Application of the Stability method in Extremal Graph Theory and related areas

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(Jinan18-A)

Calgary Conference, 1969 Lake Louise

Introduction

Extremal graph theory and Ramsey theory were among the early and fast developing branches of 20th century graph theory. We shall survey the early development of Extremal Graph Theory, including some sharp theorems.

Strong interactions between these fields:
Here everything influenced everything

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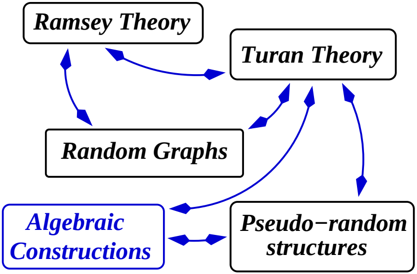

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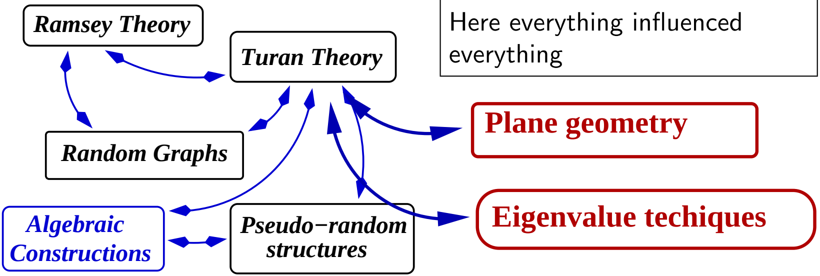
Here everything influenced everything



Why are extremal problems interesting?

- Interesting on its own

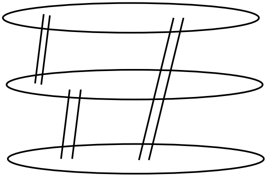
- Strong connection to Ramsey Theory
- A deep and wide theory, with may new phenomena
- Applicable: Pigeon hole principle
- Lead to important new tools
  — Using finite geometries
  — Using random graphs
  — Szemerédi Regularity Lemma
  — Property testing
  — Graph limits
  — ...

Turán type graph problems

MANTEL 1903 (?) K_3

ERDŐS: C_4: Application in combinatorial number theory.

The first **finite geometrical construction** (Eszter Klein)



Turán theorem. *(1940)*

$$e(G_n) > e(T_{n,p}) \quad \Longrightarrow \quad K_{p+1} \subseteq G_n.$$

Unique extremal graph $T_{n,p}$.

General question:

Given a family $\mathcal{L}$ of forbidden graphs, what is the maximum of $e(G_n)$ if G_n does not contain subgraphs $L \in \mathcal{L}$?

Main Line:

Some central theorems

assert that for ordinary graphs the general situation is almost the same as for K_{p+1}.

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

- The extremal graphs S_n are very similar to $T_{n,p}$.
- the almost extremal graphs are also very similar to $T_{n,p}$.

The meaning of "VERY SIMILAR":

- One can delete and add $o(n^2)$ edges of an extremal graph S_n to get a $T_{n,p}$.
- One can delete $o(n^2)$ edges of an extremal graph to get a p-chromatic graph.

Extremal graphs

The “metric” $\rho(G_n, H_n)$ is the minimum number of edges to change to get from G_n a graph isomorphic to H_n.

Notation.

$\mathbf{EX}(\mathbf{n}, \mathcal{L})$: set of extremal graphs for $\mathcal{L}$.

Theorem (Erdős-Sim., 1966)

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

If $S_n \in \mathbf{EX}(\mathbf{n}, \mathcal{L})$, *then*

$$\rho(T_{n,p}, S_n) = o(n^2).$$

ERDŐS-STONE-SIM..

The answer depends on the minimum chromatic number:

Let

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

$$\mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right) \binom{n}{2} + o(n^2),$$

Meaning?

Classification of extremal problems

- nondegenerate: $p > 1$
- degenerate: $\mathcal{L}$ contains a bipartite L
- strongly degenerate: $T_\nu \in \mathcal{M}(\mathcal{L})$

where $\mathcal{M}$ is the decomposition family.

Product conjecture

Theorem 1 separates the cases $p = 1$ and $p > 1$:

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2) \iff p = p(\mathcal{L}) = 1$$

$p = 1$: degenerate extremal graph problems

CONJECTURE (SIM.)

If

$$\mathbf{ex}(n, \mathcal{L}) > e(T_{n,p}) + n \log n$$

and $S_n \in \mathbf{EX}(\mathbf{n}, \mathcal{L})$, *then* S_n *can be obtained from a* $K_p(n_1, \dots, n_p)$ *only by adding* $o(n^2)$ *edges.*

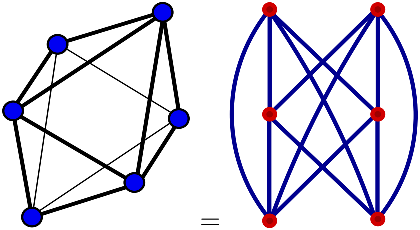
This would reduce the general case to degenerate extremal graph problems.

Example: Octahedron Theorem

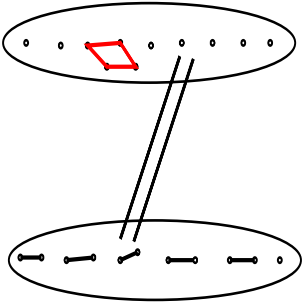
Theorem (Erdős-Sim.)

For O_6, the extremal graphs S_n are “products”: $U_m \otimes W_{n-m}$ where U_m is extremal for C_4 and W_{n-m} is extremal for P_3. for $n > n_0$.

→ ErdSimOcta



Excluded: octahedron



extremal = product

Structural stability of the extremal graphs

ERDŐS-SIM. Theorem.

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $L \not\subseteq G_n$ for any $L \in \mathcal{L}$ and

$$e(G_n) \geq \left(1 - \frac{1}{p}\right) \binom{n}{2} - \delta n^2,$$

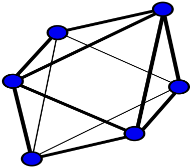
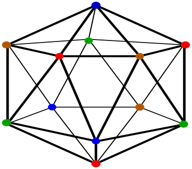
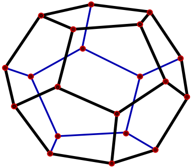
then

$$\rho(G_n, T_{n,p}) \leq \varepsilon n^2$$

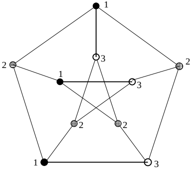
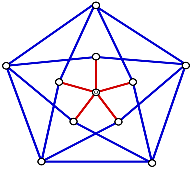
Applicable and gives also exact results

Further examples of TURÁN:

Octahedron, Icosahedron, Dodecahedron, Path P_k

Later: Petersen graph, Grötzsch

M. Simonovits:

How to solve a Turán type extremal graph problem? (linear decomposition), Contemporary trends in discrete mathematics (Stirin Castle, 1997), pp. 283–305, Amer. Math. Soc., Providence, RI, 1999.

