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Cycles of Even Length in Graphs

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In this paper we solve a conjecture of P. Erdös by showing that if a graph G^n has n vertices and at least $100kn^{1+1/k}$ edges, then G contains a cycle C^{2l} of length $2l$ for every integer $l \in [k, kn^{1/k}]$. Apart from the value of the constant this result is best possible. It is obtained from a more general theorem which also yields corresponding results in the case where G^n has only $cn(\log n)^\alpha$ edges ($\alpha \geq 1$).

0. NOTATION

The graphs considered in this paper are finite and have neither loops nor multiple edges. The number of edges of a graph G will be denoted by $e(G)$. The number of vertices will be either denoted by $v(G)$ or indicated by a superscript; thus G^n is always a graph on n vertices. C^k denotes the cycle of length k.

1. INTRODUCTION

P. Erdös, in [4], published without proof the following

THEOREM. *There exists a c_k and an $n_0(k)$ such that, if*

$$e(G^n) > c_k n^{1+1/k} \quad \textit{and} \quad n > n_0, \tag{1}$$

then

$$C^{2k} \subset G^n.$$

Later, Erdös asked whether (1) implies

$$C^{2l} \subset G^n \text{ for every integer } l \in [k, n^{1/k}].$$

(In [5] he proved a weaker form of this conjecture for $k = 2$). We shall prove

THEOREM 1. *If*

$$e(G^n) > 100kn^{1+1/k}, \tag{2}$$

then

$$C^{2l} \subset G^n \text{ for every integer } l \in [k, kn^{1/k}].$$

Remark 1. It is reasonable to conjecture the existence of a function f such that, for all sufficiently large n, there is a graph S^n with $[f(k)\, n^{1+1/k}]$ edges that does not contain a C^{2k}; this is known to be the case for $k = 2, 3$, and 5 ([3], [7], [1], [8]). Therefore (at least for these values of k), condition (2) cannot be replaced by

$$e(G^n) > f(k)\, n^{1+1/k}.$$

In this sense our theorem is sharp.

On the other hand, if Z^n is the union of approximately $(1/k)n^{1-1/k}$ complete graphs on $[kn^{1/k}]$ vertices, then

$$e(Z^n) \approx kn^{1+1/k};$$

but Z^n contains no cycle of length greater than $kn^{1/k}$. Therefore, if $e(G) \approx kn^{1+1/k}$, the existence of a C^{2l} in G^n for $l = [kn^{1/k}]$ cannot be ensured, and this again shows the sharpness of our theorem.

Remark 2. In particular (for the case $k = 2$), Theorem 1 tells us that the order of magnitude of $e(G^n)$ which forces G^n to contain a C^4 also forces G^n to contain all the even cycles C^{2l}, $l = 2, 3,..., 2n^{1/2}$. A similar phenomenon is established in a paper of J. A. Bondy [2], where it is shown that, if G^n has enough edges to force a triangle (that is, if $e(G^n) > (n^2/4)$), then G^n must contain all cycles C^l, $l = 3, 4,..., [(n + 3)/2]$.

Theorem 1 is an easy consequence of a slightly more general theorem.

THEOREM 1*. *Let* $E = e(G^n)$. *Then* $C^{2l} \subset G^n$ *for every integer* $l \geq 2$ *satisfying*

$$l \leq \frac{E}{100n}, \qquad ln^{1/l} \leq \frac{E}{10n}.$$

Besides Theorem 1, another consequence of Theorem 1* is

THEOREM 2. *There exists a function g such that, if*

$$e(G^n) \geqslant g(\epsilon)\, n(\log n)^{1+\epsilon},$$

then

$$C^{2l} \subset G^n \qquad \textit{for every integer} \quad l \in \left[\frac{\log n}{\epsilon \log \log n}, (\log n)^{1+\epsilon}\right].$$

2. BASIC LEMMAS

A coloring (not necessarily proper) of the vertices of a graph G is *t-periodic* if the end-vertices of any (simple) path of length t in G have the same color.

LEMMA 1. Let t be a positive integer, and let G be a connected graph for which

$$e(G) \geqslant 2tv(G). \tag{3}$$

Then the number of colors in any t-periodic coloring of G is at most two.

Proof. (i) First we show that any graph G with $e(G) \geqslant 2tv(G)$ contains two adjacent vertices joined by two vertex-disjoint paths, each of length at least t. The technique we use is due to Pósa.

