ON THE STRUCTURE OF EDGE GRAPHS II

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This note is a sequel to **[1]**. First let us recall some of the notations. Denote by $G(n, m)$ a graph with n vertices and m edges. Let $K_d(r_1, \ldots, r_d)$ be the complete d-partite graph with r_i vertices in its i-th class and put $K_d(t) = K_d(t, \ldots, t)$, $K_d = K_d(1)$.

Given integers $n \geqslant d (\geqslant 2)$, let $m_d(n)$ be the minimal integer with the property that every $G(n, m)$, where $m \geqslant m_d(n)$, contains a K_d. The function $m_d(n)$ was determined by Turán **[5]**. It is easily seen that

$$m_d(n) = \frac{d-2}{2(d-1)} n^2 + o(n).$$

In this note we are interested in the maximal value of t, depending on the integers n, d $(2 \leqslant d \leqslant n)$ and on a positive number c, such that every $G(n, m)$ contains a $K_d(t)$ provided

$$m \geqslant \left(\frac{d-2}{2(d-1)} + c\right) n^2.$$

We denote this maximal value by $g(n, d, c)$. Naturally, we may and will always suppose that $c < 1/(2(d-1))$. Erdős and Stone **[3]** proved the rather surprising fact that if n is large enough then $g(n, d, c) \geqslant (l_{d-1}(n))^{\frac{1}{2}}$, where l_s denotes the s times iterated logarithm. However, this estimate turns out to be rather far from best possible. For fixed d and c $\left(c \leqslant 1/(2(d-1))\right)$ the correct order of $g(n, d, c)$ was determined by Bollobás and Erdős **[1]**, who proved that there are constants $c_2 > c_1 > 0$ such that

$$c_1 \log n \leqslant g(n, d, c) \leqslant c_2 \log n.$$

More precisely, they showed that there are positive constants γ_d, γ_d^*, depending on d, such that

$$\gamma_d \, c \log n \leqslant g(n, d, c) \leqslant \gamma_d^* \frac{\log n}{-\log c}. \qquad (1)$$

The main aim of the paper is to show that, for a fixed value of d, the upper bound in (1) gives the correct order of $g(n, d, c)$ for all $c < 1/(2(d-1))$ and sufficiently large values of n.

Denote by $[x]$ the integer part of x.

THEOREM. (a) *There is an absolute constant $\alpha > 0$ such that if $0 < c < 1/d$ and*

$$m > \left(1 - \frac{1}{d} + c\right) \frac{n^2}{2},$$

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then every $G(n, m)$ contains a $K_{d+1}(t)$, where

$$t = \left[\frac{\alpha \log n}{d \log (1/c)}\right]. \tag{2}$$

(b) *Given an integer $d \geqslant 1$ there exists a constant $\varepsilon_d > 0$ such that if $0 < c < \varepsilon_d$ and $n \geqslant n(d, c)$ is an integer then there exists a graph $G(n, m)$ satisfying* (2) *which does not contain a $K_{d+1}(t)$ with*

$$t = \left[5 \frac{\log n}{\log (1/c)}\right].$$

Remarks. 1. The ratio of the upper and lower bounds of $g(n, d+1, c)$ given by the theorem does not depend on c. However, it does depend on d. We conjecture that the upper bound gives the correct order, i.e. $[-\log n/4d \log c]$ can be replaced by $[\gamma \log n/\log c]$, where γ (< 0) is an absolute constant.

2. The following result can be proved analogously to the theorem.

There exist constants $\delta = \delta(d) > 0$ and $\varepsilon = \varepsilon(d) > 0$ such that if (2) holds and $n > n(d, c)$ then every $G(n, m)$ contains a $K_{d+1}(a, \ldots, a, b)$ for every $a < \varepsilon \log n$ and $b \leqslant n\, 2^{-\delta\alpha}$. This is sharp in the sense that it fails if δ is sufficiently small.

3. Our final remark concerns r-graphs for $r > 2$. Denote by $G^r(n, m)$ an r-graph with n vertices and m r-tuples. Let $K_p^r(t)$ be the complete p-partite r-graph whose classes consist of t vertices. (An r-tuple is in this graph if and only if its elements belong to different classes.) Put $K_p^r = K_p^r(1)$.

The following problem was posed by Turán about thirty years ago. Given an integer $p > r$, determine the minimal positive number $c_{r,p}$ such that for every $\varepsilon > 0$ and sufficiently large n every graph $G^r(n, m)$ contains a K_p^r provided

$$m \geqslant (c_{r,p} + \varepsilon) \binom{n}{r}.$$

None of these values $c_{r,p}$ is known and the problem seems to be very difficult. However, it is possible that without actually determining $c_{r,p}$ one can prove a result analogous to the theorem.

