6
Extremal Graph Theory

M. SIMONOVITS

1. Introduction

Many problems in graph theory involve optimization, and therefore could be called "extremal problems". Here we take a more restricted meaning of the term, concentrating on what can be called "Turán-type extremal problems". In such a problem, a graph of given order has a certain type of subgraph prohibited, and one is to determine the maximum number of edges possible in the graph. Turán's original problem asked for the maximum number of edges in any graph of order n which does not contain the complete graph K_p.

How Turán arrived at this problem is quite interesting. In 1935 Erdős and Szekeres rediscovered and proved Ramsey's theorem, after which Turán turned the problem around and looked at it in a different way. In Ramsey's problem one assumes that a graph does not contain t independent vertices and tries to ensure that it contains K_p; in Turán's problem one assumes that a graph has a certain number of edges and tries to ensure the same result. We shall see that, whereas Ramsey's problem remains essentially unsolved, Turán's problem has a nice solution. (For a survey of Ramsey graph theory, see **ST1**, Chapter 13.)

GRAPH THEORY, 2
ISBN 0 12 086202 6

Ideally, for a given class of prohibited graphs, one can determine the extremal graphs—that is, those graphs which have the maximum number of edges but do not contain a prohibited graph, as in the case of Turán's result. This is frequently too difficult, and so one tries to find just the maximum number of edges the graph can have, or even just bounds for this number. However, there is considerably more to extremal graph theory than merely results of this nature, but in a survey such as this a great deal of selection must take place.

In Section 2 we present Turán's theorem and give two proofs of it, along with some generalizations, and then in Section 3 we move on to the general problem.

Extremal graph problems behave differently when the minimum chromatic number of the forbidden graphs is 2 and when it is larger. Problems of the former type are called *degenerate*, whereas those of the latter type are called *non-degenerate*. Section 4 gives a few theorems concerning non-degenerate problems in which the extremal graphs can be fairly well described. Section 5 deals with "perturbation problems" such as chromatic perturbation, minimum valency perturbation, and Ramsey perturbation. In Section 6, we consider degenerate extremal problems and their connection with non-degenerate problems; in fact, many non-degenerate problems can be reduced to degenerate ones.

The lower bounds of degenerate extremal graph problems are often obtained by finite geometrical constructions, as described in Section 7. Section 8 deals with the use of random graphs as a non-constructive method of obtaining lower bounds, whereas Section 9 treats some hypergraph problems.

If a graph has more edges than an extremal graph, for a given prohibited family, then it must clearly contain a prohibited subgraph. It is an interesting phenomenon that it generally contains many of them; Section 10 discusses these "supersaturated graphs". Section 11 deals with digraph extremal problems, and the last section is on some applications in analysis and geometry.

In selecting these topics, we have used several criteria for the inclusion of results. Some were chosen on the basis of being especially interesting or of being typical of a particular area of extremal graph theory; others, such as the reduction theorem (Section 6), Szemerédi's uniformization lemma (Section 8), and Erdős' hypergraph lemma (Section 9), were chosen because of their usefulness in establishing other results. We have also tried to indicate various techniques—for example, "chopping off a nice subgraph", symmetrization, progressive induction, the method of finite geometries, and the probabilistic method. The proofs which we have included were also chosen because they indicate basic ideas or illustrate techniques. For lack of space, many interesting results (and proofs) could not be included; for a fuller treatment of some areas, the reader is referred to Erdős' *Selected Works* [**39**], Bollobás' excellent book [**8**], and Simonovits [**82**].

Before proceeding with our survey, we present some of the notation and terminology to be used in this chapter. In a few instances, this differs from usage in other chapters.

Generally, our graphs are simple and undirected, but on occasion we consider variations such as multigraphs, hypergraphs and digraphs. The number of vertices in a graph G may be indicated by $|V(G)|$, or given in a superscript such as G^n (denoting an arbitrary graph of order n). However, for well-defined graphs such as K_p, the order still appears as a subscript. The number of edges in G is denoted by $e(G)$, and for disjoint subsets X and Y of the vertex-set, $e(X, Y)$ denotes the number of edges joining X and Y. We further say that X and Y are *completely joined* if every vertex of X is adjacent to every vertex of Y.

If G and H are disjoint graphs, then $G + H$ denotes their disjoint union, whereas $G \times H$ denotes the graph obtained by completely joining $V(G)$ and $V(H)$. (Note that this notation in particular differs from that of other authors.)

The **Turán graph** $T_{n,d}$ is defined to be the complete d-partite graph $K_{n_1,\ldots,n_d}$ of order n in which the partite sets are as nearly equal as possible—that is, $\Sigma n_i = n$, and $|n_i - (n/d)| < 1$ (see Fig. 1). We note that, among all d-colorable graphs of order n, $T_{n,d}$ has the maximum number of edges.

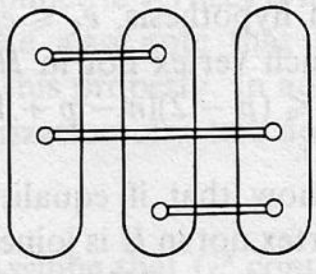

Fig. 1

2. Turán's Theorem

As we mentioned in the Introduction, from a consideration of Ramsey's problem Turán arrived at the following question:

Question 2.1. What is the maximum number of edges a graph G^n can have without containing the complete graph K_p?

His answer was published in 1941 [**90**] (see also [**91**]):

Theorem 2.1 (Turán's Theorem). *Among the graphs of order n which do not contain the complete graph K_p, there exists exactly one with the maximum number of edges—namely, $T_{n,p-1}$.*

In addition to proving this result, Turán's renowned paper established some other theorems and posed other questions. Since the other results are closer to Ramsey-type theorems than to our topic, we omit them, but we note that the questions he posed include his famous hypergraph problem, which is still unsolved and is one of the most intriguing problems in this field (see Section 9). Another problem posed by Turán (in a letter to Erdős) was the version of his problem for paths rather than complete graphs. Soon after this, numerous results on problems of a similar nature were found, and a new branch of graph theory began to flourish. From a historical point of view, it is interesting to speculate on why Turán's paper stimulated these deeper investigations, even though the special case for K_3 had been published in 1907 and Erdős [**28**] had considered the question for the circuit graph C_4 in connection with a problem in number theory.

Turán's original proof, which uses a method we call *chopping off a nice subgraph*, is quite straightforward:

Sketch of Turán's proof. The proof uses induction on n. Assume that G^n is a graph containing no complete graph K_p, and assume, without loss of generality, that G^n has a subgraph $H = K_{p-1}$. Let e_1, e_2, e_3 denote the numbers of edges in H, edges in $G^n - V(H)$, and edges joining vertices in H to vertices not in H, respectively, so that $e(G^n) = e_1 + e_2 + e_3$. Now $e_1 = \binom{p+1}{2}$, and by the induction hypothesis, $e_2 \leqslant e(T_{n-p+1,p-1})$. Furthermore, since G^n contains no K_p, each vertex not in H is joined to at most $p-1$ vertices of H, and hence $e_3 \leqslant (p-2)(n-p+1)$ (see Fig. 2). These results imply that $e(G^n) \leqslant e(T_{n,p-1})$.

All that remains is to show that if equality holds, then $G^n = T_{n,p-1}$. Clearly, in this case, each vertex not in H is joined to exactly $p-2$ vertices of H. The vertices of G (including those of H) can thus be partitioned into $p-1$ classes according to which $p-2$ vertices of H they are adjacent to. One can

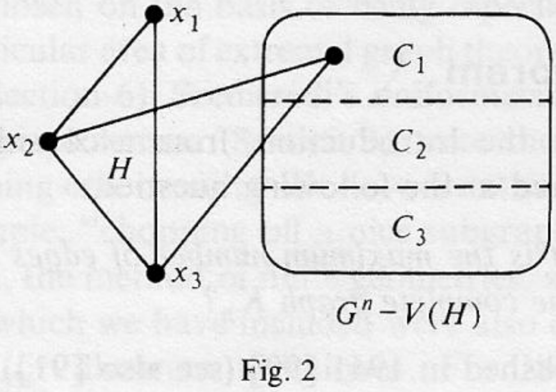

Fig. 2

easily see that the vertices in each class are independent, so that G^n is the complete $(p-1)$-partite graph defined by these classes, and hence that $G^n = T_{n,p-1}$. ‖

We now turn to a second proof, due to Zykov [**93**]. Its basic advantage is that it avoids an explicit induction, but the main reason why we choose to present it is to demonstrate one of the most powerful methods of extremal graph theory, the method of *symmetrization*. (We note that the Motzkin–Straus proof [**69**] is in a sense a variation of Zykov's.)

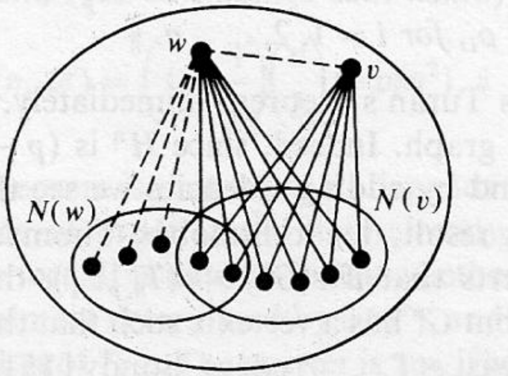

Fig. 3

Two vertices in a graph are called **symmetric** if they have precisely the same neighbors. In the operation of **symmetrization of a vertex *w* to a vertex *v***, all edges at w are deleted and then w is made adjacent to all neighbors of v (see Fig. 3). Clearly, after this symmetrization, v and w are symmetric. Further, the symmetry of symmetric vertices is not affected by the symmetrization of other vertices. We also note that symmetrization in a graph containing no K_p preserves this property. In addition, if $\rho(w) \leqslant \rho(v)$, and if vw is not in G, then symmetrization of w to v does not reduce the number of edges.

Sketch of Zykov's proof. Assume that G^n contains no complete graph K_p, and let v_1 be a vertex of maximum valency. Symmetrize to v_1 all those vertices not adjacent to v_1 (one by one, or all at once), and call this entire set of vertices S_1 (including v_1). The resulting graph G_1^n has at least $e(G^n)$ edges, and the vertices of S_1 are all adjacent to all other vertices. Repeat this procedure using a vertex v_2 of maximum valency in $G_1^n - S_1$ to obtain a graph G_2^n with two sets S_1 and S_2 of independent vertices joined to all other vertices, and with $e(G_2^n) \geqslant e(G_1^n)$. If this procedure is iterated until all vertices have been used, then the result G_r^n is a complete r-partite graph $K_{n_1,\ldots,n_r}$. During this symmetrization no K_p can occur, and so $r \leqslant p-1$. Hence $e(G^n) \leqslant e(G_r^n) \leqslant e(T_{n,p-1})$. The proof is complete once it is determined that, if $e(G^n) = e(T_{p,n-1})$, then $G^n = T_{p,n-1}$. ‖

In subsequent sections, we shall see many different generalizations of Turán's theorem; in most of these, the prohibited graph is something other than K_p. To conclude the present section, we mention a few direct generalizations. The first of these is due to Erdős, but because the result originally appeared in Hungarian, we refer the reader to Bondy and Murty [**16**].

Theorem 2.2 (Erdős' Theorem). *Let G^n be a graph which does not contain K_p, and which has valencies $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_n$. Then there exists a $(p-1)$-chromatic graph H^n (which thus contains no K_p) whose valencies $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_n$ satisfy $\sigma_i \geqslant \rho_i$, for $i = 1, 2, \ldots, n$.* ‖

This result implies Turán's theorem immediately, except for the uniqueness of the extremal graph. Indeed, since H^n is $(p-1)$-colorable, we have $e(H^n) \leqslant e(T_{n,p-1})$, and by adding valencies we see that $e(G^n) \geqslant e(H^n)$.

Another surprising result, due to Bollobás, Thomason, Erdős and Sós (see [**14**] and [**52**]), asserts that if $e(G^n) > e(T_{n,p-1})$, then not only is there a complete graph K_p, but G^n has a vertex x such that the graph G^m spanned by its neighbors has at least $e(T_{m,p-2})$ edges. Bondy [**15**] improved this result, as follows:

Theorem 2.3. *Assume that $e(G^n) > e(T_{n,p-1})$, and that x is a vertex of maximum valency m. Then the graph G^m spanned by the neighbors of x has more than $e(T_{m,p-2})$ edges.* ‖

We note that the earlier result covered the case of equality, whereas Theorem 2.3 does not (for the simple reason that the conclusion does not follow). We also observe that these results imply Turán's theorem.

3. General Theory

In this section, we present the asymptotic solution to the general extremal problem, which is the following:

General Extremal Problem. Given a family $\mathscr{L}$ of forbidden subgraphs, find those graphs G^n which contain no graph in $\mathscr{L}$ and have the maximum number of edges.

The collection of these graphs G^n forms the set of **extremal graphs** for $\mathscr{L}$. It is denoted by $\mathrm{EX}(n, \mathscr{L})$, and the number of edges of a graph in this set is denoted by $\mathrm{ex}(n, \mathscr{L})$. The problem is considered to be "completely solved" if all the extremal graphs have been found, at least for $n > n_0(\mathscr{L})$. Quite often this is too difficult, and we must be content with finding $\mathrm{ex}(n, \mathscr{L})$, or even good bounds for it.