In the case where each vertex has valence at least $2t$, we can find such a θ-graph in the following way. Let a longest path in G be $x_1 \ldots x_m$. Then x_1 is adjacent only to vertices of this path, say to $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$, where

$$2 = i_1 < i_2 < \cdots < i_r \text{ and } r \geqslant 2t.$$

The path $x_1 x_2 \cdots x_{i_{2t}}$ together with the edges $x_1 x_{i_t}$ and $x_1 x_{i_{2t}}$ form the desired θ-graph.

The general case, when there may be vertices of valence less than $2t$, can now be proved by induction on $v(G)$. For $0 < v(G) \leqslant 4t$ it is trivial that (3) cannot be satisfied, and so there is nothing to prove here. If $v(G) = 4t + 1$, G must be complete and clearly contains a θ-graph of the desired type. Suppose now that every graph that satisfies (3) and has $k \geqslant 4t + 1$ vertices contains such a θ-graph, and let G be a graph on $k + 1$ vertices with some vertex x of valence less than $2t$. Then

$$e(G - x) > e(G) - 2t \geqslant 2tv(G) - 2t = 2tv(G - x).$$

Thus, by the induction hypothesis, $G - x$ contains a θ-graph of the desired type and hence so also does G.

(ii) Let the three cycles of such a θ-graph be C_1, C_2, C_3 with lengths l_1, l_2, l_3, respectively. Clearly, the restrictions of the t-periodic coloring of G to the θ-graph and to each cycle C_i are also t-periodic. Let t_i be the smallest integer such that C_i is t_i-periodic, $i = 1, 2, 3$. It is easy to see that any period on one cycle induces the same period on the other cycles and therefore

$$t_1 = t_2 = t_3;$$

also, $t_i \mid l_i$, $i = 1, 2, 3$. If C_3 is the longest of the three cycles, then

$$l_1 + l_2 - l_3 = 2.$$

Setting $t_i = t^*$, $i = 1, 2, 3$, we find that $t^* \mid 2$ and hence that $t^* = 1$ or $t^* = 2$. Therefore, the number of colors in the θ-graph is at most two.

(iii) Because G is connected, each vertex of G is joined to some vertex of this θ-graph by a path of length kt, for some integer k, and hence has the same color as this vertex. It follows that the number of colors in the whole graph G is also at most two. This completes the proof of the lemma.

It is, in fact, easy to show that either G is bipartite with the natural coloring (trivially a 2-periodic coloring), or else G is unicolored.

LEMMA 2. *Let G^n be a bipartite graph in which every vertex has valence at least $s = \max\{5ln^{1/l}, 50l\}$. Then G^n contains a C^{2l}.*

Proof. Choose an arbitrary vertex x of G^n and let V_i be the set of vertices at distance i from x. Since G^n is bipartite, each set V_i is an independent set.

Suppose that G^n contains no C^{2l}. We shall show that this implies that, for $1 \leqslant i \leqslant l$,

$$\frac{|V_i|}{|V_{i-1}|} \geqslant \frac{s}{5l}, \tag{4}$$

thus leading to the contradiction that $v(G^n) > n$, (since $s \geqslant 5ln^{1/l}$ and, consequently, $|V_i| \geqslant n^{1/l} |V_{i-1}|$).

We prove (4) by induction on i. It is trivial for $i = 1$ since the vertex x has valence at least s. Suppose that it is true for $i - 1$. Let $H_1, H_2, \ldots, H_q$ be the components of the subgraph H of G^n induced by $V_{i-1} \cup V_i$, and let W_j be the set of vertices of H_j that are on level $i - 1$, that is, in V_{i-1} (see Figure 1).

A path $x_1x_2 \cdots x_m$ in G^n will be called *monotonic* if the distance between x and x_i is monotonic. (This means that a monotonic path crosses any level at most once.)

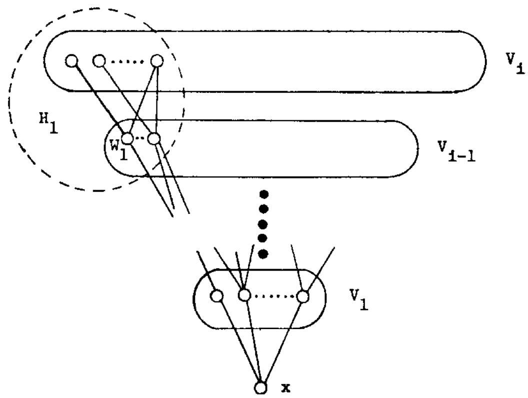

FIGURE 1

We shall show that $e(H_1) < 4lv(H_1)$. This is trivial if W_1 has just one vertex, so assume that W_1 has at least two vertices. Let $a \in V_p$ be a vertex of G^n such that

(i) there are two monotonic paths P_1, P_2 joining a to W_1 which have just the vertex a in common,

(ii) p is the minimum subject to (i).

