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Paul Turán's influence in Combinatorics

Miklós Simonovits

Abstract. This paper is a survey on the topic in extremal graphs theory influenced directly or indirectly by Paul Turán. While trying to cover a fairly wide area, I will try to avoid most of the technical details. Areas covered by detailed fairly recent surveys will also be treated only briefly. The last part of the survey deals with random ± 1 matrices, connected to some early results of Szekeres and Turán.

Keywords. Extremal graph theory, hypergraphs, regularity lemma, quasi-randomness, applications, random matrices.

AMS classification. Primary 05C35, Secondary 05C65.

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Supported by ERC-AdG 228005 and OTKA 101536 and !!! Ervin.

1. Preface

Paul Turán was one of my professors who had the greatest influence, – not only on me, on my way of thinking of Mathematics, of doing Mathematics, but – on my whole mathematical surrounding.

Once I read that Hilbert was the last polyhistor in Mathematics. This meant that after him not too many people had an overview over the whole Mathematics. I do not really know if this is true or not: I know only that "most" of the mathematicians I know concentrate basically on one or two fields, while some of my professors, like Erdős, Turán, and Rényi were covering several parts of Mathematics. I think of Turán as a polyhistor in Mathematics.

YES: Today only the best can excel in more than one branch. Turán was one of them. His main work, his most important results concern primarily number theory, interpolation and approximation theory, the theory of polynomials and algebraic equations, complex analysis, and Fourier analysis. He invented a new method in analysis, called the power sum method [369], giving interesting results in themselves and applicable in several distinct branches of Mathematics. His results in combinatorics and graph theory were definitely *not his most important results*, still they were very important in graph theory. He has found theorems becoming later the roots of whole theories. Definitely this is the case with his – today already classical – graph theorem. Paul Erdős wrote [121] that

> Turán had the remarkable ability to write perhaps only one paper or to state one problem in various fields distant from his own; later others would pursue his idea and a new subject would be born.
>
> In this way Turán initiated the field of extremal graph theory. He started this subject in 1941 (see [358] and [359])…

I should also mention here that – though the big breakthrough in the application of probabilistic methods in combinatorics is due to Erdős, – Turán's new proof of the Hardy-Ramanujan theorem [356] (later becoming the root of statistical number theory) and the Szekeres-Turán proof of the existence of "almost Hadamard matrices" [347] were important contributions.

I have just written that Paul Turán greatly influenced our way of thinking. Both Erdős and Turán quite often set out from some particular problem and then built up a whole theory around it. However, for Turán the motivation seemed to be much more important. When he spoke about Mathematics, he went a long distance to explain why that problem he was speaking of was interesting for him. My impression was that he preferred building theories, at the same time was cautious not to build too general theories that might seem to be already vacuous.

I shall explain this through some "stories".[1]

(a) I started working in extremal graph theory, basically at the end of my first year – as a student – at the Eötvös Loránd University. This happened as follows: Vera Sós (the wife of Turán) was our lecturer in "Mathematical Analysis" and in "Combinatorics and Graph Theory". (Our group of 26 first-year honour students in Mathematics had nine 50 minute lectures with her weekly. A year earlier she also taught combinatorics to the group of Bollobás.) After our first year she was definitely our most popular lecturer. The second semester Vera decided to start a so called "special lecture" on Graph Theory, as a continuation of her "introductory course". Most probably most of the dedicated students in Mathematics attended this course. Here she spoke – among others, – about Turán's hypergraph conjecture. Next week three of us, (independently?) Katona, Nemetz, and myself told her that we have proved some theorems in connection with Turán's hypergraph conjecture. Vera suggested to write them up, in Hungarian, in the Matematikai Lapok, in a joint paper. First Katona and I wrote up the paper, but that was not good enough for Vera, so Katona and Nemetz rewrote it, and finally the paper [215] appeared and became one of our most cited papers.[2] Having finished the paper, I continued working on these types of questions, while Katona and Nemetz went into other directions. So I proved several theorems which today would be called Turán type results. I wrote them up in a "student paper" and submitted it to the "Students Research Society" (Matematikai Diákkör) whose "professor" leaders were András Hajnal and Vera Sós those days. Most probably I won some prize, and the question was if to publish my new results in some mathematical journal, say in Acta Math. Acad. Hungarica. However, a little later Vera Sós informed me that "unfortunately" Gábor Dirac had just published a paper on related topics [100]. So my 2$^{\text{nd}}$ paper was "killed".