It turns out that a parameter related to the chromatic number plays a decisive role in many extremal graph theorems. The **subchromatic number** $\psi(\mathscr{L})$ of $\mathscr{L}$ is defined by

$$\psi(\mathscr{L}) = \min\{\chi(L): L \in \mathscr{L}\} - 1.$$

The following result [**46**] is an easy consequence of the Erdős–Stone theorem [**55**] (see below):

Theorem 3.1 (The Erdős–Simonovits Theorem). *If $\mathscr{L}$ is a family of graphs with subchromatic number p, then*

$$\mathrm{ex}(n, \mathscr{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2). \ \|$$

The meaning of this theorem is that $\mathrm{ex}(n, \mathscr{L})$ depends only very loosely on $\mathscr{L}$; up to an error term of order $o(n^2)$, it is already determined by the minimum chromatic number. It is natural to ask whether the *structure* of the extremal graphs is also almost determined by $\psi(\mathscr{L})$, and therefore whether it must be very similar to that of $T_{n,p}$. The answer is yes. This is expressed by the following results of Erdős and Simonovits [**35**], [**36**], [**72**]:

Theorem 3.2 (The Asymptotic Structure Theorem). *Let $\mathscr{L}$ be a family of prohibited graphs with subchromatic number p. If S^n is any graph in* $\mathrm{EX}(n, \mathscr{L})$, *then it can be obtained from $T_{n,p}$ by deleting and adding $o(n^2)$ edges. Furthermore, if $\mathscr{L}$ is finite, then $\rho_{\min}(S^n)/n = 1 - p^{-1} + o(1)$.* ‖

The structure of extremal graphs is fairly stable, in the sense that the almost-extremal graphs have almost the same structure as the extremal graphs. This is expressed in our next result:

Theorem 3.3 (The First Stability Theorem). *Let $\mathscr{L}$ be a family of prohibited graphs with subchromatic number p. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_ε such that, if G^n contains no $L \in \mathscr{L}$, and if, for $n > n_\varepsilon$,*

$$e(G^n) > \mathrm{ex}(n, \mathscr{L}) - \delta n^2,$$

then G^n can be obtained from $T_{n,p}$ by changing at most εn^2 edges. ‖

These theorems are interesting in themselves. However, it is important to note that they are also widely applicable. To illustrate this, we show how the last statement of Theorem 3.2 can be deduced from Theorem 3.3.

In order to facilitate this, we introduce some new terms. Consider a partition $S_1, S_2, \ldots, S_p$ of the vertex-set of G^n, and the p-partite graph $H^n = K_{s_1, \ldots, s_p}$ corresponding to this partition of $V(G^n)$, where $s_i = |S_i|$. An edge vw is called an **extra edge** if it is in G^n but not in H^n, and is a **missing edge** if it is in H^n but not in G^n. For given p and G^n, the partition $S_1, S_2, \ldots, S_p$ is called **optimal** if the number of missing edges is minimum. Finally, for a given vertex v, let $a(v)$ and $b(v)$ denote the numbers of missing and extra edges at v.

Sketch proof of Theorem 3.2. The first part of the theorem follows from Theorems 3.3 and 3.1 by simply setting $G^n = S^n$. For the second part, take the optimal partition $R_1, R_2, \ldots, R_p$ of $V(S^n)$, and assume that R_1 has minimum order. Then $|R_1| \leqslant n/p$ and, by Theorem 3.3, $\sum_{v \in R_1} b(v) = o(n^2)$. If r denotes the maximum order of a graph in $\mathscr{L}$, take r vertices $v_1, v_2, \ldots, v_r$ with $\sum_1^r b(v_i)$ minimum. Clearly, for some $c > 0$, $|R_1| > cn$. Thus

$$\sum_{i=1}^{r} b(v_i) \leqslant \frac{r}{|R_1|} \sum_{v \in R_1} b(v) = o(n).$$

(This is what we called a "trivial averaging argument".)

Now we apply the symmetrization method in a slightly modified form: for an arbitrary fixed vertex v in S^n, delete all the incident edges and then join it to all vertices adjacent to all of $v_1, v_2, \ldots, v_r$. The resulting graph $\tilde{S}^n$ can contain no $L \in \mathscr{L}$. It follows that $e(S^n) \leqslant e(\tilde{S}^n)$, and hence

$$\rho(v) \geqslant \left|\bigcap_{i=1}^{r} \operatorname{nbd}(v_i)\right| \geqslant \left|\bigcup_{j=2}^{r} R_j\right| - \sum_{i=1}^{r} b(v_i) \geqslant n - \frac{n}{p} - o(n). \,\|$$

In the remainder of this section we formulate a sharper variant of the stability theorem, and then return to the particular problem considered in the Erdős–Stone theorem.

One can ask whether further information on the structure of prohibited subgraphs yields better bounds on $\operatorname{ex}(n, \mathscr{L})$ and further information on the structure of extremal graphs. At this point, we need a definition. Let $\mathscr{L}$ be a family of forbidden subgraphs, and let $p = \psi(\mathscr{L})$ be its subchromatic number. The **decomposition** $\mathscr{M}$ of $\mathscr{L}$ is the family of graphs M with the property that, for some $L \in \mathscr{L}$ of order r and subchromatic number p, L contains M as an induced subgraph and $L - V(M)$ is $(p-1)$-colorable. In other words, $L \subseteq M \times K_{p-1(r)}$, and M is minimal with this property. The following result is due to Simonovits [**72**]:

Theorem 3.4 (The Decomposition Theorem). *Let $\mathscr{L}$ be a forbidden family of graphs with $\psi(\mathscr{L}) = p$ and decomposition $\mathscr{M}$. Then every extremal graph $S^n \in \operatorname{EX}(n, \mathscr{L})$ can be obtained from a suitable $K_{n_1,\ldots,n_p}$ by changing $O(\operatorname{ex}(n, \mathscr{M}))$ edges. Furthermore, $n_i = (n/p) + O(\operatorname{ex}(n, \mathscr{M})/n)$, and $\rho_{\min}(S^n) = (1 - p^{-1})n + O(\operatorname{ex}(n, \mathscr{M})/n)$.* $\|$

It follows from this theorem that, with $s = [n/p]$, $\operatorname{ex}(n, \mathscr{L}) = e(T_{n,p}) + O(\operatorname{ex}(S, \mathscr{M}))$. This result is sharp: put edges into the first class of a $T_{n,p}$ so that they form a $G^r \in \mathrm{EX}(r, \mathscr{M})$; the resulting graph contains no $L \in \mathscr{L}$, and has $e(T_{n,p}) + \mathrm{ex}(r, \mathscr{M})$ edges.

A second stability theorem can be established using the methods of [72]:

Theorem 3.5 (The Second Stability Theorem). *Let $\mathscr{L}$ be a forbidden family of graphs with $\psi(\mathscr{L}) = p$ and decomposition $\mathscr{M}$, and let $k > 0$. Further, let $G^n \in \mathrm{EX}(n, \mathscr{L})$, let $S_1, \ldots, S_p$ be its optimal partition, and let $G_i = \langle S_i \rangle$. Then, if $e(G^n) \geqslant \mathrm{ex}(n, \mathscr{L}) - k\,\mathrm{ex}(n, \mathscr{M})$, the following results hold:*

(i) G^n can be obtained from $\times\, G_i$ by deleting $O(\mathrm{ex}(n, \mathscr{M}))$ edges;

(ii) $e(G_i) = O(\mathrm{ex}(n, \mathscr{M})) + O(n)$, and $|V(G_i)| = (n/p) + O(\sqrt{\mathrm{ex}(n, \mathscr{M})})$;

(iii) for any constant $c > 0$, the number of vertices v in G_i with $a(v) \geqslant cn$ is only $O(1)$, and the number of vertices with $b(v) \geqslant cn$ is only $O(\mathrm{ex}(n, \mathscr{M})/n) + O(1)$;

(iv) let $L \in \mathscr{L}$, with $|L| = r$, and let A_i be the set of vertices v in S_i for which $b(v) \leqslant (n/2pr)$; then the graph $\langle A_i \rangle$ contains no L. ‖

This theorem is useful in applications. The deepest part is the first part of *(iii)*. This implies *(iv)*, which in turn implies all the other statements. A proof is sketched in [77], where the theorem was needed.

Now let $m = m(n, c)$ be the largest integer such that, if $e(G^n) \geqslant e(T_{n,p}) + cn^2$, then G^n contains the regular $(p+1)$-partite graph $K_{p+1(m)}$. The *Erdős–Stone theorem* asserts that $m \to \infty$ for fixed c. Moreover, $m > L_p(n)$, where L_p denotes the p-times-iterated logarithm. This implies Theorem 3.1, as follows:

Proof of Theorem 3.1 (using the Erdős–Stone theorem). Since each $L \in \mathscr{L}$ is not p-colorable, $L \notin T_{n,p}$. Hence,

$$\mathrm{ex}(n, \mathscr{L}) \geqslant e(T_{n,p}) = \left(1 - \frac{1}{p}\right)\binom{n}{p} + O(n).$$

On the other hand, there is some $L_0 \in \mathscr{L}$ with $\chi(L_0) = p + 1$. Let $m = |V(L_0)|$. The Erdős–Stone theorem asserts that

$$\mathrm{ex}(n, K_{p+1(m)}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

Hence, since $L_0 \subseteq K_{p+1(m)}$, we have

$$\mathrm{ex}(n, \mathscr{L}) \leqslant \mathrm{ex}(n, K_{p+1(m)}) \leqslant \left(1 - \frac{1}{p} + o(1)\right)\binom{n}{2}. \;\|$$

One can ask how large is $m = m(n, c)$, defined above? This was determined by Bollobás, Erdős, Simonovits, Chvátal and Szemerédi [10], [12], [94]. Observe that $m(n, c)$ also depends on p:

Theorem 3.6. *There exist constants c and c′ such that*

$$c \cdot \frac{\log n}{\log(1/c)} \leqslant m(n, c) \leqslant c' \cdot \frac{\log n}{\log(1/c)} \cdot \|$$

The first breakthrough here was that the p-times iterated log was replaced by $K \log n$, where K is a constant; the second breakthrough was the dependence on c—namely, that the multiplicative constant is $O(-1/\log c)$.

Inverse Extremal Problems

To conclude this section, we discuss some "inverse" extremal problems. These are problems in which we have a sequence of graphs $\{S^n\}$ and we wish to find prohibited graphs for which $\{S^n\}$ is a sequence of extremal graphs. On the one hand, such theorems have their own interest; on the other hand, they often show that some result on the structure of extremal graphs (for certain types of extremal problems) cannot be improved.

Theorem 3.7. *Let $\{S^n\}$ be a sequence of graphs (not necessarily defined for all positive integers n). If there exists a family $\mathscr{L}$ for which each S^n is extremal, then $e(G^n) \leqslant e(S^n)$ if $G^n \subseteq S^m$ for some m. Conversely, if $\{S^n\}$ satisfies this condition, then there exists a family $\mathscr{L}$ for which every S^n is extremal—namely, the family of those graphs which are not contained in any S^n.*

Proof. First assume that $\{S^n\}$ is extremal for $\mathscr{L}$, and that $G^n \subseteq S^m$. Then G^n contains none of the prohibited subgraphs—that is, $e(G^n) \leqslant e(S^n)$. This proves the first part. Conversely, if $\{S^n\}$ satisfies the given condition, and if $\mathscr{L}_S$ is the family of graphs contained in no S^n, then each S^n is extremal for $\mathscr{L}_S$. Now take a G^n containing no prohibited subgraph. Then $G^n \notin \mathscr{L}_S$—that is, some $S^m \supseteq G^n$. But then $e(G^n) \leqslant e(S^n)$, which means that S^n is indeed extremal. ‖

Our next result, although not very deep, is quite important:

Theorem 3.8. *For every family $\mathscr{L}$, $\text{ex}(n, \mathscr{L})/\binom{n}{2}$ is decreasing as $n \to \infty$.*

Proof. For a fixed S^m we take all $\binom{m}{n}$ subgraphs of order n, $G_1, \ldots, G_t$. Each edge of S^m belongs to $\binom{m-2}{n-2}$ of these. Hence,

$$\binom{m-2}{n-2} e(S^m) \leqslant \sum_{i<t} e(G_i) \leqslant \binom{m}{n} e(S^n),$$

which implies that

$$e(S^m) \Big/ \binom{m}{2} \leqslant e(S^n) \Big/ \binom{n}{2}. \ \|$$

We note that this proof works equally well for digraphs, hypergraphs and other structures.

Our next inverse extremal theorem [72], [77] is much deeper:

Theorem 3.9 (The $T_{n,p}$ Theorem). *A family $\mathscr{L}$ has the Turán graph $T_{n,p}$ as an extremal graph (for n sufficiently large) if and only if some $L \in \mathscr{L}$ has an edge e with $\chi(L - e) = \psi(\mathscr{L}) = p$. Furthermore, if $T_{n,p}$ is extremal for $\mathscr{L}$ for infinitely many values of n, then (again for n sufficiently large) it is the only extremal graph.*

This theorem is a special case of Theorems 5.1 and 5.3. The reason why we give a direct proof here is that this proof uses "progressive induction" in perhaps its simplest form, and it is therefore an apt illustration of this useful method. Additionally, it will be a second application of the "chop off a nice subgraph" principle (see Turán's original proof in Section 2).

The idea of progressive induction is that we have an assertion which satisfies the induction step: if we know it for n, then it follows for $n + 1$. However, it is not valid, *or at least we do not know it*, for small values of n, and proving it for some initial n_0 is as difficult as proving it for every n. Theorem 3.9 is of this type. We introduce a norm $D(n)$ measuring how far what we know is from what we conjecture, and we prove that this $D(n)$ is decreasing unless $D(n) = 0$, in which case the result already follows.

Proof of Theorem 3.9. One part of the theorem is trivial: if $\chi(L - e) \geqslant p + 1$ for every $L \in \mathscr{L}$ and every $e \in E(L)$, then adding one edge to $T_{n,p}$ yields a graph which contains no prohibited L. Thus $T_{n,p}$ is not extremal for $\mathscr{L}$.

Now assume that $\chi(\hat{L}) = p + 1$, but $\chi(\hat{L} - e) = p$. Let $\{S^n\}$ be a sequence of extremal graphs for $\mathscr{L}$. Since $T_{n,p}$ contains no forbidden subgraph, $e(S^n) \geqslant e(T_{n,p})$. We introduce

$$D(n) = e(S^n) - e(T_{n,p}),$$

to measure the "distance between our knowledge and the conjecture". We shall prove that, for $n > n_0$,

(∗) either $T_{n,p}$ is the only extremal graph, or there is an $n' < n$ for which $D(n') > D(n)$ and $n' \to \infty$ as $n \to \infty$.

