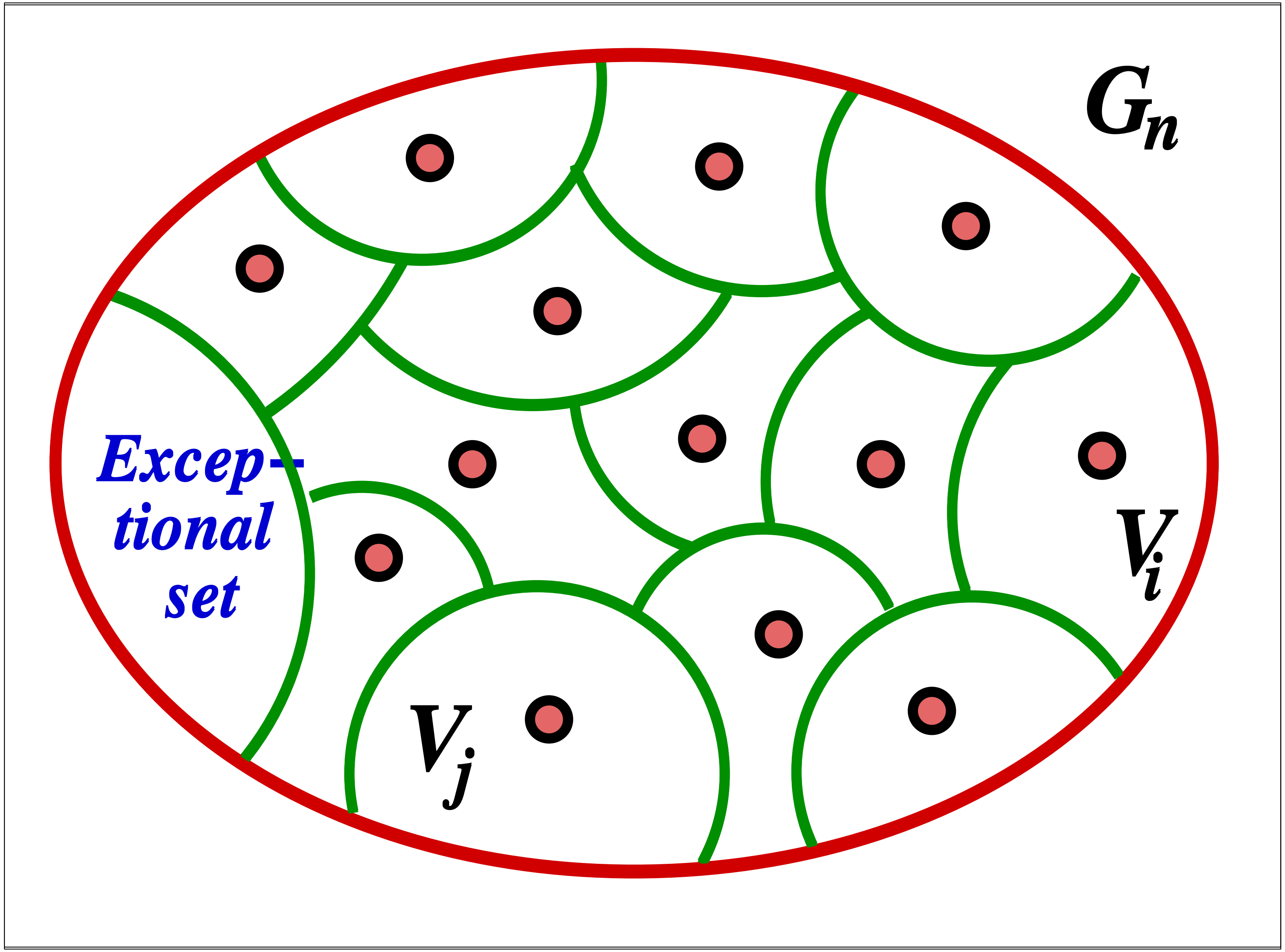
The Cluster graph, illustrated:



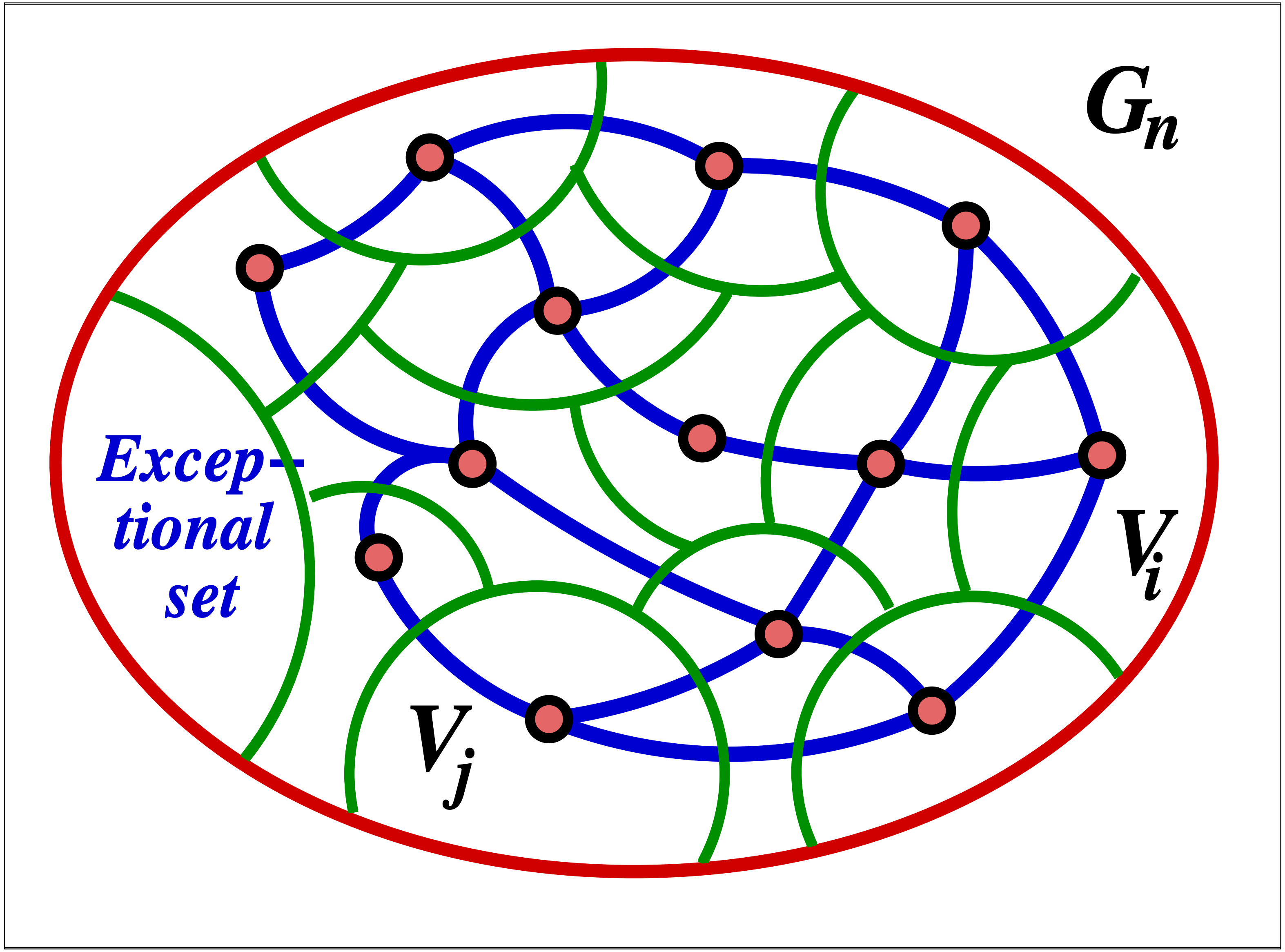
The Cluster graph, illustrated:



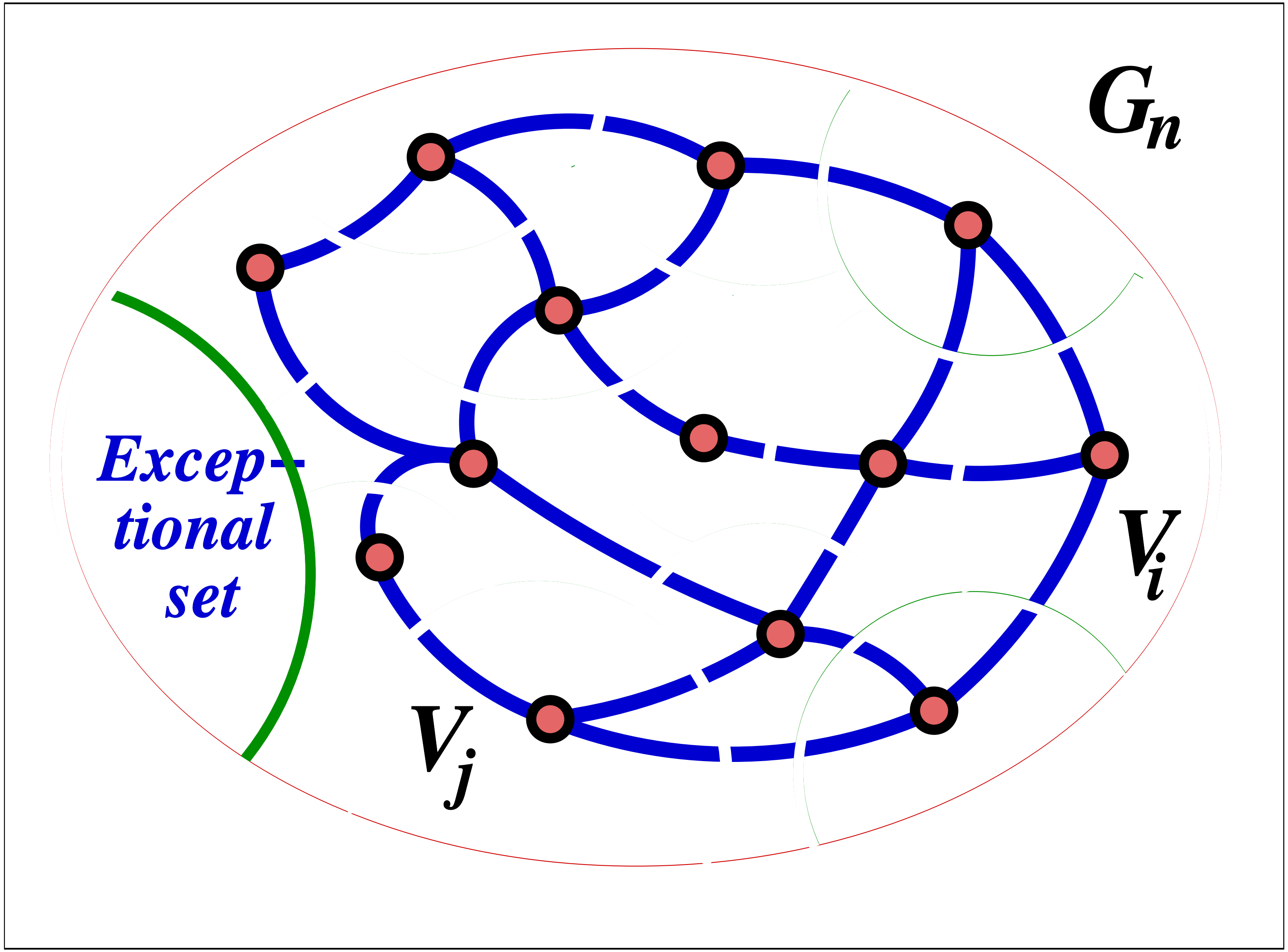
The Cluster graph, illustrated:



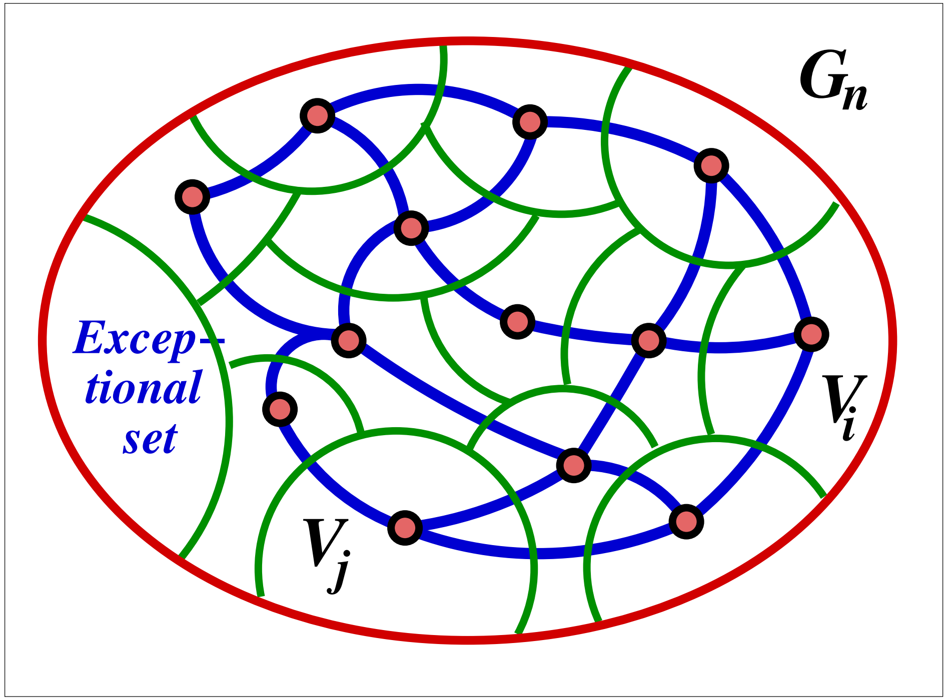
The Cluster graph, illustrated:



The Cluster graph, illustrated:



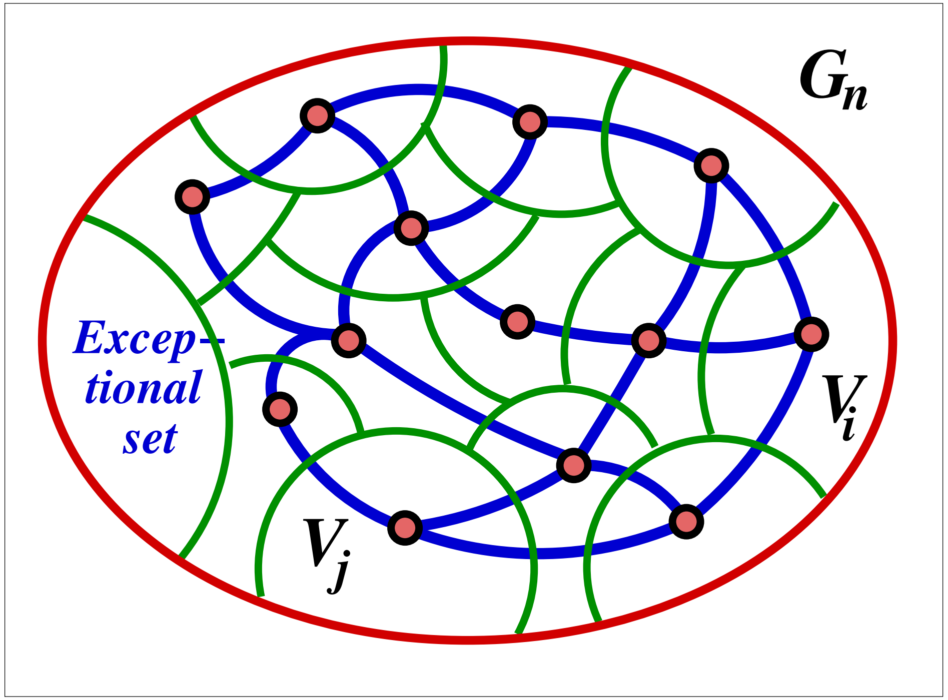
How to prove Erdős-Stone?



- No K_{p+1} in the Reduced graph H_k
- Apply Turán's theorem
- Estimate the edges of the original graph:

$$e(G_n) \le e(H_k)m^2 + 3\varepsilon n^2.$$

How to prove Stability?



- No K_{p+1} in the Reduced graph H_k
- Apply Turán's theorem with stability (Füredi)
- Estimate the edges of the original graph

Ramsey-Turán problems

Theorem (Szemerédi). → $\boxed{\textit{SzemRT}}$

If G_n *does not contain* K_4 *and* $\alpha(G_n) = o(n)$ *then*

$$e(G_n) = \frac{n^2}{8} + o(n^2).$$

How to prove this?

- Use Regularity Lemma
- Show that the reduced graph does not contain K_3.
- Show that the reduced graph does not contain

$$d(V_i, V_j) > \frac{1}{2} + \varepsilon$$

Blowup Lemma

KOMLÓS, G. SÁRKÖZY, SZEMERÉDI: → BLOWUP

Good to prove the existence of spanning subgraphs

- Pósa-Seymour conjecture,…

(A, B) is (ε, δ)**-super-regular** if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$e(X, Y) > \delta|X||Y|,$$

and

$$deg(a) > \delta|B| \quad \text{for all} \quad a \in A,$$

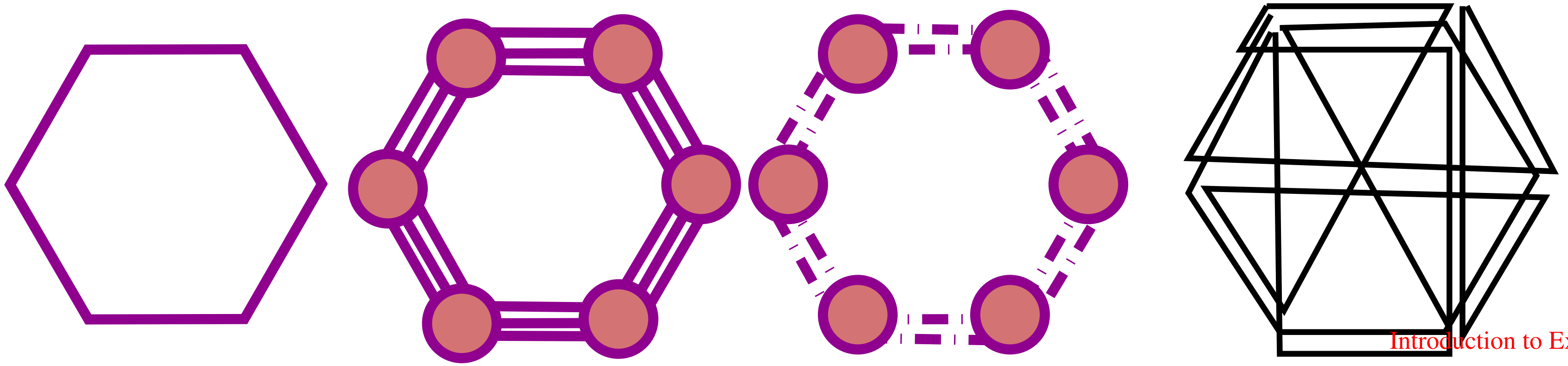
$$\text{and} \quad deg(b) > \delta|A| \quad \text{for all} \quad b \in B.$$