First we show that each vertex of W_1 is joined to a by a monotonic path. For $y \in W_1$ is joined to x by a monotonic path P_3 and, by the minimality of p, P_3 must intersect P_1 in some vertex z. The path consisting of the section of P_3 between y and z and the section of P_1 between z and a is a monotonic path from y to a. This is illustrated in Figure 2.

We now assign colors red and blue to the vertices of W_1 in such a way

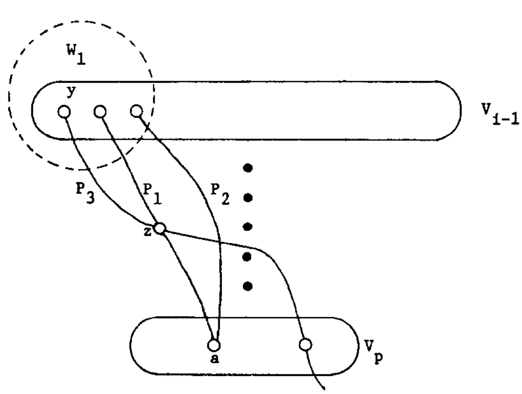

FIGURE 2

that, if two vertices have different colors, then they are joined to a by vertex-disjoint monotonic paths. This is done as follows. Each vertex of W_1 that can be joined to a by a monotonic path disjoint from P_2 is colored red; all other vertices of W_1 are colored blue. To see that this coloring has the required property, let x_1 and x_2 be vertices of W_1 colored red and blue, respectively, let P_1' be a monotonic path from x_1 to a disjoint from P_2, and let P_2' be a monotonic path from x_2 to a. Moving along P_2' from x_2 towards a, let v be the first vertex of $(P_1' \cup P_2) - a$ encountered (see Figure 3). Because x_2 has the color blue, such a v exists; v cannot belong

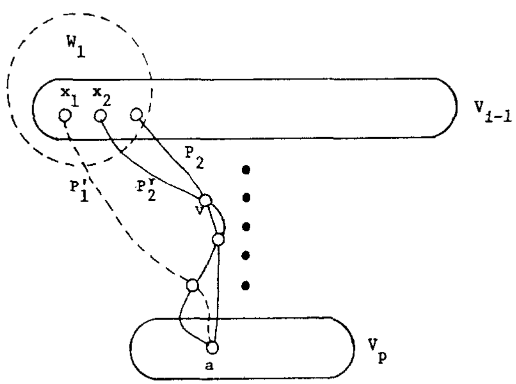

FIGURE 3

to P_1' for then the section of P_2' between x_2 and v together with the section of P_1' between v and a would constitute a monotonic path from x_2 to a disjoint from P_2, contradicting the assumption that x_2 is colored blue. But then $v \in P_2$ and we have a monotonic path $x_2P_2'vP_2a$ disjoint from P_1'.

We now color the vertices of H_1 in V_i green and show that this coloring of H_1 is t-periodic with $t = 2(l - i + p + 1)$. For, since t is even, if one end-vertex of a path of length t in H_1 is green, then so is the other. Also, there can be no path of length t joining a red and a blue vertex, because, if a red x_1 were joined to a blue x_2 by such a path, this path together with vertex-disjoint monotonic paths from x_1 to a and from x_2 to a would form a cycle of length $2l$. Therefore, the coloring of H_1 is indeed t-periodic. Since three colors are used in this coloring, Lemma 1 implies that

$$e(H_1) < 2tv(H_1) < 4lv(H_1).$$

Arguing similarly for $H_2, \ldots, H_q$, we obtain

$$e(H_j) < 4lv(H_j), \ j = 1, \ldots, q,$$

and, since the H_j are the components of H,

$$e(H) < 4lv(H). \tag{5}$$

Let H^* denote the subgraph of G^n induced by $V_{i-2} \cup V_{i-1}$. Then, similarly,

$$e(H^*) < 4lv(H^*), \tag{6}$$

and, by the induction hypothesis,

$$\frac{|V_{i-1}|}{|V_{i-2}|} \geqslant \frac{s}{5l}. \tag{7}$$

But, clearly, since each vertex of G^n has valence at least s,

$$e(H) + e(H^*) \geqslant s\,|V_{i-1}|.$$

Therefore, by (5) and (6),

$$\begin{aligned} &4l(|V_{i-1}| + |V_i| + |V_{i-2}| + |V_{i-1}|) \\ &\quad = 4l(v(H) + v(H^*)) > e(H) + e(H^*) \geqslant s\,|V_{i-1}| \end{aligned}$$