The statement (∗) implies that, for $n > n_1$, $T_{n,p}$ is the only extremal graph. Indeed, for $n < n_0$, $D(n)$ is bounded, by K (say). By (∗), $D(n) \leqslant K$, for every n. Let N_j be defined so that, if $n > N_j$, then $n' > N_{j-1}$, and $N_0 = n_0$. It is trivial to show by induction on j that, for $n > N_j$, either $D(n) = 0$ or $D(n) \leqslant K - j$. Hence, for $n_1 = N_{K+1}$ (since $D(n) \geqslant 0$), $T_{n,p}$ is the only extremal graph.

As a matter of fact, in our proof, $n' = n - pt$ for $t = 3|V(\hat{L})|$, and instead of using the fact that S^n contains no $L \in \mathscr{L}$, we use only the fact that, if $\hat{\hat{L}}$ is the graph obtained from $T_{pr,p}$ by adding one edge, then $\hat{L} \subseteq \hat{\hat{L}}$. Thus

$S^n \not\supseteq \hat{\hat{L}}$. To prove (*), we choose an arbitrary extremal graph S^n, and using the Erdős–Stone theorem and the fact that $e(S^n) \geqslant e(T_{n,p})$, we fix a "nice subgraph" $T_{pt,p} = K^* \subseteq S^n$, and similarly, a subgraph $T_{pt,p} \subseteq T_{n,p}$. Put

$$S^{n-pt} = S^n - K^*, \qquad T_{n-pt,p} = T_{n,p} - T_{pt,p}, \qquad e_S = e(K^*, S^{n-pt}),$$

and

$$e_T = e(T_{pt,p}, T_{n-pt,p}) = (n-pt)(p-1)t.$$

Since $\hat{\hat{L}} \not\subseteq S^n$, K^* is an induced subgraph of S^n and each vertex of S^{n-pt} is joined to at most $(p-1)t$ vertices of K^*. Therefore,

$$e(S^n) = e(K^*) + e_S + e(S^{n-pt}) \geqslant e(K^*) + (n-pt)(p-1)t + e(S^{n-pt}).$$

This implies that $D(n) \leqslant D(n-pt)$, as follows:

$$\begin{aligned} D(n-pt) - D(n) &= (e(S^{n-pt}) - e(T_{n-pt,p})) - (e(S^n) - e(T_{n,p})) \\ &\geqslant (e(T_{n,p}) - e(T_{n-pt,p})) - (e(S^n) - e(S^{n-pt})) \\ &\geqslant e_S - e_T \geqslant 0. \end{aligned}$$

The only thing left is to check that, if $D(n) = D(n-pt)$, then $S^n = T_{n,p}$. Clearly, each vertex of S^{n-pt} is joined to exactly $p-1$ classes of K^* in this case. We partition $V(S^n)$ into sets $S_1, \ldots, S_p$ by putting into S_j the vertices of S^n not joined to the jth class of $K^* = T_{pt,p}$. The vertices of S_j are independent since otherwise $\hat{\hat{L}} \subseteq S^n$. Thus S^n is a p-colorable graph, and $e(S^n) \geqslant e(T_{n,p})$. This shows that $S^n = T_{n,p}$, completing the proof. ∥

One reason why we could not apply ordinary induction is that the theorem does not hold for small values of n. Another reason is that the Erdős–Stone theorem cannot be applied for small n. One of the important points of the proof was that "each vertex of S^{n-pt} is joined to at most $(p-1)t$ vertices of K^*". This was the "chopping off a nice subgraph", which means selecting $U^m \subseteq S^n$ such that each x in $S^n - U^m$ is joined only to a small number of vertices of U^m.

4. Non-degenerate Extremal Problems

According to Theorem 3.2, if S^n is an extremal graph for a family $\mathscr{L}$ of subchromatic number p, then S^n can be obtained from an appropriate complete p-partite graph by deleting and adding $o(n^2)$ edges. However, in many cases we find that there is no need to delete edges: S^n can be obtained from some $K_{n_1,\ldots,n_p}$ by adding only $o(n^2)$ edges. In other words, S^n is the join of p almost-equal factors. Assume now that $S^n = \times_{i\leqslant p} G^{n_i}$, where $n_i =$

$(n/p) + o(n)$ (if n is sufficiently large). Then the graphs G^{n_i} are extremal graphs for the degenerate extremal graph problems in which the ith prohibited family is

$$\mathscr{M}_i = \{M\colon \text{there exists } L \in \mathscr{L} \text{ for which } L \subseteq M \times \bigtimes_{j \neq 1} G^{n_j}, \text{ and } |V(M_i)| \leqslant |V(L_i)|\}.$$

Clearly, the entire decomposition $\mathscr{M}$ of $\mathscr{L}$ is in $\mathscr{M}_i$, and $\psi(\mathscr{M}) = 1$; hence the extremal problem of $\mathscr{M}_i$ is indeed degenerate. The reader can easily check that G^{n_i} is extremal for $\mathscr{M}_i$. In this sense we have reduced the non-degenerate extremal graph problem of $\mathscr{L}$ to the p degenerate problems of $\mathscr{M}_1, \ldots, \mathscr{M}_p$. If $\mathscr{L}$ is finite, then $\mathscr{M}_i$ is also finite, and $\max_{M \in \mathscr{M}_i}|M_i| = \max_{L \in \mathscr{L}}|L|$.

There are some problems with this reduction method, however, and we mention three difficulties here.

First, it does not always work, since in some cases the extremal graph is not a join. However, in all known instances when this happens, there is always a tree or forest in the decomposition, and this motivates the following conjecture (see **[81]**):

Conjecture 4.1. *If $\mathscr{L}$ contains just one graph and its decomposition contains no tree or forest, then every extremal graph is a join*: $S^n = \bigtimes_{i \leqslant p} G^{n_i}$, *where* $n_i = (n/p) + o(n)$, *for* $n > n_0$.

Secondly, it can happen that the $\mathscr{M}_i$ also depend on n, and this is unpleasant.

Thirdly, even if the reduction can be achieved, the resulting degenerate extremal problems may be too difficult. This is perhaps where a breakthrough is most needed—in the solution of degenerate problems. In part, this is why three sections of this survey (Sections 6–8) are devoted to such problems.

We now turn to a specific example. Turán once asked for the value of $\text{ex}(n, \mathscr{L})$ for each of the five regular polyhedra. For the tetrahedron K_4, the answer is of course provided by Turán's theorem. For the other four, the solution is considerably more difficult, and we treat only the octahedron $O_6 = K_{2,2,2}$ in this section; the dodecahedron and icosahedron will be considered in the next section, and the cube in Section 6. We note that for all but the cube, the extremal graph is a join of almost equally large graphs. The following theorem is due to Erdős and Simonovits **[48]**:

Theorem 4.1 (Octahedron Theorem). *If S^n is an extremal graph for the octahedron O_6, for n sufficiently large, then there exist extremal graphs G_1 and G_2 for the circuit C_4 and the path P_3 such that $S^n = G_1 \times G_2$ and $|V(G_i)| = \frac{1}{2}n + O(n^{\frac{3}{4}})$, $i = 1, 2$.* ‖

Note that G_2 consists of independent edges, and that, if G_1 does not contain C_4 and G_2 does not contain P_3, then $G_1 \times G_2$ does not contain O_6. Thus, if we replace G_1 by any H_1 in $\mathrm{EX}(|V(G_1)|, C_4)$ and G_2 by any H_2 in $\mathrm{EX}(|V(G_2)|, P_3)$, then $H_1 \times H_2$ is also extremal for O_6. It follows that this theorem is indeed a reduction theorem.

We shall see in the next section that the extremal graphs for the dodecahedron and icosahedron are also products, even though their decompositions contain trees.

5. Perturbation Problems

The idea behind perturbation problems is to consider some property of extremal graphs for a family $\mathscr{L}$, and try to determine whether it is important. More formally, given a family $\mathscr{L}$ of graphs and a graphical property $\mathscr{T}$, we let $\mathrm{ex}(n, \mathscr{L}, \mathscr{T})$ denote the maximum number of edges in a graph G^n which contains $L \in \mathscr{L}$ and does *not* have property $\mathscr{T}$, and we let $\mathrm{EX}(n, \mathscr{L}, \mathscr{T})$ denote the family of graphs attaining this maximum. Of course, a property we consider is one that is held by graphs in $\mathrm{EX}(n, \mathscr{L})$ (and usually an obvious one). If $\mathrm{ex}(n, \mathscr{L}, \mathscr{T})$ is significantly smaller than $\mathrm{ex}(n, \mathscr{L})$, then we may conclude that $\mathscr{T}$ is an important property in the original problem.

In this section we consider three types of perturbation problems (the third only briefly):

(a) chromatic perturbation in Turán-type extremal problems;

(b) Ramsey perturbation;

(c) Zarankiewicz perturbation.

Chromatic Perturbation in Turán-type Problems

One feature of Turán's extremal graph $T_{n,p}$ is that it is p-chromatic. In this subsection we consider a number of results, found in Simonovits [77], related to chromatic properties of extremal graphs. The first shows that the chromatic property of Turán graphs is an important feature:

Theorem 5.1 (Chromatic Perturbation Theorem). *Let $\mathscr{L}$ be a family of graphs with subchromatic number p. If, for $n > n_0$, $T_{n,p}$ is the only extremal graph for $\mathscr{L}$, then there exists a constant K such that, if G^n contains no prohibited subgraphs and $e(G^n) \geqslant \mathrm{ex}(n, \mathscr{L}) - (n/p) + K$, then G^n is p-chromatic.* ||

This theorem shows that the almost-extremal graphs for $\mathscr{L}$ are very much like $T_{n,p}$, and that each can be transformed into a complete p-partite graph by changing $O(n)$ edges. This is a consequence of the following result:

Theorem 5.2. *Assume that $\mathscr{L}$ is a family for which $T_{n,p}$ is the only extremal graph for $n > n_0(\mathscr{L})$ and that $\mathscr{T}$ is the property of being t-colorable. Then there is an integer-valued positive function $g(t, \mathscr{L})$ such that, for $t \geqslant p$,*

$$\operatorname{ex}(n, \mathscr{L}, \mathscr{T}) = \operatorname{ex}(n, \mathscr{L}) - g(t, \mathscr{L})\frac{n}{p} + O(1). \parallel$$

This result can be deduced from a still more general theorem of Simonovits [77]. Its formulation requires two further definitions.

First, the **chromatic condition** $\mathscr{A}_{s,t}$ is the property that a graph G^n has if every subgraph of order $n - s + 1$ has chromatic number at least t. (A more general concept is defined in [77].)

Secondly, a graph G^n is in the class $\mathscr{G}(n, p, r)$ of **very symmetric graphs** if it has a set S of at most r vertices for which $G^n - S$ is the join of p graphs G_i of order m_i, where $|m_i - (n/p)| \leqslant r$. Furthermore, each G_i consists of a number of copies of a graph H_i of order at most r such that a vertex v in S is adjacent to corresponding vertices in the various copies. Figure 4 shows a typical graph in $\mathscr{G}(n, 3, 5)$.

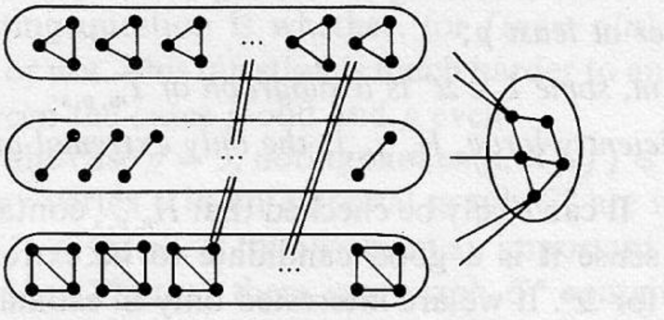

Fig. 4

Theorem 5.3 (The Very Symmetrical Extremal Graph Theorem). *Let $\mathscr{L}$ be a finite family of forbidden subgraphs with subchromatic number p. Assume that $\hat{L}_1$ is a subgraph of $P_k \times K_{p-1(k)}$, where k is the maximum order of a graph in $\mathscr{L}$ and P_k is the path of length $k - 1$. Then, for any chromatic condition $\mathscr{A}$, there exists $r = r(k)$ such that, for n sufficiently large, $\mathscr{G}(n, p, r)$ contains an extremal graph S^n of $\operatorname{EX}(n, \mathscr{L}, -\mathscr{A})$, where $-\mathscr{A}$ denotes the negation of property $\mathscr{A}$.* $\parallel$

We note that, as a consequence, under the conditions of this theorem there is an r for which $\mathscr{G}(n, p, r)$ contains an extremal graph for $\mathscr{L}$ itself.

The significance of this theorem lies in the fact that it implies many other theorems of extremal graph theory, including the icosahedron extremal theorem. As the proof of Theorem 5.3 is rather long and uses a number of technical tricks, we omit it, and prove instead the following result. Here, $H_{n,p,s}$ denotes the graph $K_{s-1} \times T_{n-s+1,p}$ (see Fig. 5). Also, $T_{m,p,s}$ is the graph obtained from $T_{m,p}$ by adding s independent edges within one partite class of maximum order.

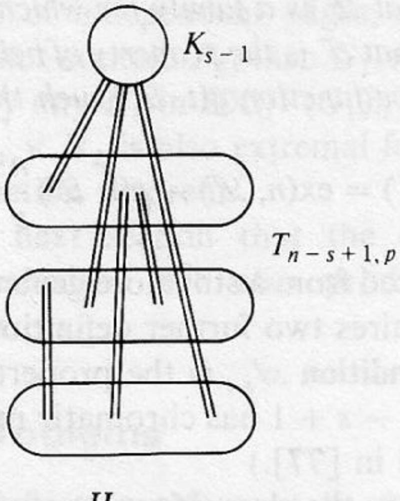

Fig. 5

Theorem 5.4 (The $H_{n,p,s}$ Theorem). *Let $\mathscr{L}$ be a family of graphs and let s be a positive integer, such that:*

(*i*) *for every $L \in \mathscr{L}$, deleting $s-1$ edges always leaves a graph with chromatic number at least p;*

(*ii*) *for some m, some $L \in \mathscr{L}$ is a subgraph of $T_{m,p,s}$.*

Then, for n sufficiently large, $H_{n,p,s}$ is the only extremal graph of $\mathscr{L}$.