Decomposition family of $\mathcal{L}$

$\mathcal{M}$: Those (minimal) graphs M that cannot be put into the first graph of $T_{n,p}$ without getting an $L \in \mathcal{L}$.

Methods: How to prove a complicated but sharp result?

- Progressive induction
- Using a general method on some particular classes of excluded graphs → SimDM
- Using Stability of the extremal graphs

M. Simonovits:

A method for solving extremal problems in graph theory, Theory of Graphs, Proc. Colloq. Tihany, (1966), (Ed. P. Erdős and G. Katona) Acad. Press, N.Y., 1968, pp. 279–319.

Several surveys

M. Simonovits:

Extremal graph theory, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory II., Academic Press, London, 1983, pp. 161–200.

Here: 3 types of stability arguments

The essence:

The **almost extremal** items are very similar to the **extremal** ones.

1. Progressive induction
2. $\mathcal{P} - \mathcal{Q}$-stability
3. Using "Ready-made stability theorems", like Erdős-Sim. or Lovász-Sim..

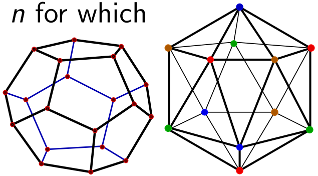
L. Lovász and M. Simonovits:

On the number of complete subgraphs of a graph II, Studies in Pure Math. (dedicated to P. Turán) (1983) 458–495 Akadémiai Kiadó+Birkhäuser Verlag.

Progressive Induction

- Induction would be easy but the initial step is difficult
- Extremal sequence (S_n).
  Distance function $\Delta(S_n, \mathcal{P})$, integer.
- Either $S_n \in \mathcal{P}$ or there is an $m < n$ for which

$$\Delta(S_n, \mathcal{P}) < \Delta(G_m)$$



and $m > \log n$, say.

Conclusion

Then there is an n_0 such that $S_n \in \mathcal{P}$ for $n > n_0$.

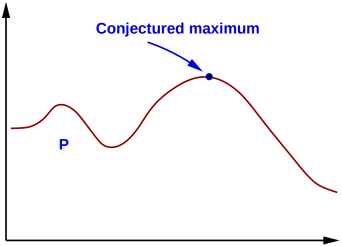
Success for the Platonic cases

Dodecahedron, Icosahedron

What is the method of $\mathcal{P} - \mathcal{Q}$-stability?

Useful for many graphs and several hard hypergaph problems.

- We wish to optimize $f(n, \mathcal{P})$.



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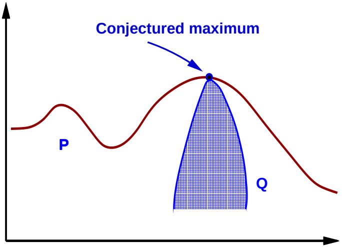

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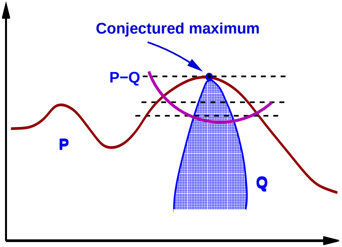

- We wish to optimize $f(n, \mathcal{P})$.
- We find a related property $\mathcal{Q}$.

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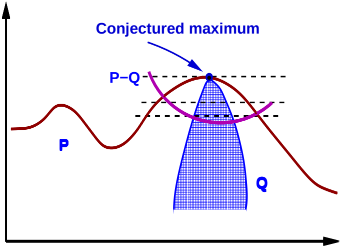


- We find a related property $\mathcal{Q}$.
- We prove that

$$\max_n f(n, \mathcal{P}) > \max_n f(n, \mathcal{P} - \mathcal{Q})$$

- Therefore the maximum can be found in $\mathcal{Q}$.

This is much easier.

Why does Stability help?

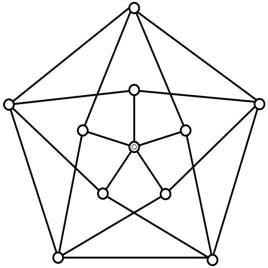
In all these examples it is much easier to optimize the number of edges for $\mathcal{Q}$.

Examples: Critical edge

Theorem (Critical edge)

If $\chi(L) = p + 1$ and L contains a color-critical edge, then $T_{n,p}$ is the (only) extremal for L, for $n > n_1$. *[If and only if]*

SIM., (ERDŐS)



GRÖTZSCH GRAPH

Complete graphs
Odd cycles

The Universe

Extremal problems can be asked (and are asked) for many other object types.

- Mostly simple graphs
- Digraphs → Brown-Harary, Brown, Erdős, Simonovits
- Multigraphs → Brown-Harary, Brown, Erdős, Simonovits
- Hypergraphs → Turán, . . .
- Geometric graph → Pach, Tóth, Tardos
- Integers → Erdős, Sidon,Szemerédi,. . .
- groups → BabaiSos → Gowers
- other structures

Main setting: Universe

- Integers
- Groups

- **Graphs**
- Digraphs
- **Hypergraphs**
- Directed Multihypergraphs

Universe:

We fix some type of **structures**, like graphs, digraphs, or r-uniform hypergraphs, integers, and a family $\mathcal{L}$ of forbidden substructures, e.g. cycles C_{2k} of $2k$ vertices.

A Turán-type extremal (hyper)graph problem

asks for the maximum number $\mathbf{ex}(n, \mathcal{L})$ of (hyper)edges a (hyper)graph can have under the conditions that it does not contain any **forbidden** substructures.

The general problem

Given a **universe**, and a structure $\mathbb{A}$ with two (natural parameters) n and e on its objects G.
Given a property $\mathcal{P}$.

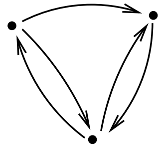
$$\mathbf{ex}(n, \mathcal{P}) = \max_{n(G)=n} e(G).$$

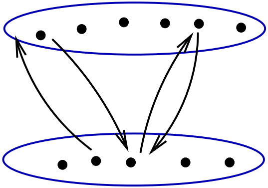
Determine $\mathbf{ex}(n, \mathcal{P})$ and
describe the **EXTREMAL STRUCTURES**

Examples: A digraph theorem

We have to assume an upper bound s on the multiplicity. (Otherwise we may have arbitrary many edges without having a K_3.) Let $s = 1$.