Blowup Lemma

Theorem 2. *Given a graph R_r and $\delta, \Delta > 0$, there exists an $\varepsilon > 0$ such that the following holds. N = arbitrary positive integer,*

- *replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \dots, V_r$.*
- *Construct two graphs on the same $V = \cup V_i$. $R(N)$ is obtained by replacing all edges of R with copies of $K_{N,N}$,*
- *and a sparser graph G is constructed by replacing the edges of R with (ε, δ)-super-regular pairs.*

If H with $\Delta(H) \le \Delta$ is embeddable into $R(N)$ then it is already embeddable into G.



Other Regularity Lemmas

- FRIEZE-KANNAN
Background in statictics, more applicable in algorithms
- LOVÁSZ-B. SZEGEDY: Limit objects, continuous version
- ALON-FISCHER-KRIVELEVICH-M. SZEGEDY:
Used for property testing

- ALON-SHAPIRA: property testing is equivalent to using Regularity Lemma

Szemerédi's Lemma for the Analyst

This is the title of a paper of L. LOVÁSZ and B. SZEGEDY
Hilbert spaces, compactness, covering

Hypergraph regularity lemmas

This topic is important but completely neglected here.

- Frankl-Rödl
- Frankl-Rödl 2.
- F. Chung
- A. Steger
- Rödl, Skokan, Nagle, Schacht,. . .
- Gowers, Tao,. . .

Part III: Some recent results

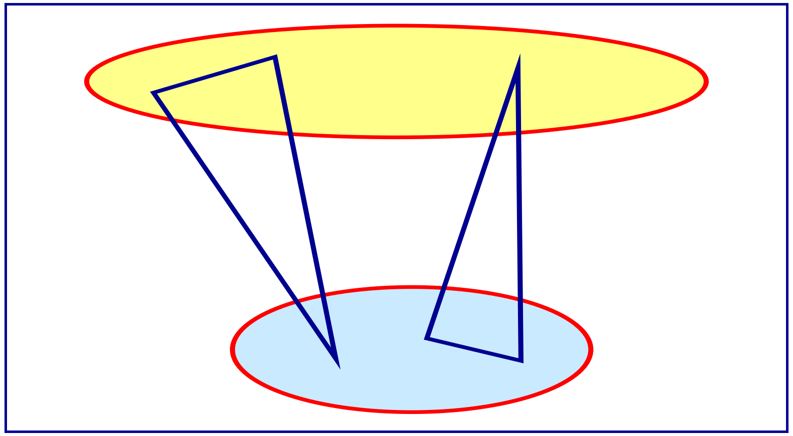
- Erdős-Sós conjecture on trees
- 3-coloured RAMSEY theorem for cycles
- Some hypergraph results