Sketch of proof. It can easily be checked that $H_{n,p,s}$ contains no graph in $\mathscr{L}$, so that in this sense it is a good candidate to be extremal. Let S^n be an extremal graph for $\mathscr{L}$. If we are interested only in estimating $e(S^n)$, then by Theorem 3.5 we may assume that $S^n \in \mathscr{G}(n, p, r)$. Since some $T_{m,p,s}$ contains some forbidden L (and hence $S^n \notin T_{m,p,s}$), we conclude that the "blocks" of S^n cannot have edges. It follows that they form a Turán graph. A simple calculation yields that $e(S^n) = e(H_{n,p,s}) + O(1)$. This is nearly what is desired, but a bit weaker, so we introduce the chromatic condition $\mathscr{A} = \mathscr{A}_{s,p}$. Applying Theorem 3.5 to a graph F^n in $\mathrm{EX}(n, \mathscr{L}, \mathscr{A}) \cap \mathscr{G}(n, p, r)$ (rather than to S^n), one can easily show that $e(F^n) < e(H_{n,p,s})$ for n sufficiently large. This implies that, for such n, all extremal graphs for $\mathscr{L}$ have property $-\mathscr{A}$, and it follows that $H_{n,p,s}$ is the only extremal graph. ‖

Immediate applications of this theorem are a result of Moon [67] and the dodecahedron extremal theorem [77], to which we now turn. One can easily see that the dodecahedron graph D_{20} is a subgraph of $T_{24,2,6}$, and that if any five edges are deleted from D_{20}, then the result is still 3-chromatic. We can therefore apply the $H_{n,p,s}$-theorem with $p = 2$ and $s = 6$ to obtain the following result:

Theorem 5.5 (Dodecahedron Extremal Theorem). *For n sufficiently large, $H_{n,2,6}$ is the only extremal graph for the dodecahedron graph.* ‖

Ramsey Perturbation

A second important feature of Turán graphs is that they contain large independent sets. In this subsection, we consider what happens if we exclude this. Incidentally, we note that these problems arise from some applications (mentioned in Section 12).

Let us fix a function f, and denote by $\mathrm{ex}(n, \mathscr{L}, f)$ the maximum number of edges in a graph G^n containing no $L \in \mathscr{L}$ and at most $f(n)$ independent vertices. Our interest here will be primarily in the case in which $T_{n,p}$ is the only extremal graph, and even then specifically when $\mathscr{L} = \{K_{p+1}\}$.

Clearly, for $f(n) = n$, we are back to Turán's theorem—in fact, that is the case even for $f(n) = \lceil n/(p-1) \rceil$. On the other hand, if $f(n)$ is either a constant or tends to ∞ very slowly, then the set of graphs will, by Ramsey's theorem, be empty.

The first interesting question is thus whether $\mathrm{ex}(n, K_p, f)$ is significantly smaller than $\mathrm{ex}(n, K_p)$ when $f(n) = \lfloor n/(p-1) \rfloor - cn$. Erdős and Sós [**51**] proved that, in fact, for every $c > 0$ there exists $c' > 0$ such that $\mathrm{ex}(n, K_p, f) \leqslant \mathrm{ex}(n, K_p) - c'n^2$, so the answer is *yes*.

The next interesting question is whether, for $f(n) = o(n)$, $\mathrm{ex}(n, K_p, f)$ is very much smaller, or not. This question is much harder to answer, and there is a difference between the cases p odd and p even.

We first consider the case $p = 3$, noting that $\mathrm{ex}(n, K_3, f) \leqslant \frac{1}{2}nf(n)$. Indeed, the neighbors of any vertex v in an extremal graph G^n are independent, so $\rho(v) \leqslant f(n)$ and $e(G^n) \leqslant \frac{1}{2}nf(n)$. It follows from an important "probabilistic" construction of Erdős [**29**] that there is a graph G^n containing no K_3 for which the vertex-independence number $\alpha(G^n) \leqslant f(n)$. Furthermore, we have $e(G^n) \geqslant cnf(n)/\log n$. Therefore, in this sense, the case of K_3 is easy and not worth investigating further.

Beyond this, Erdős and V. T. Sós solved the general odd case, showing that the exclusion of both K_{2r+1} and $o(n)$ independent vertices is roughly the same as excluding K_{r+1}:

Theorem 5.6 (The Erdős–Sós Theorem). *There is a constant $c > 0$ such that, if $g(n) = c\sqrt{n}\log n$ and $g(n) \leqslant f(n) = o(n)$, then*

$$\mathrm{ex}(n, K_{r+1}) \leqslant \mathrm{ex}(n, K_{2r+1}, g) \leqslant \mathrm{ex}(n, K_{2r+1}, f) \leqslant \mathrm{ex}(n, K_{r+1}) + o(n^2)$$

$$= \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}. \quad \|$$

We note only a construction for the lower bound: take a Turán graph $T_{n,r}$, and add $c\sqrt{n}\log n$ edges in each class so that no triangle occurs nor are there $c\sqrt{n}\log n$ independent vertices, and the resulting graph contains neither K_{2r+1} nor $c\sqrt{n}\log n$ independent vertices.

The even case is deeper [**41**], and was proved originally only for K_4 (where the upper bound was established by Szemerédi [**87**], and the lower bound by Bollobás and Erdős [**11**]).

Theorem 5.7 (The Erdős–Hajnal–T. Sós–Szemerédi Theorem).

$$\mathrm{ex}(n, K_{2k}, o(n)) = \tfrac{1}{2}\left(\frac{3k-5}{3k-2}\right)n^2 + o(n^2). \parallel$$

In other recent work (see [**41**]), it has been shown that in the general case, $\mathrm{ex}(n, L, o(n))$ actually depends on the arboricity of L. The arboricity plays a role similar to that of the chromatic number in the Erdős–Simonovits theorem (Theorem 3.1); however, the analogy is not complete.

Zarankiewicz Perturbation

Given a family $\mathscr{L}$ the Zarankiewicz problem which corresponds to Turán's problem asks for the determination of $\mathrm{dex}(n, \mathscr{L}) = \max\{\rho_{\min}(G^n)\colon G^n \not\supseteq L \in \mathscr{L}\}$—that is, the maximum minimum valency of G^n. Since $\rho_{\min}(G^n) \leqslant (2/n)e(G^n)$, Theorem 3.1 implies that

$$\mathrm{dex}(n, \mathscr{L}) \leqslant \left(1 - \frac{1}{p}\right)n + o(n).$$

On the other hand, $T_{n,p}$ shows that this is sharp. This means that in solving Turán's problem, we have also solved Zarankiewicz's problem. Earlier in this section we saw that chromatic perturbation in Turán's theorem changes the maximum only negligibly—that is, by $O(n)$. The chromatic perturbation in Zarankiewicz-type problems is interesting because it changes the maximum by cn^2 (counted in edges!)—see [**3**]:

Theorem 5.8 (The Andrásfai–Erdős–T. Sós Theorem).

$$\max\{\rho_{\min}(G^n)\colon \chi(G^n) \geqslant p,\quad K_p \notin G^n\} = \left(1 - \frac{1}{p - \frac{4}{3}}\right)n + O(1). \parallel$$

For other results, see Erdős and Simonovits [**49**].

6. Degenerate Extremal Problems

An extremal problem is called **degenerate** if there is at least one bipartite prohibited graph—that is, if $p = 1$. The basic difference between degenerate and non-degenerate problems is that $\mathrm{ex}(n, \mathscr{L}) = o(n^2)$ for the former, and $\mathrm{ex}(n, \mathscr{L}) \geqslant \lfloor \frac{1}{4}n^2 \rfloor$ for the latter.

We have seen that in many cases the extremal graphs can be decomposed into the product of extremal graphs for some other degenerate extremal problems. Such was the case, for example, in the octahedron and dodecahedron theorems. This, and other things, suggest that a major breakthrough is badly needed in the area of degenerate extremal graph problems. Also, this area has perhaps the greatest connection with other branches of combinatorics, such as random graphs and finite geometries. We therefore devote the next three sections to this topic—and even the section on hypergraphs will be related to it. For further results on degenerate extremal graph problems, see Simonovits **[82]**.

One important degenerate extremal theorem is the following result of Kővári, T. Sós and Turán **[62]**. We include a sketch of the proof since it is both typical, and yet one of the simplest in this area.

Theorem 6.1. *For* $r \leqslant s$, $\operatorname{ex}(n, K_{r,s}) \leqslant \frac{1}{2}(s-1)^{1/r} n^{2-(1/r)} + O(n)$.

Sketch of proof. We count r-stars $K_{r,1}$ in a graph G^n containing no $K_{r,s}$. Every set of r vertices is the r-set of at most $s-1$ r-stars, so that the total number is at most $(s-1)\binom{n}{r}$. On the other hand, if $\rho_1, \ldots, \rho_n$ are the valencies in G^n, then the number of r-stars equals $\sum_{i=1}^{n} \binom{\rho_i}{r}$. Extending $\binom{x}{r}$ to all $x > 0$ by

$$\binom{x}{r} = \begin{cases} x(x-1)\ldots(x-r+1)/r! & \text{for } x \geqslant r-1, \\ 0, & \text{for } x < r-1, \end{cases}$$

we have a convex function. Jensen's Inequality implies that, if m is the number of edges in G^n, then

$$n\binom{2m/n}{r} \leqslant \sum \binom{\rho_i}{r} \leqslant (s-1)\binom{n}{r},$$

and the result follows by an easy calculation. ‖

A theorem which asserts that a graph G^n contains very many graphs L from a family $\mathscr{L}$ is called a *theorem on supersaturated graphs*. Such theorems are not only interesting in themseleves, but also are often useful in establishing other extremal results. At this point it is worthwhile deriving such a result for complete bipartite graphs, obtained by Erdős and Simonovits **[50]**:

Theorem 6.2. *If* G^n *is a graph with* m *edges then, for any* r *and* s, *there exists a number* $c_{r,s}$ *such that* G^n *contains at least* $\lfloor c_{r,s} m^{rs}/n^{2rs-r-s} \rfloor$ *copies of* $K_{r,s}$.

Sketch of proof. Let R be a set of r vertices of G^n, and let $\beta(R)$ denote the number of r-stars in G^n with R as r-set. As in the preceding proof, $n\binom{2m/n}{r} \leqslant \sum \binom{\rho_i}{r}$. Furthermore, the number of $K_{r,s}$'s on R is just $\binom{\beta(R)}{s}$, and each is counted $s!$ or $2(s!)$ times, depending on whether $r \neq s$ or $r = s$. If the number of $K_{r,s}$'s in G^n is N and the number of $K_{r,1}$'s is M, then, using Jensen's Inequality and the fact that $\sum_R \beta(R) = M$, we have

$$N \geqslant \frac{1}{2s!}\sum_R \binom{\beta(R)}{s} \geqslant c_1 \binom{n}{s}\binom{M/\binom{n}{r}}{s} \geqslant c_2 \frac{M^{rs}}{n^{2rs-r-s}}.$$

for some constants c_1 and c_2. ‖

We now turn to a reduction result of Erdős and Simonovits [**47**] which has several applications. Let L be a bipartite graph with partite sets X and Y, and let $L\langle t\rangle$ denote the graph obtained by completely joining one partite set of $K_{t,t}$ to X and the other to Y. The following result was obtained by Erdős and Simonovits:

Theorem 6.3 (Reduction Theorem). *If L is a bipartite graph with* $\mathrm{ex}(n, L) = O(n^{2-a})$, *then* $\mathrm{ex}(n, L\langle t\rangle) = O(n^{2-b})$, *where* $b^{-1} = a^{-1} + t$. ‖

The first application of Theorem 6.3 is to the cube problem. As we mentioned in Section 4, Turán asked for the value of $\mathrm{ex}(n, Q_3)$, where Q_3 denotes the 3-dimensional cube. We note that Q_3 is the result of deleting four independent edges (a 1-factor) from $K_{4,4}$. Let H be the graph obtained by deleting just three independent edges from $K_{4,4}$. Then $H = L\langle 1\rangle$, where L is the circuit graph C_6 (see Fig. 6). Now $\mathrm{ex}(n, C_6) = O(n^{\frac{4}{3}})$ (by a later result, Theorem 6.11), so that $\mathrm{ex}(n, H) = O(n^{\frac{8}{5}})$ by Theorem 6.3. We therefore have the following result, which we conjecture is sharp:

Corollary 6.4 (Cube Theorem). $\mathrm{ex}(n, Q_3) \leqslant O(n^{\frac{8}{5}})$. ‖

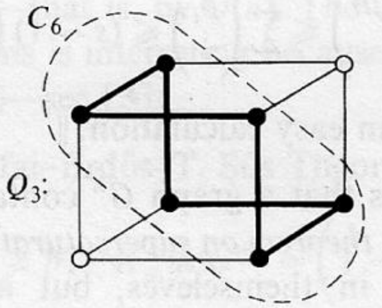

Fig. 6

A second application is to trees. By Theorem 6.7 below, $\mathrm{ex}(n, L) = O(n)$ if L is a tree, so we have the following result:

Corollary 6.5. *For any tree L,* $\mathrm{ex}(n, L\langle t-1\rangle) = O(n^{2-(1/t)})$. ‖

One special case of this result is the Kővári–T. Sós–Turán theorem, since for $L = K_2$, we have $L\langle t-1\rangle = K_{t,t}$. As a second interesting special case, Erdős and Simonovits proved an earlier conjecture of Erdős:

Corollary 6.6. $\mathrm{ex}(n, Q_3 - e) = O(n^{\frac{3}{2}})$. ‖

It is easy to show that if a graph G^n has minimum valency at least $r-1$, then it contains every tree T^r (by induction on r). A further induction on n yields the result that

$$\mathrm{ex}(n, T^r) \leqslant (r-2)n.$$

Concerning circuits, on the other hand, Erdős proved, using probabilistic methods (see Corollary 8.3), that for some constant $c_k > 0$,

$$\mathrm{ex}(n, \{C_3, \ldots, C_{2k}\}) > c_k/n^{2k/(2k-1)}.$$

We therefore have the following result:

Theorem 6.7. $\mathrm{ex}(n, \mathscr{L}) = O(n)$ *if and only if* $\mathscr{L}$ *contains a tree or forest.* ‖

Faudree and Schelp [**56**] found the exact extremal numbers for paths, and used these results to prove some Ramsey theorems. Here, we state only the following earlier and weaker result of Erdős and Gallai [**40**]:

Theorem 6.8. $\mathrm{ex}(n, P_r) \leqslant \frac{1}{2}(r-2)n$, *with equality if n is a multiple of $r-1$.* ‖

We note that the same inequality holds for the star $K_{1,r-1}$; these results led Erdős and T. Sós to conjecture that $\mathrm{ex}(n, T^r) \leqslant \frac{1}{2}(r-2)n$ for every tree T^r. The disjoint union of complete graphs K_{r-1} shows that $\mathrm{ex}(n, T^r) \geqslant \frac{1}{2}(r-2)n - O(1)$, but the upper bound was troublesome until a recent result of Ajtai, Komlós and Szemerédi:

Theorem 6.9. *There exists r_0 such that, for $r > r_0$,* $\mathrm{ex}(n, T^r) \leqslant \frac{1}{2}(r-2)n$. ‖

Certainly circuits are among the most important special graphs, and for all circuits beyond a certain length, Erdős and Gallai [**40**] established the following result:

Theorem 6.10. $\mathrm{ex}(n, \{C_r : r \geqslant t\}) = \frac{1}{2}(t-1)n + O(1)$. ‖

By excluding just one circuit length, however, we get entirely different results. For odd circuits, the $T_{n,p}$-theorem implies that $\mathrm{ex}(n, C_{2k+1}) = \lfloor \frac{1}{4}n^2 \rfloor$ for $n > n_k$. Therefore, the even circuits are the interesting ones, and the following result was established by Erdős (unpublished, see [**17**]):

Theorem 6.11 (The Even Circuits Theorem). $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$. ‖

It is worth noting that Erdős showed that excluding just one even circuit has essentially the same effect as excluding all smaller circuits as well; this is far from trivial! Erdős never published a proof of his result; however, it has been generalized by Bondy and Simonovits [**17**]:

Theorem 6.12. *Let G^n be a graph with m edges, and let t satisfy* $2 \leqslant t \leqslant m/100n$ *and* $tn^{1/t} \leqslant m/10n$. *Then G^n contains C_{2t}.* ‖

Corollary 6.13. *If G^n has at least* $100kn^{1+1/k}$ *edges, then it contains C_{2t} for every* $t \in [k, kn^{1/k}]$. ‖

Clearly this corollary implies Erdős' theorem. That theorem is known to be sharp for C_4, C_6 and C_{10}. The upper end of the interval in the corollary is also sharp, apart from the constant 100: take the disjoint union of complete graphs of order $200kn^{1/k}$. We make the following conjecture:

Conjecture 6.1. $\mathrm{ex}(n, C_{2k}) \geqslant c_k n^{1+1/k}$. *Moreover,* $\mathrm{ex}(n, C_{2k})/n^{1+1/k}$ *converges to a positive limit.*

In drawing this section to a close, we note that not many recursion theorems are known. One is the following rather trivial result:

Theorem 6.14. *If L' is obtained from a graph L by appending a tree at some vertex, then* $\mathrm{ex}(n, L') = \mathrm{ex}(n, L) + O(n)$. ‖

Another result, due to Faudree and Simonovits [57], has as a consequence another generalization of Erdős' theorem:

Theorem 6.15. *Let $C_{k,p}$ denote the graph of order* $2 + (k-1)p$ *in which two vertices are joined by p paths of length k. Then* $\mathrm{ex}(n, C_{k,p}) = O(n^{1+1/k})$. ‖

We conclude with a degenerate-graph conjecture of Erdős and Simonovits [47]:

Conjecture 6.2. *For every degenerate family $\mathscr{L}$, there is a rational number* $c \in [1, 2)$ *such that* $\mathrm{ex}(n, \mathscr{L})/n^c$ *converges to a positive limit.*

7. Finite Geometries

The method of finite geometrical constructions is very important and powerful in combinatorics. In particular, it is often the best way to obtain lower bounds. It is for this reason that we include this section, giving detailed explanations of this proof technique rather than including (say) ten more theorems.

We give three constructions: the first two show that the Kővári–T. Sós–Turán theorem (Theorem 6.1) is sharp for both $K_{2,2}$ and $K_{3,3}$, and the third shows that Erdős' theorem on even circuits is sharp for C_6.

The Erdős–Rényi–T. Sós–Brown Construction [45], [18]

From Theorem 6.1, we know that $\mathrm{ex}(n, C_4) \leqslant \frac{1}{2}\sqrt{n^3} + o(\sqrt{n^3})$, but is this result sharp? In analyzing the proof, we realize that if it is sharp (that is, if there are infinitely many orders of graphs G^n not containing C_4 and having $\approx\frac{1}{2}\sqrt{n^3}$ edges), then almost all valencies are $\approx\sqrt{n}$, and every pair of vertices must have a common neighbor (and no pair has two). This suggests that the neighborhoods $N(x)$ behave much like the lines in a projective plane, in that the following statement "almost" holds: any two vertices lie in a common set, and any two sets intersect in one vertex.

Construction 7.1. Let p be a prime, and construct a graph as follows: the vertices are the p^2 pairs (x, y) of residues (modulo p), and (x, y) is joined to (a, b) by an edge if $ax + by = 1$. (This graph may contain loops, but we simply delete them.) With $n = p^2$, the resulting graph H_n has the necessary properties to be sharp for $K_{2,2}$ in Theorem 6.1:

(a) for a given pair (a, b), there are p solutions to $ax + by = 1$, so that, even after the loops are deleted, there are at least $\frac{1}{2}p(p-1)$ edges in H_n, and hence $e(H_n) \geqslant \frac{1}{2}n^{\frac{3}{2}} - n$;

(b) if H_n had a 4-circuit with vertices (a, b), (u, v), (a', b') and (u', v'), then the two equations $ax + by = 1$ and $a'x + b'y = 1$ would have two solutions, which is impossible. Hence the construction gives the infinitely many graphs needed to show sharpness.

The Brown Construction

The geometric idea behind the above construction is to join a point of the plane to the points of its "polar" (with respect to the unit circle), and then to use the fact that two lines intersect in at most one point. In contrast, the Brown construction for the sharpness of Theorem 6.1 when $p = 3$ and $q \geqslant 3$ uses the fact that, if points of 3-space at distance 1 are joined, then the resulting infinite graph G does not contain $K_{3,3}$. This is easily seen as follows: suppose G does contain $K_{3,3}$. Then the three points of one color class cannot be collinear since no point is equidistant from three collinear points. On the other hand, only two points are equidistant from three points on a circle, and so $K_{3,3}$ cannot occur.

Brown's construction **[18]**, **[19]** is the following:

Construction 7.2. Let p be a prime of the form $4k - 1$, and construct a graph B_n whose vertices are the triples (x, y, z) of residue classes (modulo p) and whose edges join vertices (x, y, z) and (x', y', z') if

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = 1.$$

The graph B_n then contains no $K_{3,3}$ and has $\frac{1}{2}n^{\frac{5}{3}} + o(n^{\frac{5}{3}})$ edges. This last fact follows from a theorem of Lebesgue asserting that, for fixed (x', y', z'), the edge-defining equation has $p^2 - p$ solutions. The proof that B_n does not contain $K_{3,3}$ was sketched above, but here the geometric language must be translated into the language of analytic geometry. It then carries through to the given finite affine space. There is just one place where care must be taken: it is indeed true that the sphere $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ contains no three collinear points, but number theory must be used to show this since the corresponding result does not hold for $p=4k+1$.

The Benson Construction

In the preceding section, we asserted that Erdős' theorem on even circuits is sharp for C_4, C_6 and C_{10} (and is conjectured to be sharp in all cases). For C_4, the sharpness follows from Construction 7.1. For C_6, it can be deduced from the Benson construction (see **[5]** and **[83]**), which follows.

We begin with some heuristic calculations. Consider the 4-dimensional finite geometry $GF(p,4)$ over the field of order p—that is, the 5-tuples with $\mathbf{x}$ and $\mathbf{y}$ considered identical if there exists $\lambda \neq 0$ such that $\mathbf{x}=\lambda\mathbf{y}$. Let $\mathbf{A}$ be the matrix

$$\mathbf{A}=\begin{bmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{bmatrix}.$$

Clearly, $\mathbf{A}$ is regular. Furthermore the equation $\mathbf{xAx}^T=0$ defines a surface S in which each point is in $\approx p$ lines. For $\mathbf{x}$ and $\mathbf{y}$ on S, the line $\mathbf{xy}$ consists of the points $\mathbf{z}=a\mathbf{x}+(1-a)\mathbf{y}$, and lies entirely in S if both $\mathbf{yAy}^T=0$ and $\mathbf{xAy}^T=0$. The number of points on S is $\approx p^3$, so that y can be chosen in $\approx p^2$ ways. But, since $\mathbf{xy}$ has $\approx p$ points different from $\mathbf{x}$, the number of lines containing a given point $\mathbf{x}$ is $\approx p$. We observe that the number of lines is also $\approx p^3$.

Construction 7.3. Let G_n denote the bipartite graph whose sets of vertices are the sets of points and lines of the surface S described above, with adjacency in G_n corresponding to incidence in S.

Clearly, $e(G_n)\approx \frac{1}{2}pn=(\frac{1}{2}+o(1))n^{\frac{4}{3}}$. Furthermore, G_n contains no circuits of length 3, 4, 5 or 7. (For the odd cases this is because G_n is bipartite, and the existence of a 4-circuit would imply that two points of S are on two different lines.) Now suppose that G_n contains a 6-circuit $v_1w_1v_2w_2v_3w_3v_1$. Then S must contain the three lines v_1v_2, v_2v_3, and v_3v_1, and so it must contain the plane $\langle v_1v_2v_3\rangle$. But this is impossible, for if we apply a coordinate transformation $\mathbf{T}$ with v_1, v_2 and v_3 as the first three base vectors, we get the matrix

$$\mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}^T = \begin{bmatrix} 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix},$$

since $\mathbf{v}_i\mathbf{A}\mathbf{v}_j^T = 0$. But then $\mathbf{A}'$ cannot be regular, contradicting the regularity of $\mathbf{A}$. Hence G^n cannot contain C_6 either.

We note that a justification of this construction must include a careful check that any results of ordinary linear algebra which are used do indeed carry over to finite fields. (It is trivial that det $\mathbf{A}' = 0$ and that det $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}^T =$ det $\mathbf{A}$, so that det $\mathbf{A} = 0$.)

In concluding this section, we note that finite geometry constructions can also be used in hypergraph extremal problems (see [**24**], [**25**] and [**76**]).

8. Random Graphs

The theory of random graphs is an interesting, important, and rapidly-developing subject. Applications of probabilistic methods have proved effective not only in graph theory, but in coding theory, analysis, and other areas of mathematics. We shall not go into details of the theory of random graphs here, but refer the reader to Chapter 7 of this book. Rather, we restrict ourselves to simple applications of random graph methods to extremal theory, concluding with a brief description of what we call *pseudo-random graphs*.

The Erdős–Rényi Threshold Theorem

The random graph distribution used here is that in which, for each n, an integer E_n between 0 and $\binom{n}{2}$ is fixed, and each graph G^n with E_n edges has the same probability p_n, given by

$$p_n = \binom{\binom{n}{2}}{E_n}^{-1}.$$

Theorem 8.1 (The Erdős–Rényi Threshold Theorem). *Let $\mathscr{L} = \{L_1, \ldots, L_t\}$ be a family of graphs, and let*

$$c = c(\mathscr{L}) = \min_j \min_{H \subseteq L_j} \frac{|V(H)|}{e(H)}.$$

Further, let $\{E_n\}$ *be a sequence of integers with* $0 \leqslant E_n \leqslant \binom{n}{2}$ *and let* G^n *be a graph of order n with* E_n *edges. Then the probability that* G^n *contains a member of* $\mathscr{L}$ *tends to*

(a) 0, *if* $E_n = o(n^{2-c})$;

(b) 1, *if* $E_n/n^{2-c} \to \infty$. ‖

(In extremal graph theory, only part (a) of this theorem is used.)

For an example, we consider $L = K_{r,s}$, in which case $c = r^{-1} + s^{-1}$. Then, for c_0 sufficiently small, the probability that a graph G^n with $c_0 n^{2-c}$ edges does not contain $K_{r,s}$ is positive. It follows that

$$\mathrm{ex}(n, K_{r,s}) \geqslant c_0 n^{2-(1/r)-(1/s)}.$$

Comparing this with the Kővári–T. Sós–Turán theorem (Theorem 6.1), we see that that theorem is in a sense best possible if r is fixed while $s \to \infty$.

The next result, proved implicitly by Erdős, is sharper. It uses a method which might be called the *expected number method*, or the *altering a random graph method*.

Theorem 8.2. *Let* $\mathscr{L} = \{L_1, \ldots, L_t\}$ *be a family of graphs, and let*

$$(*) \qquad c = \min_j \min_{H \subseteq L_j} \frac{|V(H)| - 2}{e(H) - 1}.$$

Then, for some $c_0 > 0$, $\mathrm{ex}(n, \mathscr{L}) \geqslant c_0 n^{2-c}$.

Proof. Consider G^n as a labeled graph in which each edge occurs independently with the same probability p, For each L_j, choose a subgraph H_j which attains the inner minimum in $(*)$, and let

h_j denote the order of H_j,

e_j denote the number of edges in H_j,

α_j denote the number of copies of H_j in K_{h_j},

and

β_j denote the expected number of copies of H_j in G^n.