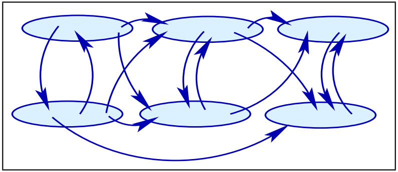
L:





$$\mathbf{ex}(n, L) = 2\mathbf{ex}(n, K_3) \qquad (n > n_0?)$$

Many extremal graphs: We can combine arbitrary many oriented double Turán graph by joining them by single arcs.



W. G. Brown, and M. Simonovits:

Extremal multigraph and digraph problems, Paul Erdős and his mathematics, II (Budapest, 1999), pp. 157–203, Bolyai Soc. Math. Stud., 11, János Bolyai Math. Soc., Budapest, 2002.

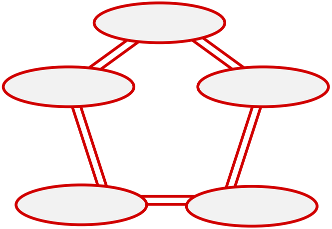
Examples:

Erdős

Prove that each triangle-free graph can be turned into a bipartite one deleting at most $n^2/25$ edges.

The construction shows that this is sharp if true.

Partial results: Erdős-Faudree-Pach-Spencer



Erdős-Győri-Sim.

Atypical question?

Turán's approach

In which other way can we ensure a large $K_k \subseteq G_n$?

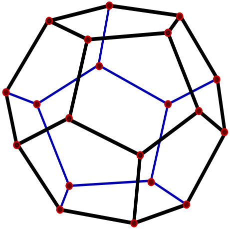
E.g., if $e(G_n)$ is large?

Later Turán used to say: Ramsey and his theorems are applicable because they are generalizations of the Pigeon Hole Principle.

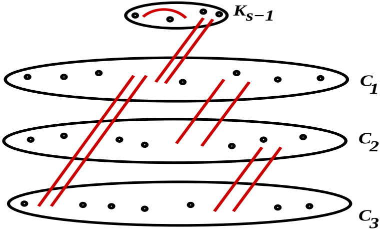
Turán asked for several other sample graphs L to determine $\mathbf{ex}(n, L)$:

- Platonic graphs: Icosahedron, cube, octahedron, dodecahedron.
- path P_k

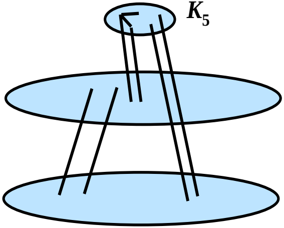
Dodecahedron Theorem (Sim.)



Dodecahedron: D_{20}



$H(n,d,s)$

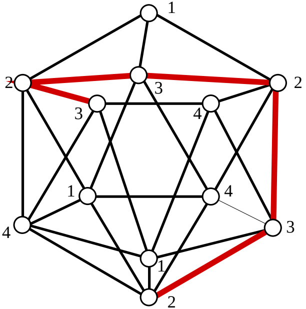


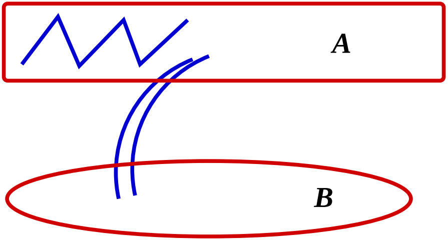
$H(n,2,6)$

For D_{20}, $H(n,2,6)$ is the (only) extremal graph for $n > n_0$.

($H(n,2,6)$ cannot contain a D_{20} since one can delete 5 points of $H(n,2,6)$ to get a bipartite graph but one cannot delete 5 points from D_{20} to make it bipartite.)

Example: the Icosahedron





If B contains a P_6, then G_n contains an icosahedron

The decomposition class is: P_6.

In some sense the Icosahedrom problem is different from the others: the stability is missing?

Application in combin. number theory

Erdős (1938): → ErdTomsk
Maximum how many integers $a_i \in [1, n]$ can be found under the condition: $a_i a_j \neq a_k a_\ell$, unless $\{i, j\} = \{k, \ell\}$?

This lead Erdős to prove:

$$\mathbf{ex}(n, C_4) \leq cn\sqrt{n}.$$

The first finite geometric construction to prove the lower bound (Eszter Klein)

Crooks tube

First "attack":

The primes between 1 and n satisfy Erdős' condition.

Can we conjecture $$g(n) \approx \pi(n) \approx \frac{n}{\log n}?$$

YES!

Proof idea: If we can produce each non-prime $m \in [1, n]$ as a product:

$$m = xy, \text{ where } x \in X, \ y \in Y,$$

then

$$g(n) \leq \pi(n) + \mathbf{ex}_B(X, Y; C_4).$$

where $\mathbf{ex}_B(U, V; L)$ denotes the maximum number of edges in a subgraph of $G(U, V)$ without containing an L.

Degenerate vs Non-degenerate problems

Theorem (Erdős)

$$\mathbf{ex}^*(n, C_4) \leq 3n\sqrt{n} + O(n).$$

Theorem (Erdős-Kővári-T. Sós-Turán)

$$\mathbf{ex}(n, K(a,b)) \approx \frac{1}{2}\sqrt[a]{b-1} \cdot n^{2-\frac{1}{a}} + O(n).$$

1

Z. Füredi and M. Simonovits:

The history of degenerate (bipartite) extremal graph problems, Erdős Centennial, (2013) pp. 169–264 Springer arXiv

Kővári-T. Sós-Turán theorem

One of the important extremal graph theorems is that of Kővári, T. Sós and Turán, → KovSosTur

solving the extremal graph problem of $K_2(p, q)$.

Theorem (Kővári–T. Sós–Turán)

Let $2 \leq p \leq q$ be fixed integers. Then

$$\mathbf{ex}(n, K(p, q)) \leq \frac{1}{2}\sqrt[p]{q-1}\, n^{2-1/p} + \frac{1}{2}pn.$$

Is the exponent $2-(1/p)$ sharp?

Conjecture (KST is Sharp)

For every integers p, q,

$$\mathbf{ex}(n, K(p,q)) > c_{p,q}\, n^{2-1/p}.$$

Known for $p=2$ and $p=3$: Finite geometric constructions

Erdős, Rényi, V. T. Sós, → ErdRenyiSos

W. G. Brown → BrownThom

Random methods: → ErdRenyiEvol

$$\mathbf{ex}(n, K(p,q)) > c_p n^{2-\frac{1}{p}-\frac{1}{q}}.$$

Füredi on $K_2(3,3)$: The Brown construction is sharp.

Kollár-Rónyai-Szabó: $q > p!$. Commutative Algebra constr.

Alon-Rónyai-Szabó: $q > (p-1)!$.