Anyway, slightly later I met Turán, and tried to inform him of my results, starting in a "very abstract way". Basically I defined a monotone property $\mathcal{P}$ and maximized the number of edges in the family of n-vertex graphs of property $\mathcal{P}$. Turán suggested to take the simpler but equivalent formulation that "We have a finite or infinite family of excluded subgraphs…". Even today I stick to this "more transparent" formulation.

(b) Actually, the first time I met Turán – as a mathematics professor – was slightly earlier. In the first semester Vera Sós taught us Analysis, however, one day she got a flu, had fever, had to stay home. So her husband, Turán came in to give the lecture, on the Lagrange Mean Value Theorem. Despite the fact that those days Vera was our favourite lecturer, I was shocked by the spellbinding style of Turán, while speaking of this relatively simple theorem.

Actually, I heard some opinions, according to which Turán was excellent for the

[1] Telling stories is a very dangerous thing: the reader may think that I promised to write of Paul Turán and instead I am speaking of Vera Sós, or even worse, of myself. No, No, No: I am speaking of our excellent professors, Turán, Erdős, Vera Sós, András Hajnal, Rényi, Gallai…

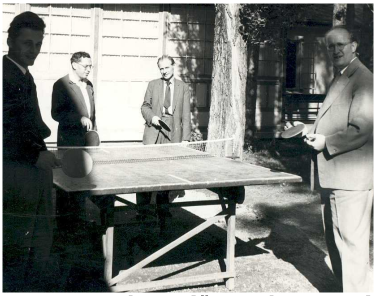
[2] This was the first paper of mine and of Nemetz, and the second one of Katona, who finished his fourth year at the university at that time.

best students but sometimes difficult to follow for the less gifted ones[3]. The reason for this was that he not only proved the theorems but (a) explained the background very carefully and (b) explained what would fail if we tried to prove it in some other ways.

(c) When I became a third year student, I started learning Function Theory (Theory of Complex Analytic functions), from Kató Rényi, the wife of Alfréd Rényi. I enjoyed her lectures very much and having finished this two-semester course, for some reason I dropped in to the Mathematical Institute[4]. There I met Gábor Halász and asked, what he was doing there. He answered that in 10 minutes there would be a seminar of Turán in Number Theory and Complex Analysis, and he would give a lecture there. I happened to be free, so I decided to attend Gábor's lecture. I enjoyed that whole atmosphere and the Mathematics there so much that I became a regular participant of the "Turán seminar" for many-many years. And that was partly due to Halász, but primarily to Turán. The seminar was interactive, very friendly, anyone could ask any (relevant mathematical) questions, to help one to understand the details, and the background...

(d) Several years later, as an assistant professor, once I entered Turán's office. He was reading a letter, which informed him about some new results (about the convergence properties of power series on the unit complex disk). He started explaining it to me. I asked him why that result was interesting and the answer was very convincing. Actually, I was "slightly frightened": I felt that Turán could convince me of any mathematical result being interesting, if he felt it interesting.

Kató Rényi, Turán, Vera Sós, Erdős (and somebody covered by Vera?)



Knapowski, Erdős, Szekeres, and Turán

We are often asked: what is the secret of that the Hungarian Mathematics is so good. Of course, we have standard answers to this, despite the fact that the question itself may be slightly dangerous.

It is nice to hear that our Mathematics is outstanding, but at the same time one should keep checking, in which areas can one be satisfied and where we have to do something to improve the "Hungarian Mathematics".

[3] Unfortunately Turán have not given regular Number Theory courses those years. Here the "gifted" would mean the best 10 students in our group.