Hypergraph extremal problems

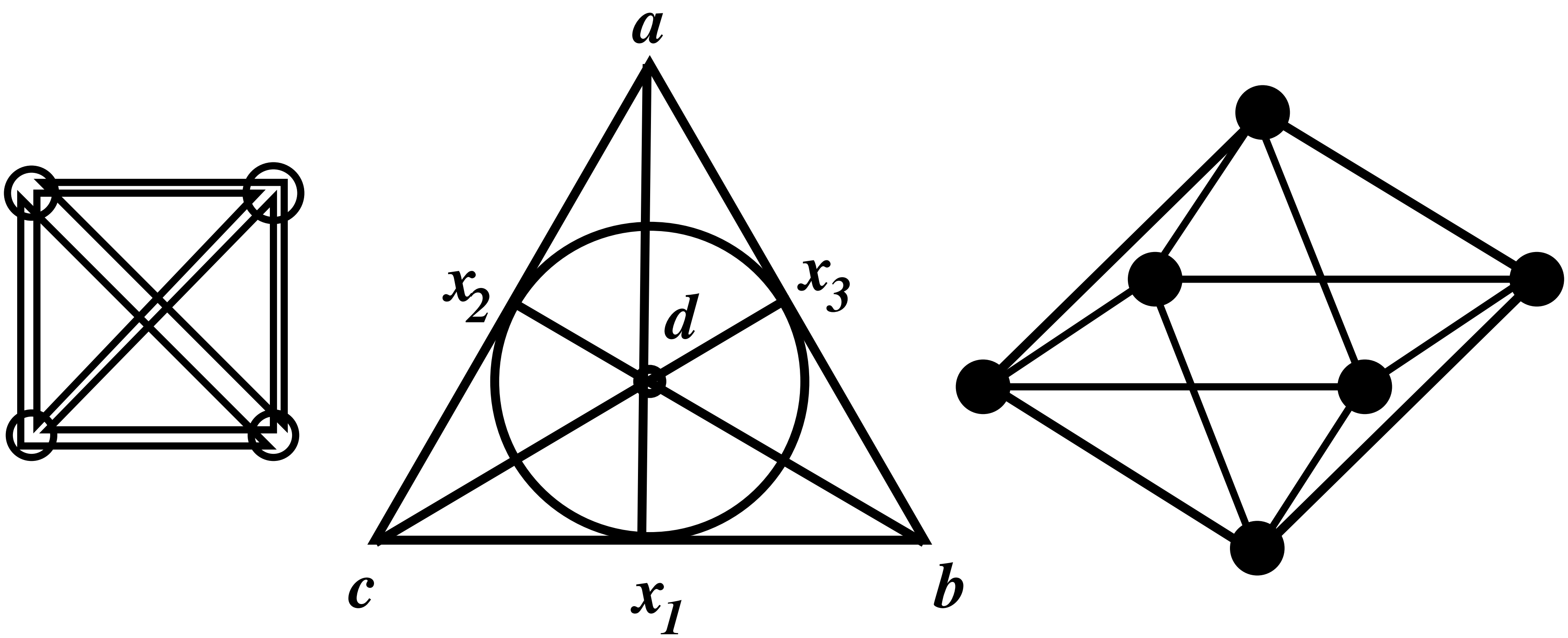
3-uniform hypergraphs: $\mathcal{H}_n^{(3)} = (V, \mathcal{H})$

$\chi(\mathcal{H}_n^{(3)})$: the minimum number of colors needed to have in each triple 2 or 3 colors.

Bipartite 3-uniform hypergraphs:



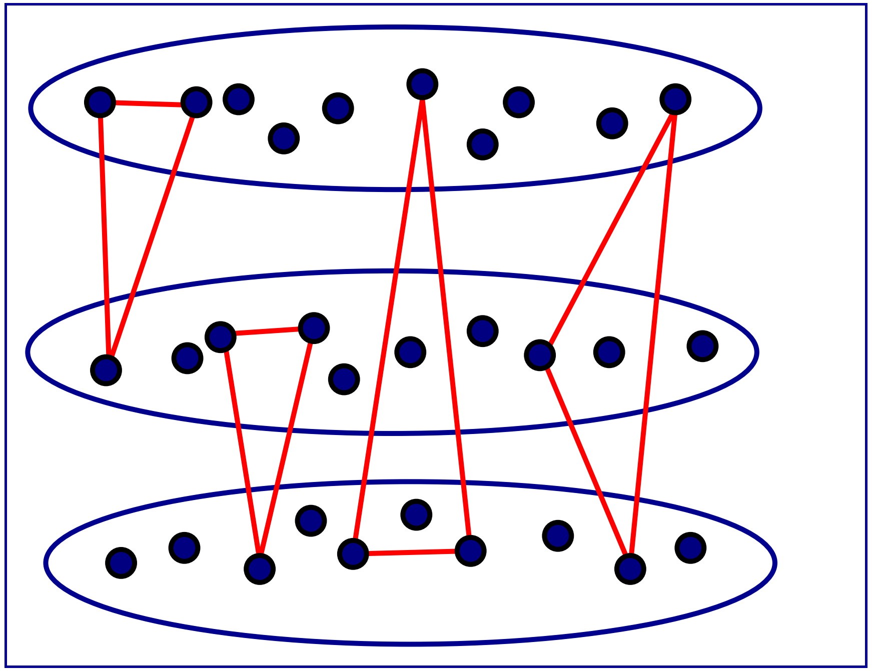
THE EDGES INTERSECT BOTH CLASSES

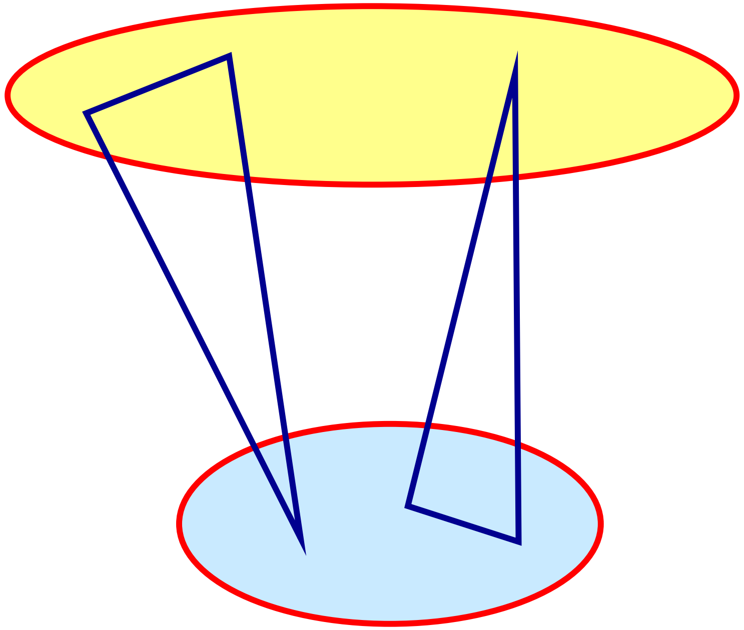


Complete 4-graph, || Fano configuration, || octahedron graph

The famous Turán conjecture

Conjecture 2 (Turán). *The following structure is the (? asymptotically) extremal structure for $K_4^{(3)}$:*

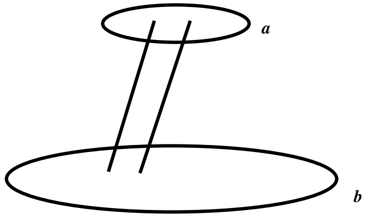




For $K_5^{(3)}$ one conjectured extremal graph is just the above "complete bipartite" one!

Two important theorems

Kővári-T. Sós-Turán theorem. *Let $2 \le a \le b$ be fixed integers. Then*

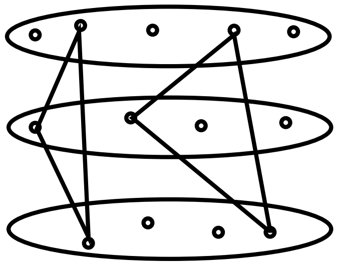


$$\mathbf{ex}(n, K(a,b)) \le \frac{1}{2}\sqrt[a]{b-1}\,n^{2-\frac{1}{a}} + \frac{1}{2}an.$$

→ KovSosTur

Erdős theorem.

$$\mathbf{ex}(n, K_r^{(r)}(m,\ldots,m)) = O(n^{r-(1/m^{r-1})}).$$



How to apply this?

Call a hypergraph extremal problem (for k-uniform hypergraphs) degenerate if

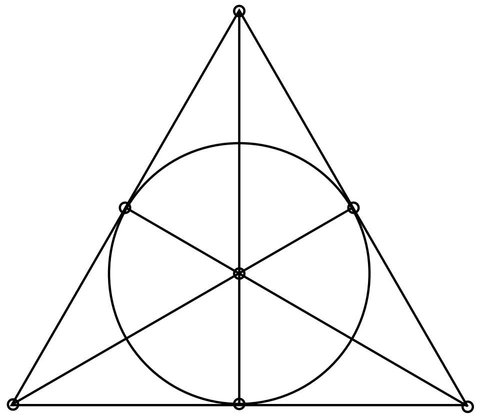
$$\mathbf{ex}(n, \mathcal{L}) = o(n^k).$$