Unknown:

- Missing lower bounds: Constructions needed

- "Random constructions" do not seem to give the right order of magnitude when $\mathcal{L}$ is finite

We do not even know if Pr1

$$\frac{\mathbf{ex}(n, K(4,4))}{n^{5/3}} \to \infty.$$

- Partial reason for the bad behaviour:

Lenz Construction

Degenerate problems

Given a family $\mathcal{L}$ of forbidden graphs,

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2).$$

if and only if there is a bipartite graph in $\mathcal{L}$.
Moreover, if $L_0 \in \mathcal{L}$ is bipartite, then

$$\mathbf{ex}(n, \mathcal{L}) = O(n^{2-2/v(L_0)}).$$

Proof. Indeed, if a graph G_n contains no $L \in \mathcal{L}$, then it contains no L_0 and therefore it contains no $K_2(p, v(L_0) - p)$, yielding an $L \subseteq G_n$. . ∎

Supersaturated Graphs: Degenerate

Prove that if

$$E = e(G_n) > c_0 n^{2-(1/p)},$$

then the number of $K_{p,q}$'s in G_n

$$\#K(p,q) \geq c_{p,q} \frac{E}{n^2}^{pq}$$

The meaning of this is that an arbitrary G_n having more edges than the (conjectured) extremal number, must have – up to a multiplicative constant, – at least as many $K_{p,q}$ as the corresponding random graph,

see conjectures Erdős and Sim. and of Sidorenko

Supersaturated, Non-Degenerate

If

$$e(G_n) > \mathbf{ex}(n, L) + cn^2,$$

then G_n contains $\geq c_L n^{v(L)}$ copies of L

This extends to multigraphs, hypergraphs, directed multihypergraphs.

Brown-Simonovits

→ BrownSimDM

Bondy-Simonovits

Theorem (Even Cycle: C_{2k})

ex$(n, C_{2k}) \le c_1 k n^{1+(1/k)}$.

Conjecture (Sharpness)

Is this sharp, at least in the exponent? The simplest unknown case is C_8,

It is sharp for C_4, C_6, C_{10}

Could you reduce k in $c_1 k n^{1+(1/k)}$?

YES: Boris Bukh and Zilin Jiang: basically: $k \to \sqrt{k \log k}$

An annoying open problem

Conjecture (General, even cycles)

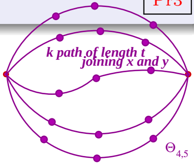
For some $c_k > 0$, $\mathbf{ex}(n, C_{2k}) > c_k n^{1+(1/k)}$. Pr2

Weakening:

Conjecture (Just for the octogon)

For some $c_4 > 0$ $\mathbf{ex}(n, C_8) > c_4 n^{5/4}$. Pr3

Weakening, other direction:



Conjecture (For Θ-graphs)

Given a k, *there exists an* $t = t(k)$ *For which some* $c_k > 0$ $\mathbf{ex}(n, \Theta_{k,t}) > c_k n^{1+\frac{1}{k}}$. Pr4

Sketch of the proof of Bondy-Simonovits: (???)

Lemma

If D is the average degree in G_n, then G_n contains a subgraph G_m with

$$d_{\min}(G_m) \geq \frac{1}{2}D \text{ and } m \geq \frac{1}{2}D.$$

- So we may assume that G_n is bipartite and regular.

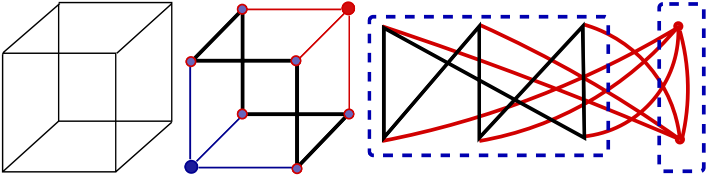
Assume also that it does not contain shorter cycles either.

Cube-reduction

Theorem (Cube, Erdős-Sim.)

$\mathbf{ex}(n, Q_3) = O(n^{8/5})$.

New Proofs: PINCHASI-SHARIR, FÜREDI, ...

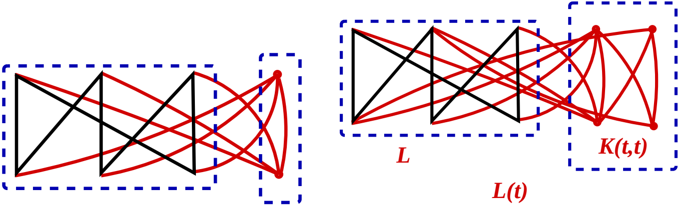


The cube is obtained from C_6 by adding two vertices, and joining two new vertices to this C_6 as above.

- We shall use a more general definition: $L(t)$.

General definition of $L(t)$:

- Take an arbitrary bipartite graph L and $K(t,t)$. 2-color them!
- join each vertex of $K(t,t)$ to each vertex of L of the opposite color



Theorem (Reduction, Erdős-Sim.)

Fix a bipartite L and an integer t.
If $\mathbf{ex}(n,L)=n^{2-\alpha}$ and $L(t)$ is defined as above then $\mathbf{ex}(n,L(t))\le n^{2-\beta}$ for

$$\frac{1}{\beta}-\frac{1}{\alpha}=t.$$

Examples

Open Problem: Pr5

Find a lower bound for $\mathbf{ex}(n, Q_8)$, better than $cn^{3/2}$.
Conjectured: $\mathbf{ex}(n, Q_8) > cn^{8/5}$.

An Erdős problem: Compactness?

We know that if G_n is bipartite, C_4-free, then

$$e(G_n) \leq \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

We have seen that there are C_4-free graphs G_n with

$$e(G_n) \approx \frac{1}{2} n^{3/2} + o(n^{3/2}).$$

Conjecture (Erdős)

Is it true that if $K_3, C_4 \not\subseteq G_n$ then

$$e(G_n) \leq \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}) \ ?$$

This does not hold for hypergraphs (Balogh) or for geometric graphs (Tardos)

Erdős-Sim., C_5-compactness:

If $C_5, C_4 \not\subseteq G_n$ then → ErdSimComp

$$e(G_n) \leq \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

Unfortunately, this is much weaker than the conjecture on C_3, C_4: excluding a C_5 is a much more restrictive condition.

Degenerate Compactness

Is it true that if $\mathcal{L}$ is a finite family of bipartite graphs then there exists an $L_0 \in \mathcal{L}$ such that

$$\frac{\mathbf{ex}(n, \mathcal{L})}{\mathbf{ex}(n, L_0)}$$

is bounded?

Rational exponents?

Conjecture (Rational exponents, Erdős-Sim.)