and so

$$|V_i| > \frac{1}{4l}((s - 8l)\,|V_{i-1}| - 4l\,|V_{i-2}|).$$

Using (7) we obtain

$$|V_i| > \frac{1}{4l}\left(s - 8l - \frac{20l^2}{s}\right)|V_{i-1}|$$

and, therefore, since $s \geqslant 50l$,

$$\frac{|V_i|}{|V_{i-1}|} > \frac{1}{4l}(s - 9l) > \frac{1}{4l} \cdot \frac{4s}{5} = \frac{s}{5l},$$

as desired.

3. Main Theorem

We are now in a position to prove Theorem 1*. First we recall its statement.

THEOREM 1*. *Let $E = e(G^n)$. Then $C^{2l} \subset G^n$ for every integer $l \geqslant 2$ satisfying*

$$l \leqslant \frac{E}{100n}, \qquad ln^{1/l} \leqslant \frac{E}{10n}. \tag{8}$$

Proof (by induction on n). For $n = 1$ the theorem is trivial, since condition (8) cannot be satisfied in this case. We now suppose that the theorem has been proved for all graphs on $n - 1$ vertices. Let G^n be a graph on n vertices and let $l \geqslant 2$ be an integer satisfying (8).

It has been shown by Erdös [6] that any graph G contains a bipartite spanning subgraph H with $e(H) \geqslant e(G)/2$; in fact H can be chosen so that each vertex has valence in H at least half its valence in G.

So let H^n be such a bipartite spanning subgraph of G^n. If each vertex of H^n has valence at least $E/2n$ then, by Lemma 2, we have that, for every integer l such that

$$\max\{5ln^{1/l}, 50l\} \leqslant E/2n,$$

H^n contains a cycle of length $2l$. Thus, in this case, Theorem 1* is proved.

So suppose now that some vertex w of H^n has valence less than $E/2n$. By the choice of H^n, w has valence less than E/n in G^n. Let $G^{n-1} = G^n - w$. By the induction hypothesis, G^{n-1} contains a cycle of length $2l$ for every integer l satisfying

$$l \leqslant \frac{e(G^{n-1})}{100(n-1)}, \qquad l(n-1)^{1/l} \leqslant \frac{e(G^{n-1})}{10(n-1)}.$$

But if l satisfies (8) with G^n, then it also satisfies (8) with G^{n-1} since,

(a) if $l \leqslant e(G^n)/100n$, then

$$l \leqslant \frac{e(G^n)}{100n} = \frac{e(G^n) - e(G^n)/n}{100(n-1)} \leqslant \frac{e(G^{n-1})}{100(n-1)}$$

(since w has valence less than $e(G^n)/n$),

(b) if $ln^{1/l} \leqslant e(G^n)/10n$, then

$$l(n-1)^{1/l} \leqslant ln^{1/l} \leqslant \frac{e(G^n)}{10n} = \frac{e(G^n) - e(G^n)/n}{10(n-1)} \leqslant \frac{e(G^{n-1})}{10(n-1)}.$$

Hence G^{n-1}, and therefore also G^n, contains a cycle of length $2l$ for every integer l satisfying (8). This completes the proof.

Perhaps, by other methods, Theorem 1* could be improved so as to be meaningful for

$$E \geqslant \frac{cn \log n}{\log \log n}.$$

However, this would then be the best possible result since, if c^* is small, there exists no fixed l such that every graph on n vertices and with $(c^*n \log n)/(\log \log n)$ edges has a cycle of length $2l$.

Remark 3. One can find an l satisfying (8) in Theorem 1* if and only if

$$E \geqslant \frac{100n \log n}{\log 10}. \tag{9}$$

If (9) holds, then (8) is satisfied for all values of l in an interval. The upper end of this interval is $E/100n$. The lower end can be determined in the following way:

For a fixed n the function $y = xn^{1/x}$ is strictly decreasing in $(0, \log n]$. Let $\phi_n(y) = x$ denote its inverse. Then $\phi_n(E/10n)$ is the lower end of our interval. $\phi_n(E/10n)$ is a transcendental function but one can easily give good approximations for it using the iteration

$$\psi_{n,1}(y) = \frac{\log n}{\log y}, \qquad \psi_{n,k}(y) = \frac{\log n}{\log y - \log \psi_{n,k-1}}.$$

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