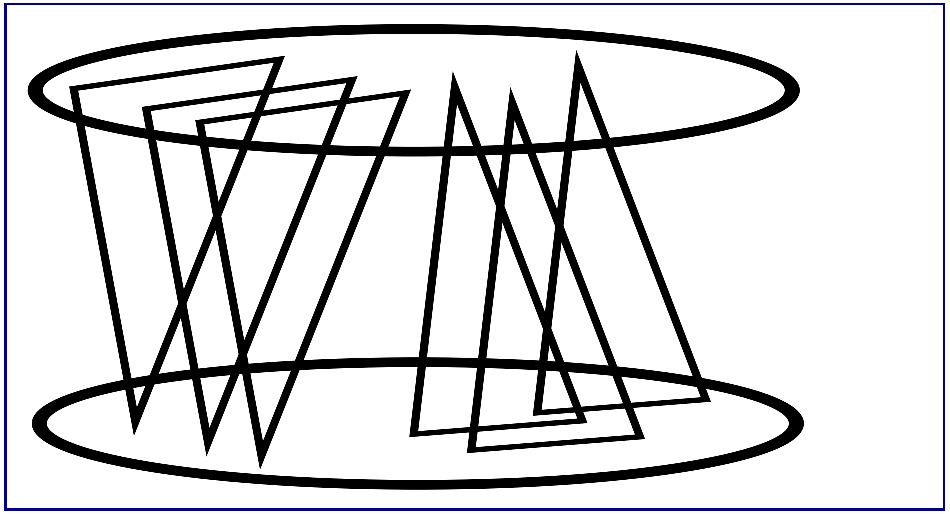
Degenerate hypergraph problems

Exercise 38. Prove that the problem of L is degenerate iff it can be k-colored so at each edge meats each of the k colors.

The T. Sós conjecture

Conjecture (V. T. Sós). *Partition $n > n_0$ vertices into two classes A and B with $||A| - |B|| \le 1$ and take all the triples intersecting both A and B. The obtained 3-uniform hypergraph is extremal for $\mathcal{F}_7$.*





The conjectured extremal graphs: $\mathcal{B}(X, \overline{X})$

Füredi-Kündgen Theorem

If M_n is an arbitrary multigraph (without restriction on the edge multiplicities, except that they are nonnegative) and all the 4-vertex subgraphs of M_n have at most 20 edges, then

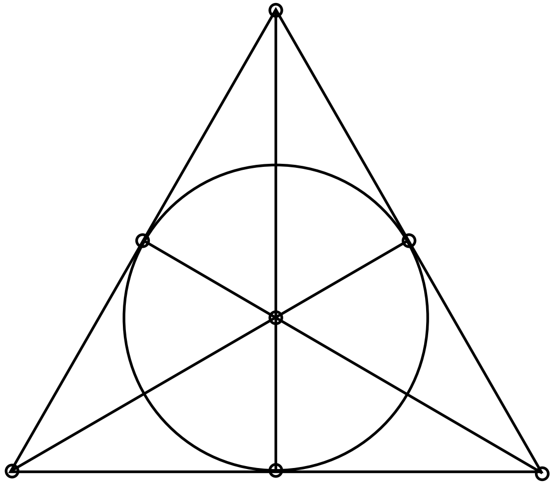
$$e(M_m) \leq 3\binom{n}{2} + O(n).$$

$\rightarrow$ FureKund

Theorem 2 (de Caen and Füredi). $\rightarrow$ *FureCaen*

$$\mathbf{ex}(n, \mathcal{F}_7) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

New Results: The Fano-extremal graphs



Main theorem. *If $\mathcal{H}_n^{(3)}$ is a triple system on $n > n_1$ vertices not containing $\mathcal{F}_7$ and of maximum cardinality, then $\chi(\mathcal{H}_n^{(3)}) = 2$.*

$$\Longrightarrow \qquad \mathrm{ex}_3(n, \mathcal{F}_7) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

Remark 1. The same result was proved independently, in a fairly similar way, by

Peter Keevash and Benny Sudakov $\rightarrow$ $\boxed{\text{KeeSud}}$.

Theorem 2. *There exist a $\gamma_2 > 0$ and an n_2 such that:*
If $\mathcal{F}_7 \not\subseteq \mathcal{H}_n^{(3)}$ and

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2} \quad \text{for each} \quad x \in V(\mathcal{H}_n^{(3)}),$$

then $\mathcal{H}_n^{(3)}$ is bipartite, $\mathcal{H}_n^{(3)} \subseteq \mathcal{H}_n^{(3)}(X, \overline{X})$. $\rightarrow$ $\boxed{\textit{FureSimFano}}$

What to read?

- Bollobás: Extremal Graph Theory
- Handbook of Combinatorics, Bollobás, Alon,. . .
- Füredi surveys (London, Zurich)
- Erdős volumes (e.g. 1999)
- Erdős papers, e.g. Art of Counting (a collection of Erdős' combinatorial papers.
- download survey papers
  – from my homepage: www.renyi.hu/ miki
  – from Yoshi's homepage,
  – from Alon's homepage,
  – from Lovász' homepage, . . .

Exercises

Exercise 39. Prove that if you attach a tree to an L containing a cycle, then for the obtained M, for large n,

$$\mathbf{ex}(n, L) = \mathbf{ex}(n, M)$$

Exercise 40. Prove that each G contains a bipartite subgraph with at least half the edges.

Exercise 41. (Győri) Prove that if G_n does not contain C_6 then it has a subgraph with roughly $\frac{1}{2}e(G_n)$ edges, not containing C_4 either.

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