Given a bipartite graph L, is it true that for suitable $\alpha \in [0, 1)$ there is a $c_L > 0$ for which

$$\frac{\mathbf{ex}(n, L)}{n^{1+\alpha}} \to c_L > 0 \quad ?$$

Or, at least, is it true that for suitable $\alpha \in [0, 1)$ there exist a $c_L > 0$ and a $c_L^* > 0$ for which

$$c_1^* \le \frac{\mathbf{ex}(n, L)}{n^{1+\alpha}} \le c_L \quad ?$$

Constructions using finite geometries

$p \approx \sqrt{n} =$ prime ($n = p^2$)
Vertices of the graph F_n are pairs: $(a, b) \mod p$.
Edges: (a, b) is joined to (x, y) if $ac + bx = 1 \mod p$.

Geometry in the constructions: the neighbourhood is a straight line and two such nighbourhoods intersect in ≤ 1 vertex.

$\Longrightarrow$ No $C_4 \subseteq F_n$

loops to be deleted
most degrees are around $\sqrt{n}$:

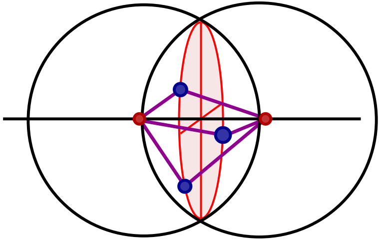
$$e(F_n) \approx \frac{1}{2} n\sqrt{n}$$

Finite geometries: Brown construction

Vertices: (x, y, z) mod p

Edges:

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = \alpha.$$



$$\mathbf{ex}(n, K(3,3)) > \frac{1}{2} n^{2-(1/3)} + o(\ldots).$$

→ BrownThom

The first missing case

The above methods do not work for $K(4,4)$.

We do not even know if

Pr6

$$\frac{\mathbf{ex}(n, K_2(4,4))}{\mathbf{ex}(n, K_2(3,3))} \to \infty.$$

One reason for the difficulty: Lenz construction:

$\mathbb{E}^4$ contains two circles in two orthogonal planes:

$$\mathcal{C}_1 = \{x^2+y^2 = \frac{1}{2},\ z = 0,\ w = 0\} \text{ and } \mathcal{C}_2 = \{z^2+w^2 = \frac{1}{2},\ x = 0,\ y = 0\}$$

and each point of $\mathcal{C}_1$ has distance 1 from each point of $\mathcal{C}_2$: the unit distance graph contains a $K_2(\infty, \infty)$.

Examples: Multigraphs, Digraphs, ...

-
- BROWN-HARARY: bounded multiplicity: r
- BROWN-ERDŐS-SIM. → BrownSimDM

$r = 2s$: digraph problems and multigraph problems seem to be equivalent:

– each multigraph problem can easily be reduced to digraph problems

– and we do not know digraph problems that are really more difficult than some corresponding multigraph problem

Examples: Numbers, . . .

- Tomsk
- Sidon sequences

- Let $r_k(n)$ denote the maximum m such that there are m integers $a_1, \dots, a_m \in [1, n]$ without k-term arithmetic progression.

Theorem (Szemerédi Theorem)

For any fixed k $r_k(n) = o(n)$ as $n \to \infty$.

History (simplified):

- K. F. Roth: $r_3(n) = o(n)$
- Szemerédi
- Fürstenberg: Ergodic proof
- Fürstenberg-Katznelson: Higher dimension
- Polynomial extension, Hales-Jewett extension
- Gowers: much more effective

Erdős on unit distances

Many of the problems in the area are connected with the following beautiful and famous conjecture, motivated by some grid constructions.

Conjecture (P. Erdős)

For every $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that if $n > n_0(\varepsilon)$ and G_n is the Unit Distance Graph of a set of n points in $\mathbb{E}^2$ then

$$e(G_n) < n^{1+\varepsilon}.$$

Szemerédi-Ruzsa

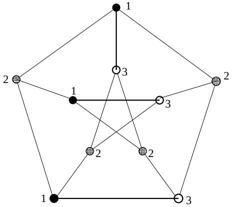
$f(n, 6, 3)$

Removal Lemma

Originally for K_3, RUZSA-SZEMERÉDI

Generaly: through a simplified example:

For every $\varepsilon > 0$ there is a $\delta > 0$:
If a G_n does not contain δn^{10} copies of the Petersen graph, then we can delete εn^2 edges to destroy all the Petersen subgraphs.

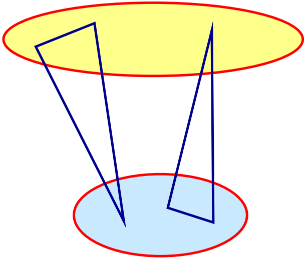


- something similar is applicable in PROPERTY TESTING.

Hypergraph extremal problems

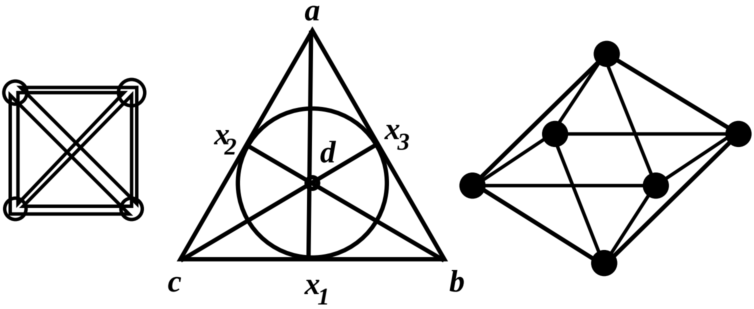
3-uniform hypergraphs: $\mathcal{H} = (V, \mathcal{H})$
$\chi(\mathcal{H})$: the minimum number of colors needed to have in each triple 2 or 3 colors.

Bipartite 3-uniform hypergraphs:



The edges intersect both classes

Three important hypergraph cases

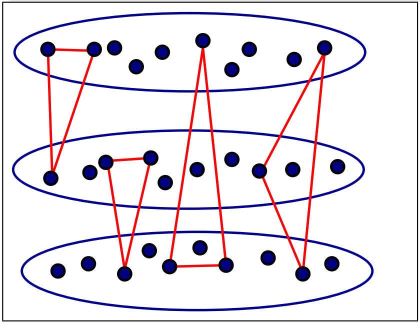


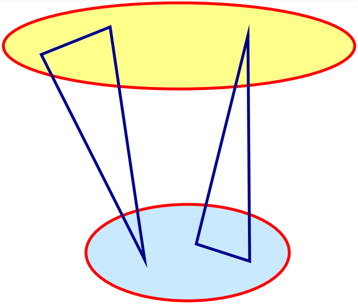
Complete 4-graph, || Fano configuration, || octahedron graph

The famous Turán conjecture

Conjecture (Turán)

The following structure is the (? asymptotically) extremal structure for $K_4^{(3)}$*:*

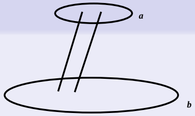




For $K_5^{(3)}$ one conjectured extremal graph is just the above "complete bipartite" one!