Conjecture. Let $2 < r < p$ and $\varepsilon > 0$. Then there exists a constant $\gamma > 0$ such that if

$$m \geqslant (c_{r,p}) + \varepsilon \binom{n}{r}$$

and n is sufficiently large then every $G^r(n, m)$ contains a $K_p^r(t)$ where

$$t = [(\gamma \log n)^{1/(r-1)}].$$

It can be deduced from the results in **[2]** that this assertion holds with

$$t = [(\gamma \log n)^{1/(p-1)}].$$

Proof of the Theorem. As (b) can be proved as Theorem 2 in **[1]**, we prove only (a).

The cardinality of a set X is denoted by $|X|$. In the proof we shall make use of the following relations that follow from Stirling's formula:

$$\binom{n}{k} \leqslant \frac{n^k}{k!} \approx \left(\frac{en}{k}\right)^k \frac{1}{\sqrt{(2\pi k)}} < \left(\frac{en}{k}\right)^k . \tag{3}$$

To simplify the calculations we shall not choose $\alpha > 0$ immediately but we shall show that if $\alpha > 0$ is a *sufficiently small absolute constant* then the result holds.

Let $G = G(n, m)$ be a graph satisfying (2). As in the proof of Theorem 1 of **[1]**, it is easily seen that G contains a subgraph H with $n' \geqslant (dc/4)^{\frac{1}{2}} n$ vertices whose every vertex has degree at least $(1-1/d+c/2)\,n'$ in H. So with a slight change of notation it suffices to prove the following proposition.

PROPOSITION. *If $0 < \beta < 1$ is a sufficiently small absolute constant and every vertex of a graph G with n vertices has degree at least*

$$(1 - 1/d + c)n \qquad (0 < c < 1/d) \tag{4}$$

then G contains a $K_{d+1}(M)$ where

$$M = \left[\beta \frac{\log n}{d \log (1/c)}\right].$$

Proof of the proposition. The proposition is obvious if $M < 1$; so we can assume without loss of generality that $M \geqslant 1$, i.e.

$$n \geqslant (1/c)^{d/\beta}. \tag{5}$$

To prove the result we use induction on d. For $d = 1$ a stronger result is proved in [1] (and it also follows from **[4]**). Suppose now that $d \geqslant 2$ and the proposition is already proved for smaller values of d.

Put

$$c' = \frac{1}{d-1} - \frac{1}{d} + c \quad \text{and} \quad p_0 = \left[\beta \frac{\log n}{(c-1) \log (1/c')}\right].$$

As the minimal degree in H is greater than $\left(1-1/(d-1)+c'\right)n$, by the induction hypothesis G contains a $K_d(p_0)$.

In the sequel we shall make use of the following simple lemma.

LEMMA. *Let X be a set of vertices of G. Put $x = |X|/d$. Denote by Y the set of those vertices of $G-X$ that are joined to at least $(-1/d+c/2)\,dx$ vertices of X. Then*

$$d(cn-2x) \leqslant 2|Y|. \tag{6}$$

Proof. Denote by S the number of edges connecting $G-X$ to X. S clearly satisfies the following inequalities:

$$dx\{(1 - 1/d + c)n - dx\} \leqslant |S| \leqslant |Y|\,dx + (n-dx - |Y|)(1 - 1/d + c/2)dx.$$

Consequently

$$\tfrac{1}{2}c\,dxn - dx^2 + \tfrac{1}{2}c\,d^2x^2 \leqslant |Y|\,(1/d - \tfrac{1}{2}c)dx,$$

and this implies (6).

Let us go on with the proof of the proposition. Put $P = (2/c)\,M$.

(a) Let us assume that G contains a $K = K_d(p_1, \ldots, p_d)$ such that

$$p_i \leqslant p+M, \quad 1 \leqslant i \leqslant d, \tag{7}$$

where

$$p = \frac{1}{d}\sum_1^d p_i,$$

and

$$P \leqslant p \leqslant P+1. \tag{8}$$

If β is sufficiently small then

$$cn > 4(P+1), \tag{9}$$

so by the lemma we can suppose that if Z is the set of vertices of $G-K$ that are joined to at least $p(d-1)+cpd/2$ vertices of K, then $|Z| \geqslant dcn/4$. A vertex of Z is joined to at least

$$p(d-1)+(cpd/2)-(d-1)\,(p+M) \geqslant (cPd/2)-(d-1)\,M = M$$

vertices of each class of K, so it is joined to a subgraph $K_z(M)$ of K. By (3) the number of $K_d(M)$ subgraphs of K is at most

$$\binom{P+1+M}{M}^d < \binom{2P}{M}^d < \left(\frac{4e}{c}\right)^{Md}$$

$$= \left(\frac{1}{c}\right)^{\beta(\log n/\log(1/c))}(4e)^{\beta(\log n/\log(1/c))} < n^{\beta}n^{2\beta} = n^{3\beta}. \tag{10}$$

If β is sufficiently small then

$$n^{3\beta} < \frac{cn}{\beta\log n} < \frac{cnd^2\log(1/c)}{4\beta\log n} = \frac{dcn}{4M} \leqslant \frac{|Z|}{M}.$$

Thus Z contains a set Z' of M vertices and K contains a $K_d(M)$ subgraph K' such that every vertex of Z' is joined to every vertex of K'. Consequently G contains a $K_{d+1}(M)$.