Note that, if K_{h_j} contains α_j copies of H_j, then K_n contains $\alpha_j\binom{n}{h_j}$ copies of H_j. For each copy H of H_j, define a random variable $k_H = k_H(G^n) = 1$ if $H \subseteq G^n$, and 0 otherwise. Since the number of copies of H_j in G^n is just $\sum_{H \subseteq K_n} k_H$, it follows that

$$\beta_j = \alpha_j \binom{n}{h_j} \sum_{H \subseteq K_n} E(k_H) = \alpha_j \binom{n}{h_j} p^{e_j}.$$

Summing over j and taking $p = c_1 n^{-c}$, we get

$$\sum_j \beta_j \leqslant t \max \alpha_j \binom{n}{h_j} p^{e_j} \leqslant t \max c_1 n^{h_j - ce_j} = tc_1 n^{2-c}.$$

Now let $a(G^n) = e(G^n) - \sum_j \beta_j$. Then, for c_1 sufficiently small, the expected value is

$$E(a(G^n)) > \tfrac{1}{2}\binom{n}{2} p > \tfrac{1}{5} c_1 n^{2-c}.$$

It follows that there exists a G^n with $a(G^n) > \frac{1}{5} c_1 n^{2-c}$. Delete an edge from each H_j in this G^n. Then the resulting graph contains no L_j, and has at least $\frac{1}{5}\binom{n}{2} p \geqslant \frac{1}{11} c_1 n^{2-c}$ edges, completing the proof. ‖

The following corollary is the result of applying the theorem to some families of circuits:

Corollary 8.3. *For some constant* c_m, $\mathrm{ex}(n, \{C_3, \ldots, C_m\}) \geqslant c_m n^{1+(m-1)^{-1}}$. ‖

Erdős' even circuits theorem asserts that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$, and this upper bound is probably sharp. The random method (that is, Theorem 8.2) yields a lower bound of $c'_m n^{1+(2k-1)^{-1}}$, a weaker result. We believe that is unlikely that Theorem 8.2 ever yields a sharp bound for a finite family.

We note that Corollary 8.3 was used in Section 6 to prove that $\mathrm{ex}(n, \mathscr{L}) = O(n)$ if and only if $\mathscr{L}$ contains a tree or forest.

The Lovász Sieve

Even though the next result has no applications in this chapter, we feel that its inclusion here is worthwhile because of its use in the theory of random graphs:

Theorem 8.4 (Lovász' Sieve Theorem). *Let G^n be a graph with maximum valency ρ. For $i = 1, \ldots, n$, let A_i be an event in a probability space A such that A_i is independent of $\{A_j : v_i v_j \text{ is in } G^n\}$. If there is $p \leqslant \frac{1}{4}\rho^{-1}$ such that $P(A_i) \leqslant p$, then*

$$P(\bar{A}_1 \cap \ldots \cap \bar{A}_n) > 0. \ ‖$$

In general, to ensure the conclusion of this theorem, one needs that $\sum P(A_i) < 1$. Of course, if all the A_i are independent, then no condition is needed. Lovász' theorem handles the in-between case: the events are not independent, but the "dependence graph" has only a few edges. It is interesting to note that the condition involves neither the number of vertices nor the number of edges, but only the maximum valency.

Pseudo-random Graphs

There are important instances in the theory of extremal and random graphs, in which graphs that are not really random at all can be regarded as being "approximately random". In our view, the significance of these "pseudo-random" graphs has been increasing, and we give two examples here. The first is due to Szemerédi [**89**], and says that if the vertex-set of a graph is partitioned in the right way, then the edges joining the sets behave in some random fashion. To facilitate the statement of the result we define the **density of edges** between two disjoint sets of vertices X and Y by

$$\delta(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

Theorem 8.5 (The Szemerédi Uniformization Lemma). *For every $\varepsilon > 0$, there exists k_ε such that, for every G^n, the vertex-sets can be partitioned into sets $V_0, V_1, \ldots, V_k$ (for some $k < k_\varepsilon$) such that each $|V_i| < \varepsilon n$, each $|V_i| = m$ for $i \geqslant 1$, and for all but at most $\varepsilon\binom{k}{2}$ pairs (i, j) with $i \neq j$, for every $X \subseteq V_i$ and $Y \subseteq V_j$ satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|\delta(X, Y) - \delta(V_i, V_j)| < \varepsilon. \,\|$$

As we have already indicated, the basic meaning of this theorem is that G^n behaves as if the edges between V_i and V_j were taken independently, at random, with probability $p_{ij} = \delta(V_i, V_j)$. Applications of this lemma (or an earlier version of it) can be found in many proofs of extremal theorems, examples being Theorems 5.7 and 9.2.

The last theorem to be mentioned in this section is due to Ajtai, Komlós and Szemerédi [**1**], [**2**]:

Theorem 8.6 (The Triangle-Perturbation Lemma). *There exist positive constants c_1 and c_2 such that if G^n contains no K_3 and its average valency is $t > c_1$, then it contains $c_2 n(\log t)/t$ independent vertices.* ||

This result is an improvement of the assertion that, if the average valency of G^n is t, then G^n contains $\lfloor n/(t+1) \rfloor$ independent vertices. The lower bound $n/(t+1)$ is achieved by the complement of $T_{n,d}$, if $n = d(t+1)$. However, this graph consists of complete components. It may (and does) happen that we need a lower bound on the independence number of G^n when $K_3 \notin G^n$. In that case it is advisable to use Theorem 8.6. (Recall that for random graphs the independence number is much larger than n/t; it is around $n(\log t)/t$.)

9. Hypergraph Extremal Problems

A hypergraph is ***r*-uniform** if every edge has r vertices. Clearly, extremal problems extend to r-uniform hypergraphs (with no multiple edges), and the corresponding definitions of $\mathrm{EX}(n, \mathscr{L})$ and $\mathrm{ex}(n, \mathscr{L})$ are obvious.

The oldest hypergraph problems are due to Turán **[90]**, **[91]**. All hypergraphs are assumed to be uniform here.

Turán's Problem. For given r, p and n, how many edges can an r-uniform hypergraph H^n have without containing the complete hypergraph of order p?

This problem seems to be extremely difficult. For the sake of brevity, we consider here only the simplest form of the corresponding conjecture, with $r = 3$ and $p = 4$.

Let $T^{(3)}_{n,3}$ be the 3-uniform hypergraph whose n vertices are divided into three sets C_1, C_2 and C_3(as nearly equal in size as possible), where a triple $\{x, y, z\}$ is an edge if no two are in the same set or if two belong to C_i and the third belongs to C_{i+1} (subscripts taken modulo 3)—see Fig. 7.

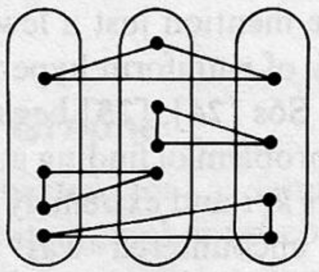

Fig. 7

Turán's conjecture is that $T^{(3)}_{n,3}$ has the maximum number of edges among the 3-uniform hypergraphs of order n not containing the complete hypergraph of order 4.

Even this case remains unsolved. If the conjecture is true, then the extremal graph is not unique. In addition to there being some trivial variations of Turán's construction, a very nice construction was found by Brown **[19]**.† Let $n = 3k$ and let C_1, C_2 and C_3 be sets each of k vertices. Further, let s vertices of each C_i be called *A-vertices*. The edges of our hypergraph $F^{(3)}_n$ on $C_1 \cup C_2 \cup C_3$ are those triples $\{x, y, z\}$ such that

(*i*) no two are in the same class C_i, and the number of A-vertices is not 1;

or (*ii*) $x, y \in C_i$, $z \in C_{i+1}$, and at least one is an A-vertex;

or (*iii*) $x, y \in C_i$, $z \in C_{i-1}$, and x and z are A-vertices.

One can easily check that $F^{(3)}_n$ contains no complete 4-hypergraph, and has exactly as many edges as $T^{(3)}_{n,3}$. (In fact, if $s = 0$, they are isomorphic.)

We now turn to the hypergraph version of the extremal problem of $K_{p,q}$ for

† Added in proof. Kostochka has found a generalization of this construction—see *Combinatorica* **2**/2 (1982), 187–192.

simple graphs (that problem being known as Zarankiewicz' problem and whose solution is the Kővári–T. Sós–Turán theorem). Let $K^{(r)}_{n_1,\ldots,n_r}$ denote the complete r-partite r-uniform hypergraph (with n_i vertices in the ith class and having those edges with one vertex from each class). The problem is to determine how many r-tuples H^n can have without containing $K^{(r)}_{r(m)}$. (As usual, $r(m)$ means that each of the r partite sets has m vertices.) Bounds for this number were found by Erdős [**34**]:

Theorem 9.1. *There exist positive numbers c and A such that*

$$n^{r-cm^{1-r}} \leqslant \mathrm{ex}(n, K^{(r)}_{r(m)}) \leqslant A\, n^{r-m^{1-r}}. \;\|$$

The proof of the upper bound of this theorem is not much more difficult than that of the Kővári–T. Sós–Turán theorem: as a matter of fact, it can be reduced to it. (The lower bound is obtained by the method of random hypergraphs.) We know much less of extremal results on hypergraphs than on ordinary graphs; here we mention just a few.

Let $\mathscr{L}_{k,t}$ denote the family of r-uniform hypergraphs with k vertices and t edges. Brown, Erdős and T. Sós [**24**], [**25**] began investigating the function $f(n, k, t) = \mathrm{ex}(n, \mathscr{L}_{k,t})$. The problem of finding good lower and upper bounds is fairly simple for some pairs k, t and extremely difficult for others. The first real difficulty which they encountered was the case $r = t = 3$, $k = 6$. Although they could not settle this problem, Ruzsa and Szemerédi [**71**] later found an astonishing result.

Let $r_k(n)$ denote the maximum number of integers in $[1, n]$ containing no arithmetic progression of length k. Szemerédi's famous theorem [**88**] asserts that $r_k(n) = o(n)$. Further, it is known (see [**4**]) that

$$n^{1-c(\log n)^{-1}} \leqslant r_3(n) \leqslant c' \frac{n}{\log\log n}.$$

Theorem 9.2 (The Ruzsa–Szemerédi Theorem). *There exists $c > 0$ such that*

$$cn\, r_3(n) < f(n, 6, 3) = o(n^2). \;\|$$

One reason, why this result is so surprising is that it implies the non-existence of an α such that

$$c_1 n^\alpha < \mathrm{ex}(n, \mathscr{L}) < c_2 n^\alpha .$$

Although we cannot prove it, we are convinced that for ordinary graphs the situation is completely different (see Conjecture 6.2).

The Problem of the Jumping Constants. Theorem 9.1 has the following consequence:

Corollary 9.3. *Let $\varepsilon > 0$, and let $\{S^n\}$ be a sequence of r-uniform hyper-*

graphs such that $e(S^n) > \varepsilon\binom{n}{r}$. *Then* S^n *contains a subhypergraph* H^m *with* $e(H^m) > (r!/r^r)\binom{m}{r}$ *and* $m \to \infty$ *as* $n \to \infty$. ‖

This means that if the edge-density of S^n is positive then, for some appropriately chosen subgraphs, this density must jump up (independently of ε) to $r!/r^r$. The problem of the jumping constants can be formulated in its most special form (with $r = 3$) as follows:

Problem 9.2. *Let* $\varepsilon > 0$ *and let* $\{S^n\}$ *be a sequence of* 3-*uniform hypergraphs for which* $e(S^n) > (\frac{1}{27} + \varepsilon)n^3$. *Is it true that there exist a constant* $c > 0$ *(independent of* ε *and* $\{S^n\}$*) and a sequence of subgraphs* $H^m \subseteq S^n$ *(where* $m \to \infty$ *as* $n \to \infty$*), for which* $e(H^m) > (\frac{1}{27} + c)m^3$?

The general problem is to characterize those constants c for which there exists an $f(c) > c$ such that, whenever the edge-density of S^n is larger than c, then there is a subsequence $H^m \subseteq S^n$ (as $m \to \infty$) for which the edge-density of H^m is at least $f(c)$.

10. Supersaturated Graphs

A graph G^n is **supersaturated** for a family $\mathscr{L}$ if $e(G^n) > \mathrm{ex}(n, \mathscr{L})$. Consequently, a supersaturated graph must contain a prohibited subgraph. What is surprising is that it frequently contains not only one but many of them.

The general problem can be stated as follows:

Problem 10.1. *For given* $\mathscr{L}$, n, *and* k, *how many copies of graphs in* $\mathscr{L}$ *must* G^n *contain if* $e(G^n) = \mathrm{ex}(n, \mathscr{L}) + k$?

The first result in this area is for K_3. It was established for $k = 1$ by Rademacher in 1941, and subsequently extended to other values of k by Erdős **[32]**. Later, Erdős **[37]** further generalized it to arbitrary complete graphs.

Theorem 10.1 (The Rademacher–Erdős Theorem). *There exists* $c > 0$ *such that, if* $1 < k < cn$ *and* $e(G^n) = \lfloor \frac{1}{4}n^2 \rfloor + k$, *then* G^n *contains at least* $k\lfloor \frac{1}{2}n \rfloor$ *copies of* K_3. ‖

The literature on supersaturated graphs is quite extensive, but for the sake of brevity, we restrict ourselves to just a few results. The next two theorems may be found in **[66]**:

Theorem 10.2 (The Bollobás–Lovász–Simonovits Theorem). *If* t *is defined by* $e(G^n) = \frac{1}{2}(1 - t^{-1})n^2$, *then* G^n *contains at least* $\binom{t}{p}(n/t)^p$ *copies of* K_p. ‖

The complete t-partite graph $K_{t(m)}$ shows that this result is sharp. The next result describes its stability:

Theorem 10.3 (The Lovász–Simonovits Stability Theorem). *For every $c > 0$, there exist $\delta > 0$ and $c' > 0$ such that, if t is defined by $e(G^n) = \frac{1}{2}(1 - t^{-1})n^2$, if $d = \lfloor t \rfloor$, and if $e(G^n) = e(T_{n,d}) + k$ for some $k \in [0, \delta n^2]$, then either G^n contains at least*

$$\binom{t}{p}\left(\frac{n}{p}\right)^p + ckn^{p-2}$$

copies of K_p, or it can be obtained from $T_{n,d}$ by changing at most $c'k$ edges. ‖

Heuristically (with $d = p - 1$), this result implies that if $e(G^n) = \text{ex}(n, K_p) + k$, then either the structure of G^n is very regular and similar to $T_{n,p-1}$, or else there is anarchy in the structure and then it has many K_ps.