Two important theorems

Theorem (Kővári-T. Sós-Turán)



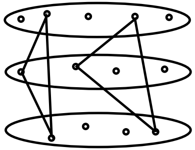
Let $2 \le a \le b$ be fixed integers. Then

$$\mathbf{ex}(n, K(a,b)) \le \frac{1}{2}\sqrt[a]{b-1} \cdot n^{2-\frac{1}{a}} + \frac{1}{2}an.$$

Theorem (Erdős)

$$\mathbf{ex}(n, K_r^{(r)}(m,\dots,m)) = O(n^{r-(1/m^{r-1})}).$$

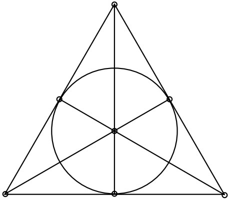
Prove that $\mathbf{ex}(n, \mathcal{L}) = o(n^k)$. iff some $L \in \mathcal{L}$ can be k-colored so at each edge meats each of the k colors.

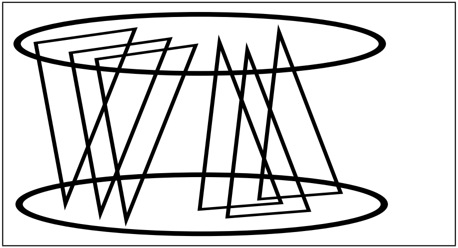


The T. Sós conjecture

Conjecture (V. T. Sós)

Partition $n > n_0$ vertices into two classes A and B with $||A| - |B|| \leq 1$ and take all the triples intersecting both A and B. The obtained 3-uniform hypergraph is extremal for $\mathcal{F}$.





The conjectured extremal graphs: $\mathcal{B}(X, \overline{X})$

Füredi-Kündgen Theorem

If M_n is an arbitrary multigraph (without restriction on the edge multiplicities, except that they are nonnegative) and all the 4-vertex subgraphs of M_n have at most 20 edges, then

$$e(M_m) \leq 3\binom{n}{2} + O(n).$$

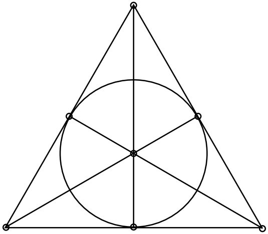
→ FureKund

Theorem (de Caen and Füredi)

→ *FureCaen*

$$\mathbf{ex}(n, \mathcal{F}) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

The Fano-extremal graphs



Theorem (Main, Füredi-Sim. / Keevash-Sudakov)

If $\mathcal{H}$ is a triple system on $n > n_1$ vertices not containing $\mathcal{F}$ and of maximum cardinality, then $\chi(\mathcal{H}) = 2$.

$$\Longrightarrow \qquad \mathrm{ex}_3(n, \mathcal{F}) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

Stability

Theorem

There exist a $\gamma_2 > 0$ and an n_2 such that:
If $\mathcal{F} \not\subseteq \mathcal{H}$ and

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2} \textit{ for each } x \in V(\mathcal{H}),$$

then $\mathcal{H}$ is bipartite, $\mathcal{H} \subseteq \mathcal{H}(X, \overline{X})$.

→ FureSimFano

Anti-Ramsey theorems

Definition

Given a colouring of the edges of a graph L, we call L **totally multicoloured** (TMC), if all the edges of L have different colours. For fixed L, an edge-coloured G is TMC if each $L \subseteq G$ is TMC. If G is not TMC, then we call it BADLY coloured. (If G is TMC, we may call it WELL-coloured.)

The original version

Given a sample graph L, and $e(G_n) = e$, How many colours X of an edge-colouring of G_n ensure at least one TMC-copy of L?

Notatition: The maximum will be denoted by $\mathbf{AR}(\mathbf{n}, \mathcal{L})$.

Reducing Anti-Ramsey to Extremal

Consider the case when $G_n = K_n$. If we take one edge from each colour, then we get a graph H_n and the condition means that it cannot contain any $L \in \mathcal{L}$. Therefore

$$\mathbf{AR}(\mathbf{n}, \mathcal{L}) \leq \mathbf{ex}(n, \mathcal{L}).$$

Improvement

For a given $\mathcal{L}$, denote by $\mathcal{L}^*$ the family of the graphs obtained from the graphs $L \in \mathcal{L}$ by deleting an edge xy from L in all the ways and then gluing the pairs of these graphs in all the possible ways by identifying $xy \in L_i$ and $xy \in L_j$.

Balanced versions, Erdős-Tuza

Given a sample graph L, and $e(G_n) = e$, How many colours X of an edge-colouring of G_n ensure at least one TMC-copy of L, if each colour is used "in an even way"????

Erdős, Tuza: Rainbow subgraphs in edge-colorings of complete graphs. Quo vadis, graph theory?, 81–88, Ann. Discrete Math., 55, North-Holland, Amsterdam, 1993.

A dual Anti-Ramsey problem

Introductory example

Given a graph G_n with

$$e(G_n) = \left[\frac{n^2}{4}\right] + 1.$$

How many colours are needed to 5-edge-colour each $C_5 \subset G_n$?

The more general version

Given a sample graph L, and graph G_n with

$$e(G_n) = \mathbf{ex}(n, L) + k.$$

How many colours are needed to $e(L)$-edge-colour each $L \subset G_n$?

Motivation

- [BEGS]: Burr, Erdős, Graham, Sós
- [BEFGS]: Burr, Erdős, Frankl, Graham, Sós

The problem seems to be very interesting on its own. It emerged in "Theoretical Computer Science". Both [BEGS] and [BEFGS] mention that their motivation actually originated from a question of **S. Berkowitz,** concerning time-space trade-offs for **Turing Machines** (models of computation), for which the Ruzsa-Szemerédi theorem [RuzsaSzem], yields some estimate but "which is still unresolved." The details can be found in the Appendix of [BEGS].