(b) By (a) we can assume without loss of generality that whenever a subgraph $K_d(p_1, \ldots, p_d)$ of G satisfies (7) then $p = (1/d)\sum_1^d p_i < P$.

Let $K = K_d(p_1, \ldots, p_d)$ be a subgraph for which p attains its maximum under the conditions (7). As G contains a $K_d(p_0)$, $M < p_0 \leqslant p < P$. Let U be the set of those vertices that are joined to at least M vertices of each class of K. If U is large, say $|U| \geqslant n^{\frac{1}{2}}$, then G contains a $K_{d+1}(M)$, just as in case (a). For by (10) the number of $K_d(M)$ subgraphs of K is at most

$$\binom{p+M}{M}^d < \binom{P+1+M}{M}^d < n^{3\beta} < \frac{n^{\frac{1}{2}}}{M} \leqslant \frac{|U|}{M},$$

provided β is sufficiently small.

Thus we can suppose that $|U| \leqslant n^{\frac{1}{2}}$. Let W be the set of vertices of $G-K$ that are joined to at least $(1-1/d+c/2)\,dp$ vertices of K. Put $V = W-U$. By the lemma, (9) and (5), for sufficiently small β we have

$$|V| \geqslant \tfrac{1}{4}\, cnd - n^{\frac{1}{2}} \geqslant \tfrac{1}{8} cnd.$$

Let us define an equivalence relation on V by putting $x \sim y$ $(x, y \in V)$ if x and y are joined to exactly the same vertices of K. Let C_i denote the i-th class of K. If $x \in V$ there exists an i_0, $1 \leqslant i_0 \leqslant d$, such that x is joined to less than M vertices of C_{i_0}. As x is joined to more than $(d-1)\,p$ vertices of K, the number of vertices of $\bigcup_{j \neq i_0} C_j$ *not* joined to x is less than

$$(d-1)\,(p+m)-\{(d-1)\,p-M\} = dM.$$

Hence the number of equivalence classes in V is less than

$$\sum_{i=1}^{d} \left\{\sum_{\lambda \leqslant M} \binom{p_i}{\lambda}\right\} \left\{\sum_{\mu \leqslant dM} \binom{pd-p_i}{\mu}\right\}$$

$$\leqslant d^2\, M^2 \binom{2p}{M} \binom{pd}{dM} \leqslant d^2\, M^2 \left(\frac{2p}{M}\right)^{M} \left(\frac{p}{M}\right)^{dM} eM^{(d+1)}$$

$$\leqslant d^2\, M^2 (2e)^{(d+1)M} \left(\frac{1}{c}\right)^{dM} < \beta^2 (\log n)^2\, n^{4\beta/\log(1/c)}\, n^{\beta}.$$

Thus (5) implies that if β is sufficiently small, the number of equivalence classes is less than $cnd/(8p+8M)$, so there exists a set V_1 of $[p+M]$ equivalent vertices.

We shall show that *there is a* $K' = K_d(q_1, \ldots, q_d)$ *subgraph in* G *that contradicts the maximality of* $K = K_d(p_1, \ldots, p_d)$. Let $x \in V_1$ and let $\bar{C}_i$ denote the set of those vertices of C_i which are joined to x. We may suppose without loss of generality that x is joined to less than M vertices of C_1, i.e. $|\bar{C}_1| \leqslant M$. Assume furthermore that $|\bar{C}_2| \leqslant |\bar{C}_j|$, $j = 3, \ldots, d$. We shall give different constructions for K' according as $|\bar{C}_2| \leqslant p$ or $|\bar{C}_2| > p$.

If $|\bar{C}_2| \leqslant p$ let the classes $C_i{}^*$ of a $K_d(q_1, \ldots, q_d)$ be defined as follows:

$$C_1{}^* = V_1, \quad C_2{}^* = C_1 \cup C_2 \quad \text{and} \quad C_j{}^* = \bar{C}_j, \quad j = 3, \ldots, n.$$

Since

$$\left|\bigcup_1^d \bar{C}_i\right| > (d-1)\,p, \quad \left|\bigcup_1^d C_i{}^*\right| > dp.$$

Furthermore, $|C_i| \leqslant p+M$. Thus this subgraph $K_d(q_1, \ldots, q_d)$ satisfies (7) and contradicts the maximality of K.

If $|C_2| > p$, select $q = [p+1]$ vertices from each $\bar{C}_j$, $j = 2, \ldots, d$ and from V_1. These vertices determine a $K_d(q)$ in G, contradicting the maximality of K.

This completes the proof of the proposition and so the proof of the theorem is also complete.

References

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