For further results on supersaturation for complete graphs, we refer the reader to [**65**] and [**66**], in which Lovász and Simonovits generalize Erdős' extension of Theorem 10.1, and to [**6**] and [**7**], in which Bollobás gives various results, including an elegrant proof of the Nordhaus–Stewart conjecture using symmetrization.

For supersaturated hypergraphs, we have the following result of Erdős and Simonovits [**50**]. Although it is not very deep, it is quite interesting and useful; its proof depends on an elementary lemma of Erdős [**38**].

Theorem 10.4. *Let L^k and G^n be r-uniform hypergraphs. For every $c > 0$, there exists $c' > 0$ such that if $e(G^n) \geqslant \text{ex}(n, L^k) + cn^r$, then G^n contains at least $c'n^k$ copies of L^k.* ‖

This theorem is sharp, in the sense that G^n cannot have more than $O(n^k)$ copies of L^k.

There is another type of result on supersaturated graphs; for this type, what is guaranteed is a richer structure instead of many copies of prohibited graphs. One example is a generalization of Turán's theorem due to Dirac [**27**]. Another is Theorem 6.15, as an extension of Theorem 6.11. Our final result in this section is a hypergraph theorem of this type, which was proved for complete hypergraphs by Erdős [**40**].

We first make a definition: if L is an r-uniform hypergraph with vertex-set $\{v_1, \ldots, v_p\}$, let $L(t)$ denote the r-uniform hypergraph with vertex-set $\{w_{ij}: 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant t\}$, where $(w_{i_1 j_1}, \ldots, w_{i_r j_r})$ is an edge if and only if $(v_{i_1}, \ldots, v_{i_r})$ is an edge of L. We then have the following result:

Theorem 10.5. $\text{ex}(n, L(t)) = \text{ex}(n, L) + o(n^r)$. ‖

11. Digraph Extremal Problems

The general problem for digraphs is the same as for graphs: *given a family $\mathscr{L}$ of prohibited digraphs, what is the maximum number of arcs which a digraph of* *order n can have without containing a digraph in* $\mathscr{L}$? As before, this maximum will be denoted by $\mathrm{ex}(n, \mathscr{L})$, and the family of digraphs attaining this maximum (that is, the family of **extremal digraphs**) will be denoted by $\mathrm{EX}(n, \mathscr{L})$.

The subject of extremal digraphs was introduced by Brown and Harary [**26**], who concentrated on exact results for small prohibited digraphs. Other exact results were obtained by Häggkvist and Thomassen [**58**]. In this brief section we shall consider only asymptotic results from the papers by Brown, Erdős, and Simonovits [**20**], [**21**], [**22**], [**23**]. (In this survey we have considered only simple (undirected) graphs and, for the sake of simplicity, we shall do the same for digraphs. This is in spite of the fact that in some cases the general result may be more interesting. For such results, in both the undirected and directed cases, see the previously-mentioned papers.)

A sequence $\{D^n\}$ of digraphs is said to be **asymptotically extremal** for a family $\mathscr{L}$ of digraphs if no D^n contains an $L \in \mathscr{L}$, and if $e(D^n) = \mathrm{ex}(n, \mathscr{L}) + o(n^2)$.

Our first, and most important, theorem states that there are always asymptotically extremal digraphs with a very elementary structure. In a sense, it is a generalization of the Erdős–Stone–Simonovits theorem (Theorem 3.1). Before stating it, we need a generalization of the Turán graphs $T_{n,p}$.

Let $\mathbf{A} = (a_{ij})$ be an $r \times r$ matrix in which each non-zero diagonal entry is 1, and each non-zero off-diagonal entry is 2. (The reason for the 2s will be indicated later.) For a vector $\mathbf{x} = (x_1, \ldots, x_r)$ with integer entries, we define the **matrix digraph** $\mathbf{A}(\mathbf{x})$ to have $n = \Sigma x_i$ vertices divided into r classes C_i, with $|C_i| = x_i$. For $i \neq j$, there are $\frac{1}{2}a_{ij}$ arcs from each vertex in C_i to each vertex in C_j. For $i = j$, if $a_{ii} = 1$, C_i is made into a transitive tournament, whereas if $a_{ii} = 0$, C_i has no internal arcs. An example is indicated in Fig. 8.

Although this definition is quite involved, these digraphs are considered elementary since they are quite natural generalizations of complete p-partite graphs. The Turán graph $T_{n,p}$ was chosen to have the most edges for a given order. We make a similar definition here.

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 0 & 0 \end{bmatrix}$$

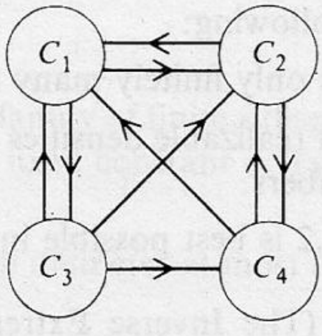

Fig. 8

For given **A** and n, a matrix digraph $\mathbf{A}(\mathbf{x})$ of order n is called **optimal** if it has the greatest possible number of arcs. In general, optimal matrix digraphs are not unique, but we let $\mathbf{A}^{(n)}$ denote a specific one. (Later it will be observed that only $O(1)$ of them satisfy a "density" property.)

Theorem 11.1 (The Fundamental Theorem for Extremal Digraphs). *For each family $\mathscr{L}$ of prohibited digraphs there is a matrix* **A** *such that $\{\mathbf{A}^{(n)}\}$ is an asymptotic extremal sequence for $\mathscr{L}$.* ‖

Now, for any $r \times r$ matrix **A** with non-negative entries, we consider the quadratic form $\mathbf{uAu}^T$ and define the **density** of **A** to be

$$(1) \qquad g(\mathbf{A}) = \max\{\mathbf{uAu}^T \colon u_i \geqslant 0 \ \text{ and } \sum_i u_i = 1\}.$$

The matrix **A** is said to be **dense** if, for every principal submatrix **B**, $g(\mathbf{B}) < g(\mathbf{A})$.

The advantage of using quadratic forms and the notion of density is this: if a_{ii} loops are added to each vertex in C_i, then $\mathbf{A}^{(n)}$ gives a digraph D^n with $\mathbf{xAx}^T$ arcs. Hence

$$(2) \qquad \mathbf{xAx}^T - 2n \leqslant 2e(\mathbf{A}(\mathbf{x})^n) \leqslant \mathbf{xAx}^T.$$

(This, incidentally, is what motivates the 2s in the matrix **A**.) Since the maximum in (1) can always be obtained by a vector with rational entries, it follows at once that, for n sufficiently large,

$$(3) \qquad \tfrac{1}{2}g(\mathbf{A})n^2 - O(n) \leqslant e(\mathbf{A}^{(n)}) \leqslant \tfrac{1}{2}g(\mathbf{A})n^2.$$

Now, a principal submatrix **B** of **A** of minimum order satisfying $g(\mathbf{B}) = g(\mathbf{A})$ is clearly dense, so that, by (3), $e(\mathbf{A}^{(n)}) - e(\mathbf{B}^{(n)}) = O(n)$, and **A** can be replaced by **B** in Theorem 11.1. Since **A** has greater order than **B**, $\mathbf{B}^{(n)}$ is a graph with simpler structure.

Corollary 11.2. *For every family of digraphs $\mathscr{L}$ there exists a dense matrix* **A** *such that $\{\mathbf{A}^{(n)}\}$ is a sequence of asymptotically extremal digraphs for $\mathscr{L}$.* ‖

Some interesting facts about our matrices **A** are established in [23], including the following:

(*i*) there are only finitely many (if any) matrices **A** of given density;

(*ii*) the set of realizable densities is well ordered under the usual ordering of the real numbers.

Corollary 11.2 is best possible in the following sense (see [22]):

Theorem 11.3. (The Inverse Extremal Theorem for Digraphs). *For any dense matrix* **A**, *there is a finite family $\mathscr{L}$ of digraphs for which $\{\mathbf{A}^{(n)}\}$ is extremal,* *and hence* $\{\mathbf{A}^{(n)}\}$ *is asymptotically extremal. Furthermore, there exists* n_0 *such that, for* $n > n_0$, *there are no other extremal digraphs for* $\mathscr{L}$, *in the sense that*

(*i*) *if* $\mathbf{B}$ *is a dense matrix for which* $\{\mathbf{B}^{(n)}\}$ *is an asymptotically extremal sequence, then* $\mathbf{B} = \mathbf{PAP}^{-1}$ *for some permutation matrix* $\mathbf{P}$;

(*ii*) *if* $\{D^n\}$ *is a sequence of digraphs which is asymptotically extremal for* $\mathscr{L}$, *then* D^n *can be obtained from* $\mathbf{A}^{(n)}$ *by changing* $o(n^2)$ *arcs.* ||

Prohibited families of digraphs also have a "compactness property" (see **[23]**):

Theorem 11.4. *Every infinite family* $\mathscr{L}$ *of prohibited digraphs contains a finite subfamily* $\mathscr{M}$ *such that*

$$\mathrm{ex}(n, \mathscr{L}) = \mathrm{ex}(n, \mathscr{M}) + o(n^2).$$

Furthermore, a sequence $\{\mathbf{A}^{(n)}\}$ *is asymptotically extremal for* $\mathscr{L}$ *if and only if it is asymptotically extremal for* $\mathscr{M}$. ||

In conclusion, we observe that there is an algorithm (see **[23]**) which finds, for an arbitrary finite family $\mathscr{L}$, all the dense matrices $\mathbf{A}$ for which $\mathbf{A}^{(n)}$ is asymptotically extremal. Again, we remind the reader that the results given here hold, or are conjectured to hold, in a much more general multi-digraph form.

12. Applications of Turán's Theorem

In a certain sense, Turán's theorem and its generalizations are closely related to the pigeonhole principle, and this is why they can be used in so many areas of graph theory and other branches of mathematics. We shall not consider here any applications to combinatorics (which are perhaps not so surprising), but only applications to geometry, potential theory and probability theory. Work in the first two areas was initiated by Turán **[92]**, and continued by Erdős, Meir, Sós and Turán **[43]** and others. The application to probability theory is due to Katona **[60]**. No proofs will be given.

Distance Distributions

Let M be a metric space, and let $\mathscr{F}$ be a family of finite subsets of M each of which has diameter at most c, for some fixed constant c. Typical examples are:

(*i*) the family of all finite subsets with diameter at most c of a closed set $D \subseteq M$;

(*ii*) the family of all subsets of a bounded set $D \subseteq M$.

We are interested in the distribution of distances $d(P_i, P_j)$ for an n-element set $\{P_1, \ldots, P_n\}$. In characterizing these distributions, we find that the "packing constants" defined below are very useful. The kth packing constant is

$$d_k = \sup_{\{P_1, \ldots, P_k\} \in \mathscr{F}} \min_{i \neq j} d(P_i, P_j).$$

Clearly, $d_{i+1} \leqslant d_i$. If M is a bounded subset of the m-dimensional Euclidean space $\mathbf{R}^m$, then $d_k \to 0$.

Observe that, by the definition of d_k, if $\{P_1, \ldots, P_k\} \in \mathscr{F}$ and if G^n is the graph defined on these vertices by joining P_i and P_j by an edge if and only if $d(P_i, P_j) > d_{k+1}$, then G^n contains no complete subgraph. Applying Turán's theorem to this G^n we obtain a slightly simplified version of Turán's distance-distribution theorem [**92**]:

Theorem 12.1. *For any $\{P_1, \ldots, P_n\} \in \mathscr{F}$, the number of distances $d(P_i, P_j) \leqslant d_{k+1}$ is at least $(n/2k)(n-k)$.* ‖

Under some quite natural additional conditions, Theorem 12.1 becomes sharp.

Potential Theory

Let $f(r)$ be a decreasing function, and let $r_{x,y}$ be the distance between x and y in $\mathbf{R}^m$. If D is a closed subset of $\mathbf{R}^m$ and μ is a mass distribution (or measure) on D, then the **generalized potential** is defined by

$$I(f) = \iint_{D \times D} f(r_{x,y})\, \mathrm{d}\mu_x\, \mathrm{d}\mu_y.$$

(In classical physics, $f(r) = -\log r$ for $m = 2$, and $f(r) = r^{2-m}$ for $m = 3, 4, \ldots$)

Theorem 12.1 immediately implies the following result (see [**92**]):

Theorem 12.2 (Turán's Potential Theorem). *If $D \subseteq \mathbf{R}^m$ is compact, if d_k is its kth packing constant, and if $f(r) \geqslant c_0$ for $r \in (0, d_2)$, then*

$$I(f) \geqslant \mu(D) \sum_{k \geqslant 2} \frac{f(d_k)}{k^2 - k}. \ \|$$

Probability Theory

We conclude with an application, due to Katona [**60**], of Turán's theorem to probability theory:

Theorem 12.3 (Katona's Inequality). *If ξ and η are vector-valued independent random variables with the same distribution, then*

$$P(|\xi + \eta| > x) \geqslant \tfrac{1}{2} P(|\xi| > x)^2. \ \|$$

References

1. M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, *J. Combinatorial Theory* (*A*) **29** (1980), 354–360; *MR***82a**:05064.
2. M. Ajtai, J. Komlós and E. Szemerédi, A dense infinite Sidon sequence, *European J. Combinatorics* **2** (1981), 1–11.
3. B. Andrásfai, P. Erdős and V. T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.* **8** (1974), 205–218; *MR***49**#4831.
4. F. Behrend, On sets of integers which contain no three terms in arithmetic progression, *Proc. Nat. Acad. Sci. USA* **32** (1956), 331–332; *MR***8**, 317.
5. C. T. Benson, Minimal regular graphs of girths eight and twelve, *Canad. J. Math.* **18** (1966), 1091–1094; *MR***33**#5507.
6. B. Bollobás, Relations between sets of complete subgraphs, *Proc. Fifth British Combinatorial Conference* (ed. C. St.J. A. Nash-Williams and J. Sheehan), Congressus Numerantium **XV**, Utilitas Math., Winnipeg, 1976, pp. 79–84; *MR***53**#195.
7. B. Bollobás, On complete subgraphs of different orders, *Math. Proc. Cambridge Phil. Soc.* **79** (1976), 19–24; *MR***54**#12573.
8. B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978; *MR***80a**:05120.
9. B. Bollobás, *Graph Theory. An Introductory Course*, Springer-Verlag, Heidelberg–New York, 1979; *MR***80j**:05053.
10. B. Bollobás and P. Erdős, On the structure of edge graphs, *Bull. London Math. Soc.* **5** (1973), 317–321; *MR***49**#129.
11. B. Bollobás and P. Erdős, On a Ramsey-Turán type problem, *J. Combinatorial Theory* (*B*) **21** (1976), 166–168; *MR***54**#12572.
12. B. Bollobás, P. Erdős and M. Simonovits, On the structure of edge graphs II, *J. London Math. Soc.* (2) **12** (1976), 219–224; *MR***58**#5358.
13. B. Bollobás, P. Erdős, M. Simonovits and E. Szemerédi, Extremal graphs without large forbidden subgraphs, *Advances in Graph Theory* (ed. B. Bollobás), *Ann. Discrete Math.* **3** (1978), 29–41; *MR***80a**:05119.
14. B. Bollobás and A. Thomason, Large dense neighbourhoods in Turán's theorem, *J. Combinatorial Theory* (*B*) **31** (1981), 111–114; *MR***82m**:05059.
15. J. A. Bondy, *J. Combinatorial Theory* (*B*) to appear.
16. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York, and MacMillan, London, 1976; *MR***54**#117.
17. J. A. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combinatorial Theory* (*B*), **16** (1974), 97–105; *MR***49**#4851.
18. W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–285; *MR***34**#81.
19. W. G. Brown, On an open problem of P. Turán concerning 3-graphs, *Studies in Pure Math.*, to appear.
20. W. G. Brown, P. Erdős and M. Simonovits, Extremal problems for directed graphs, *J. Combinatorial Theory* (*B*) **15** (1973), 77–93; *MR***52**#7952.
21. W. G. Brown, P. Erdős and M. Simonovits, On multigraph extremal problems, *Problèmes Combinatoires et Théorie des Graphes* (ed. J.-C. Bermond *et al.*), CRNS, Paris, 1978, pp. 63–66; *MR***80e**:05005.
22. W. G. Brown, P. Erdős and M. Simonovits, Inverse extremal digraph problems, *Proc. Colloq. Math. Soc. János Bolyai*, to appear.
23. W. G. Brown, P. Erdős and M. Simonovits, Algorithmic solution to extremal digraph problems, to appear.
24. W. G. Brown, P. Erdős and V. T. Sós, On the existence of triangulated spheres in 3-graphs, and related problems. *Period. Math. Hungar.* **3** (1973), 221–228; *MR***48**#2003.
25. W. G. Brown, P. Erdős and V. T. Sós, Some extremal problems on *r*-graphs, *New Directions in the Theory of Graphs* (ed. F. Harary), Academic Press, New York, 1973, pp. 53–63; *MR***50**#4376.
26. W. G. Brown and F. Harary, Extremal digraphs, *Combinatorial Theory and its Applications*, *I* (ed. P. Erdős *et al.*), North-Holland, Amsterdam, 1970, pp. 135–198; *MR***45**#8576.

27. G. A. Dirac, Extensions of Turán's theorem on graphs, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 417–422; *MR***28**#605.
28. P. Erdős, On sequences of integers no one of which divides the product of two others, and some related problems, *Mitt. Forschungsinst. Math. u. Mech. Tomsk* **2** (1938), 74–82.
29. P. Erdős, Graph theory and probability II, *Canad. J. Math.* **13** (1961), 346–352.
30. P. Erdős, On a theorem of Rademacher-Turán, *Illinois J. Math.* **6** (1962), 122–127; *MR***25**#1111.
31. P. Erdős, Über ein Extremalproblem in der Graphentheorie, *Arch. Math.* **13** (1962), 222–227; *MR***25**#2974.
32. P. Erdős, On the structure of linear graphs, *Israel J. Math.* **1** (1963), 156–160; *MR***28**#4533.
33. P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183–190; *MR***32**#1134.
34. P. Erdős, On some extremal problems in graph theory, *Israel J. Math.* **3** (1965), 113–116; *MR***32**#7443.
35. P. Erdős, Some recent results on extremal problems in graph theory, *Theory of Graphs* (ed. P. Rosenstiehl), Gordon and Breach, New York, and Dunod, Paris, 1967, pp. 117–123; *MR***37**#2634.
36. P. Erdős, On some new inequalities concerning extremal properties of graphs, *Theory of Graphs* (ed. P. Erdős and G. Katona), Academic Press, New York, 1968, pp. 77–81; *MR***38**#1026.
37. P. Erdős, On the number of complete subgraphs and circuits contained in graphs, *Časopis Pešt. Mat.* **94** (1969), 290–296; *MR***40**#5474.
38. P. Erdős, On some extremal problems on r-graphs, *Discrete Math.* **1** (1971), 1–6; *MR***45**#6656.
39. P. Erdős, *The Art of Counting* (ed. J. Spencer), The MIT Press, Cambridge, Mass., 1973; *MR***58**#27144.
40. P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–356; *MR***22**#5591.
41. P. Erdős, A. Hajnal, V. T. Sós and E. Szemerédi, On Turán-Ramsey type theorems, *Combinatorica*, to appear.
42. P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *Infinite and Finite Sets, II* (ed. A. Hajnal *et al.*), North-Holland, Amsterdam, 1975, pp. 609–627; *MR***52**#2938.
43. P. Erdős, A. Meir, V. T. Sós and P. Turán, On some applications of graph theory, III, *Canad. Math. Bull.* **15** (1972), 27–32; *MR***50**#4393.
44. P. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** (1960), 17–61; *MR***23**#A2338.
45. P. Erdős, A. Rényi and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235; *MR***36**#6310.
46. P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 51–57; *MR***34**#5702.
47. P. Erdős and M. Simonovits, Some extremal problems in graph theory, *Combinatorial Theory and its Applications, I* (ed. P. Erdős *et al.*), North-Holland, Amsterdam, 1970, pp. 377–390; *MR***46**#84.
48. P. Erdős and M. Simonovits, An extremal graph problem, *Acta Math. Acad. Sci. Hungar.* **22** (1971), 275–282; *MR***45**#1790.
49. P. Erdős and M. Simonovits, On a valence problem in extremal graph theory, *Discrete Math.* **5** (1973), 323–334; *MR***49**#7175.
50. P. Erdős and M. Simonovits, Supersaturated graphs, to appear in *Combinatorica*, 1983.
51. P. Erdős and V. T. Sós, Some remarks on Ramsey's and Turán's theorem, *Combinatorial Theory and its Applications, II* (ed. P. Erdős *et al.*), North-Holland, Amsterdam, 1970, pp. 395–404; *MR***45**#8560.
52. P. Erdős and V. T. Sós, On a generalization of Turán's theorem, *Studies in Pure Math.*, to appear.
53. P. Erdős and V. T. Sós, On Turán–Ramsey type theorems II, *Studia Sci. Math. Hungar.* **14** (1979), 27–36.

54. P. Erdős and J. Spencer, *Probabilistic Methods in Combinatorics*, Academic Press, New York, 1974; *MR*52#2895.
55. P. Erdős and A. M. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091; *MR*8, 333b.
56. R. J. Faudree and R. H. Schelp, Ramsey type results, *Infinite and Finite Sets, II* (ed. A. Hajnal *et al.*), North-Holland, Amsterdam, 1975, pp. 657–665; *MR*53#13016.
57. R. J. Faudree and M. Simonovits, On a class of degenerate extremal graph problem, *Combinatorica* **3** (1983), 97–107.
58. R. Häggkvist and C. Thomassen, On pancyclic digraphs, *J. Combinatorial Theory (B)* **20** (1976), 20–40; *MR*52#10481.
59. C. Hylton-Cavallius, On a combinatorial problem, *Colloq. Math.* **6** (1958), 59–65; *MR*21#1941.
60. G. Katona, Graphs, vectors and inequalities in probability theory, *Mat. Lapok* **20** (1969), 123–127; *MR*41#109.
61. G. Katona, T. Nemetz and M. Simonovits, On a problem of Turán in the theory of graphs, *Mat. Lapok* **15** (1964), 228–238; *MR*30#2483.
62. T. Kővári, V. T. Sós and P. Turán, On a problem of Zarankiewicz, *Colloq. Math.* **3** (1954), 50–57; *MR*16, 456a.
63. L. Lovász, On chromatic number of finite set systems, *Acta Math. Acad. Sci. Hungar.* **19** (1968), 59–67; *MR*36#3673.
64. L. Lovász, Independent sets in critical chromatic graphs, *Studia Sci. Math. Hungar.* **8** (1973), 165–168; *MR*48#8289.
65. L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph, *Proceedings of the Fifth British Combinatorial Conference* (ed. C. St.J. A. Nash-Williams and J. Sheehan), Congressus Numerantium **XV**, Utilitas Math., Winnipeg, 1976, pp. 417–429; *MR*52#13445.
66. L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph II, *Studies in Pure Math.*, to appear.
67. J. W. Moon, On independent complete subgraphs in a graph, *Canad. J. Math.* **20** (1968), 95–102; *MR*36#2529.
68. J. W. Moon and L. Moser, On a problem of Turán, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 283–286; *MR*27#1936.
69. T. S. Motzkin and E. G. Straus, Maxima for graphs and a new proof of a theorem of Turán, *Canad. J. Math.* **17** (1965), 533–540; *MR*31#89.
70. L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976), 359–364; *MR*52#10497.
71. I. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles, *Combinatorics, II* (ed. A. Hajnal and V. T. Sós), North-Holland, Amsterdam, 1978, pp. 939–945; *MR*80c:05116.
72. M. Simonovits, A method for solving extremal problems in graph theory, *Theory of Graphs* (ed. P. Erdős and G. Katona), Academic Press, New York, 1968, pp. 279–319; *MR*38#2056.
73. M. Simonovits, Extremal graph problems, *Combinatorial Structures and their Applications* (ed. R. K. Guy *et al.*), Gordon and Breach, New York, 1970, pp. 399–401; *MR*41#8247.
74. M. Simonovits, On the structure of extremal graphs, Ph.D. thesis, Budapest, 1970.
75. M. Simonovits, On colour-critical graphs, *Studia Sci. Math. Hungar.* **7** (1972), 67–81; *MR*47#4841.
76. M. Simonovits, Note on a hypergraph extremal problem, *Hypergraph Seminar* (ed. C. Berge and D. Ray-Chaudhuri), Lecture Notes in Mathematics **411**, Springer-Verlag, Berlin–Heidelberg–New York, 1974, pp. 147–151; *MR*51#2987.
77. M. Simonovits, Extremal graph problems with symmetrical extremal graphs, additional chromatic conditions, *Discrete Math.* **7** (1974), 349–376; *MR*49#2459.
78. M. Simonovits, The extremal graph problem of the icosahedron, *J. Combinatorial Theory (B)* **17** (1974), 69–79; *MR*51#7945.
79. M. Simonovits, On graphs not containing large saturated planar graphs, *Infinite and Finite Sets, III* (ed. A. Hajnal *et al.*), North-Holland, Amsterdam, 1975, pp. 1365–1386; *MR*52#201.

80. M. Simonovits, On Paul Turán's influence on graph theory, *J. Graph Theory* **1** (1977), 102–116; *MR***56**#2870.
81. M. Simonovits, Extremal graph problems and graph products, *Studies in Pure Math.*, to appear.
82. M. Simonovits, Extremal graph problems, degenerate extremal problems and supersaturated graphs, *Proc. Silver Jubilee Conf. on Combinatorics, Waterloo, 1982*, to appear.
83. R. R. Singleton, On minimal graphs of maximum even girth, *J. Combinatorial Theory* **1** (1966), 306–332; *MR***34**#1231.
84. V. T. Sós, On extremal problems in graph theory, *Combinatorial Structures and their Applications* (ed. R. K. Guy *et al.*), Gordon and Breach, New York, 1970, pp. 407–410; *MR***42**#7550.
85. J. Spencer, Turán's theorem for *k*-graphs, *Discrete Math.* **2** (1972), 183–186; *MR***45**#6668.
86. J. Spencer, Nonconstructive methods in discrete mathematics, *Studies in Combinatorics* (ed. G.-C. Rota), Carus Math. Monographs, Math. Assoc. of America, Washington, D.C.
87. E. Szemerédi, On graphs containing no complete subgraph with four vertices, *Mat. Lapok* **23** (1972), 113–116; *MR***50**#4385.
88. E. Szemerédi, On a set containing no *k* elements in arithmetic progression, *Acta Arithmetica* **27** (1975), 199–245; *MR***51**#5547.
89. E. Szemerédi, Regular partitions of graphs, *Problèmes Combinatoires et Théorie des Graphes* (ed. J.-C. Bermond *et al.*), CNRS, Paris, 1978, pp. 399–401; *MR***81i**:05095.
90. P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* **48** (1941), 436–452; *MR***8**, 284j.
91. P. Turán, On the theory of graphs, *Colloq. Math.* **3** (1954), 19–30; *MR***15**, 976b.
92. P. Turán, Applications of graph theory to geometry and potential theory, *Combinatorial Structures and their Applications* (ed. R. K. Guy *et al.*), Gordon and Breach, New York, 1970, pp. 423–434; *MR***42**#7551.
93. A. A. Zykov, On some properties of linear complexes, *Mat. Sbornik N.S.* **24 (66)** (1949), 163–188; *MR***11**, 733h = *Amer. Math. Soc. Transl.* **79**, 1952; *MR***14**, 493a.

Added in Proof

94. V. Chvátal and E. Szemerédi, On the Erdős–Stone theorem, *J. London Math. Soc.* (*2*) **23** (1981/2), 207–214; *MR***82f**:05057.