Introduction

L	$\mathbf{ex}(n,L)$	$\mathbf{ex}(n,L)+1$	$\mathbf{ex}(n,L)+cn$	$t_2(n)+\varepsilon n^2$	$\binom{n}{2}-\varepsilon n^2$
K_k	$t_{k-1}(n)$	$\binom{k}{2}$	$\binom{k}{2}$	$\binom{k}{2}$	misprint?
C_5	$t_2(n)$	cn	$\leq cn\sqrt{n}$	$> cn$	$\leq \frac{cn^2}{\log n}$
C_p $p=7,9$	$t_2(n)$	cn^2	cn^2	cn^2	cn^2

Table : Values of $\chi_S(n,e,L)$ for various graphs and values of n,e

*"If we examine the first three rows of Table **??**, we see a striking trichotomy: C_3, C_5 and all the other odd cycles behave very differently. For $L=C_3$, $\chi_S(n,e,L)$ is very small, and is not hard to determine; for $L=C_5$, $\chi_S(n,e,L)$ seems to behave in a complicated and poorly-understood way; for the other odd cycles, $\chi_S(n,e,C_k)$ is very large and good estimates are known. . . "*

Dual Anti-Ramsey theorems

As we have mentioned, in some sense the problems in [BEGS] are dual to the "original" Anti-Ramsey problems: instead of determining the **maximum** number of colours without having a TMC copy of L, we are looking for the **minimum** number of colours making possible that each copy of L is TM-coloured.[a]

[a] More precisely, we have a "host" graph U_n containing L and we try to determine the maximum number of colours used for U_n without getting a TMC copy of L, where U_n can be K_n, or a random graph $R_{n,p}$...

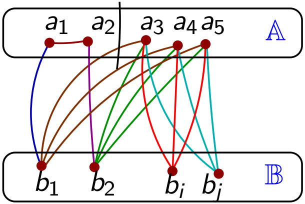
[BEGS] S. A. Burr, P. Erdős, R. L. Graham, and V. T. Sós,
Maximal antiramsey graphs and the strong chromatic number, J Graph Theory 13 (1989), 263–282.

S. A. Burr, P. Erdős, P. Frankl, R. L. Graham, and V. T. Sós,
Further results on maximal antiramsey graphs, In Graph Theory, Combinatorics and Applications, Vol. I, Y. Alavi, A. Schwenk (Editors), John Wiley and Sons, New York, 1988, pp. 193–206.

So $L = C_5$ seems to be one of the most interesting cases. Chapter 4 of [BEGS] deals with $L = C_5$. It contains four related theorems. We improve those results, find the corresponding exact bounds. Actually, the C_5-line of Table **??** is

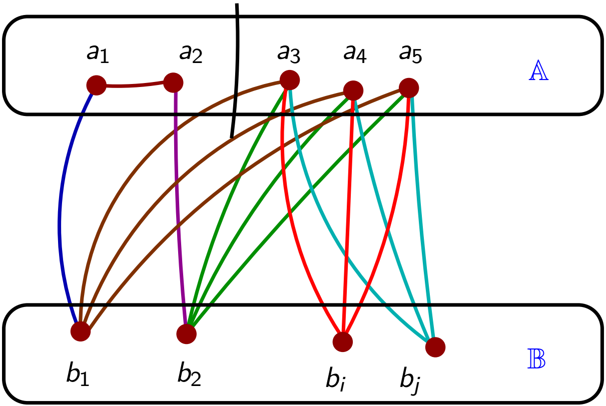
THEOREM 4.1 OF **[BEGS]**. *There exists an n_0 such that if $n > n_0$ and $e = \left[\frac{n^2}{4}\right] + 1$, then*

$$c_1 n \le \chi_S(n, e, C_5) \le \left\lfloor \frac{n}{2} \right\rfloor + 3.$$



Theorem (Erdős-Sim)

There exists a threshold n_0 such that if $n > n_0$, and a graph G_n has $\left[\frac{n^2}{4}\right] + 1$ edges and we colour its edges so that every C_5 is 5–coloured, then we have to use at least $\left\lfloor \frac{n}{2} \right\rfloor + 3$ colours.


a_1
a_2
a_3
a_4
a_5
$\mathbb{A}$
b_1
b_2
b_i
b_j
$\mathbb{B}$

Construction (Upper bound in Theorem 4.1 of [BEGS])

Consider $G_n \in \mathcal{T}_{n,2,1}$, *with two colour classes* $\mathbb{A} = \{a_1, a_2, a_3, a_4, \ldots, a_\alpha\}$ *and* $\mathbb{B} = \{b_1, \ldots, b_\beta\}$, *where* $\alpha = \left\lceil \frac{n}{2} \right\rceil$, $\beta = \left\lfloor \frac{n}{2} \right\rfloor$. G_n *has one special edge* a_1a_2, *and we colour the edges of* G_n *by* $\left\lfloor \frac{n}{2} \right\rfloor + 3$ *colours in the following way:*

$$\begin{aligned} \mathrm{X}(a_1a_2) &= \overline{0}; \\ \mathrm{X}(a_1u) &= \overline{a}_1, \quad \textit{if} \;\; u \in \mathbb{B}; \\ \mathrm{X}(a_2u) &= \overline{a}_2 \quad \textit{if} \;\; u \in \mathbb{B}; \\ \mathrm{X}(zb_t) &= \overline{b_t} \quad \textit{if} \;\; b_t \in \mathbb{B} \;\; \textit{and} \;\; z \in \mathbb{A} - \{a_1, a_2\}. \end{aligned}$$

A slightly more general Construction 3

For each $a_t \in \mathbb{A}$ fix a **permutation** $\pi_t : \mathbb{B} \to \mathbb{B}$ and

$$\text{colour } a_t b_j \text{ by } \overline{\pi_t(b_j)}$$

Good colourings $\longleftrightarrow$ Truncated Latin Squares.

Theorem (Uniqueness)

There exists an n_0 such that if $n > n_0$ and $e(G_n) = \left[\frac{n^2}{4}\right] + 1$, then the minimum number of colours, $\left\lfloor \frac{1}{2} n \right\rfloor + 3$, to TM-colour all the C_5's of G_n is attained only if G_n is a TURÁN *graph on two classes. and the colouring is described in Construction 3.*

General One-sided construction

Given p, q, ℓ, h, with $p+q=n$, $p \geq q$, $\ell \leq \binom{h}{2}$, consider a complete bipartite graph $G[\mathbb{A}, \mathbb{B}]$, where $\mathbb{A} = \{y_1, \ldots, y_p\}$, $\mathbb{B} = \{u_1, \ldots, u_q\}$ and $\mathbb{A}^* = \{y_1, \ldots, y_h\} \subset \mathbb{A}$. Embed ℓ edges $e_1, \ldots, e_\ell$ into $G[\mathbb{A}, \mathbb{B}]$ with endvertices in $\mathbb{A}^*$. Assume that each $y_t \in \mathbb{A}^*$ is covered by some e_i. For each $y_t \in \mathbb{A}$ fix a permutation $\pi_t : \mathbb{B} \to \mathbb{B}$. Let G_h be the graph defined by the edges $e_1, \ldots, e_\ell$.

1. Colour G_h in $\chi_{SI}(G_h)$ colours so that the edges of the same colour are pairwise strongly independent.
2. If $y_t \notin V(G_h)$, i.e. $t > h$, then $X(y_t u_j) = \overline{\pi_t(u_j)}$.
3. Finally,
    3.1 for $h = 2$ $(\ell = 1)$ colour $y_t u_j$ with $\overline{y}_t$ for $t = 1, 2$;
    3.2 For $h = 3$, $\ell = 2$, let $G_h = P_3 = y_1 y_2 y_3$. Then, as an exception, we may connect y_2 to $\mathbb{B}$ in one colour $\overline{y}_2$, but then any edge between y_1, y_2, y_4 and $\mathbb{B}$ are distinct: in case of this exception we use at least $3|\mathbb{B}| + 3$ colours.
    3.3 for $h \geq 4$ colour $y_t u_j$ with $\overline{(\pi_t(u_j), t)}$ for $t = 1, 2, \ldots, h$.

Results on slightly larger k

Theorem

There exists a function $\vartheta(n) \to \infty$ such that if $0 < k = \binom{h}{2} < \vartheta(n)$, then the upper bound of Theorem 4.2/ [BEGS] is sharp for $e = \left[\frac{n^2}{4}\right] + k$:

$$\chi_S(n, e, C_5) = (h+1)\left\lfloor \frac{n}{2} \right\rfloor + k.$$

Because of the monotonicity, this implies

Theorem

There exists a function $\vartheta(n) \to \infty$ such that if $0 < k \le \binom{h}{2} < \vartheta(n)$, then for $e = \left[\frac{n^2}{4}\right] + k$,

$$\chi_S(n, e, C_5) = (h+1)\left\lfloor \frac{n}{2} \right\rfloor + k + O(\sqrt{k}).$$

Let

$$\binom{h-1}{2} < k \le \binom{h}{2}, \qquad \text{i.e.} \qquad h = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$$

THEOREM 4.2 OF **[BEGS]**. *Let n be large and $e = \left[\frac{n^2}{4}\right] + k$. Define h by (above) Then*

$$\chi_S(n, e, C_5) \le (h+1)\left\lfloor \frac{n}{2} \right\rfloor + k.$$

To prove this, consider the following construction (see [BEGS])

Construction (Small k)

Let $k \ge 3$. Using the above notations, embed $G_h = K_h$ into $\mathbb{A}$ of $G[\mathbb{A}, \mathbb{B}]$. Colour each edge of G_h by distinct colours $\overline{1}, \overline{2}, \dots, \overline{k}$. For each $a_i \in \mathbb{A}$ fix a permutation $\pi_i : \mathbb{B} \to \mathbb{B}$ and colour $a_i b_j$ by $\overline{(\pi_i(j), i)}$, for $i = 1, \dots, h$. Further, for $i > h$ colour $a_i b_j$ by $\overline{\pi_i(j)}$.

Proof, First step: Almost bipartite

The first tool will be to count the triangles in G_n.

Theorem

Fix an arbitrary (huge) constant $\Omega > 0$. Let G_n be a graph with $\chi(G_n, C_5) \leq \Omega n$. Then $m(C_3, G_n) = o(n^3)$. Further, if $e(G_n) > \left[\frac{n^2}{4}\right] - o(n^2)$, then $\rho(G_n, T_n, 2) = o(n^2)$, i.e. $V(G_n)$ can be partitioned into two classes $\mathbb{A}$ and $\mathbb{B}$ of sizes $|\mathbb{A}|, |\mathbb{B}| = \frac{1}{2}n + o(n)$, so that every vertex of $\mathbb{A}$ is joined to at most $o(n)$ other vertices of $\mathbb{A}$, and every vertex of $\mathbb{B}$ is joined to at most $o(n)$ other vertices of $\mathbb{B}$.

Lovász-Simonovits Stability

In the next theorem t and d are defined by

$$e(G_n) = \left(1 - \frac{1}{t}\right)\frac{n^2}{2} \qquad \text{and} \qquad d = \lfloor t \rfloor.$$

Theorem (Lovász–Sim. [LovSimBirk])

Let $C \geq 0$ be an arbitrary constant. There exist positive constants $\delta > 0$ and a $C' > 0$ such that if $0 < k < \delta n^2$ and G_n is a graph with

$$e(G_n) = e(T_{n,p}) + k,$$

and

$$m(K_p, G_n) < \binom{t}{p}\left(\frac{n}{p}\right)^p + Ckn^{p-2},$$

then there exists a $K_d(n_1, \dots, n_d)$ such that $\sum n_i = n$, $|n_i - \frac{n}{d}| < C'\sqrt{k}$ and G_n can be obtained from $K_d(n_1, \dots, n_d)$ by changing at most $C'k$ edges.

Ruzsa-Szemerédi/Removal Lemma

Brown-Erdős-Sós: $f(n, k, \ell)$

Theorem (Ruzsa-Szemerédi)

If (G_n) is a graph sequence with $o(n^3)$ triangles, then we can delete $o(n^2)$ edges from the graph to get atriangle-free graph.

Theorem (Ruzsa-Szemerédi)

If (H_n) is a sequence of 3-uniform hypergraphs with no 6 vertices defining 3 triangles, then it has at most $o(n^2)$ triangles.

Connection to Ruzsa-Szemerédi

As a tool, we shall need one more result from [BEGS], on the case $L = P_4$, which – as we shall see – is strongly connected to the problem of determining $\chi_S(n, e, C_5)$.

Theorem 6.3 of **[BEGS]**. *For any $c > 0$,*

$$\frac{1}{n}\chi_S(n, cn^2, P_4) \to \infty.$$

In other words, if $e(G_n)$ is a graph with $> cn^2$ edges and the edges are coloured so that every P_4 is 3–coloured, then we use at least $p(c, n) \cdot n$ colours for some function $p(c, n)$ tending to ∞ . (This result is strongly connected to the theorem of Ruzsa and Szemerédi [RuzsaSzem].)
The function $p(c, n)$ will play an important role in our proofs.

In fact, the largest value e for which

$$\chi_S(n, e, P_4) \leq n$$

satisfies

$$c_1 f(n, 6, 3) \leq e(n) \leq c_2 f(n, 6, 3).$$

Lemma

If G_n contains a vertex x for which $N(x)$ contains $> cn^2$ edges, then for some $g_c(n) \to \infty$ we use at least $g_c(n) \cdot n$ colours to TM–colour all the pentagons of G_n.

Sketch of the proof

1. We show that the neighbourhood of each $x \in V(G_n)$ contains $o(n^2)$ edges.
2. Therefore $m(G_n, K_3) = o(n^3)$.
3. Applying Lovász-Sim we get that G_n is almost $T_n n, 2$.
4. We recursively delete the low-vertex vertices to get a (large) subgraph G_m in which the minimum degree is at least, say $n/3$.
5. We show (in several steps) that the extremal structure is the one described by our constructions, otherwise our G_n would need many colours.

Thank for your attention