[4] I was a student later an assistant professor, ... at the Eötvös University, while this was a Research Institute, part of the Academy, headed by Alfréd Rényi. Fortunately those days the walking distance between the two places was roughly 5 minutes.

I myself have at least three answers to this question. The first one is that in Hungary there is a very strong tradition to support talented young students in Mathematics and Physics, (and most probably, in many other fields as well). We had our KöMaL: the High School Mathematics Journal. Most of those who are today math professors in Hungary, still remember, how much we owe to it, we have gained from participating in the contests organized in this surrounding.[5] Also, there were organized math lectures and meetings, while we were high school students. This is where I first met Bollobás, Komlós, Halász, and many others when I was a second year high school student.

Yet, definitely, one of the most important factors was that we had excellent professors at the University. Excellent in Mathematics and excellent in conveying their Mathematics to us. I myself, selecting those who really influenced my Mathematics, (following the time-line) would list first Vera Sós, Paul Erdős and Paul Turán.[6]

1.1. Apologizing?

In this survey I will try to cover several areas, but not in too many details. Often I will start some topic, give a few theorems, and then refer the reader to other surveys or papers.

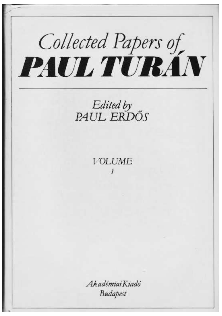
While writing this survey, I looked at several other surveys, of excellent authors, and many of them started with apologizing sentences that there was no way to try to be complete, and the author had to leave out several interesting and important results. The same applies to this survey as well. In several cases – selecting a paper – I had to restrict myself to including its first, or most characteristic results, and leave the other, at least for me very important, results to the reader. One reason for this was that I tried to write a readable survey. And the same is the reason why I was not afraid to repeat some parts: be occasionally “redundant”.

When Turán died, in 1976, his collected papers were published in a three-volume book [368], which is an annotated edition of his works in the sense that the grateful mathematical surrounding added mathematical notes to his papers. I myself was responsible for Graph Theory and Combinatorics. I wrote three mini-surveys for [368]: one on “pure extremal graph theorems”, another one on applications of extremal graph theorems in Analysis, Geometry (and Potential Theory), and the third one on “random matrices”. *This survey* includes a large part of those surveys, however, it goes much further: the new developments in the field showing Turán’s influence in Discrete Mathematics greatly surpass what I could write those days. Here I include many results

[5] Actually, Erdős and Turán learnt of each other also from this journal.

[6] If I wanted to extend this list, of course, I would add my mother, perhaps Hajós, definitely Hajnal, Gallai, and Rényi. We met Rényi relatively late, when we became third year students, however, when he started giving special lectures about Random Methods in Analysis, Random Methods in Combinatorics, Introduction to Information Theory, again, all the best students were sitting there and eagerly listening to him. He – similarly to Turán – also gave long explanations on the background of the theorems he was speaking of.

showing these new developments (and leave out certain parts covered by other surveys of this volume, see Katona e.g., [214]. I also cut short describing areas that are covered by the very recent survey papers of the Erdős Centennial volume, e.g., Gowers [189], Rödl and Schacht [303] or Füredi and myself [180],...



Of course, the most important subject covered here (where Turán's influence can be seen) is Extremal Graph Theory. One basic source to provide a lot of information is the book of Bollobás, Extremal Graph Theory [55]. There are many surveys covering distinct parts of this very large area. Among them are mines, e.g., [327], [328], [330], [332] and there is a survey by Bollobás in the Handbook of Combinatorics [51]. Of course, the Handbook contains several further chapters basic to this field, just to mention the chapters by Bondy [64] and by Alon [9]. I should also mention many excellent, more detailed further surveys related to this one, e.g., of Füredi [167], Keevash [218], Kühn and Osthus [255].

Since the very recent survey of Füredi and myself [180] covers a huge and important area of extremal graph theory, namely the so called Degenerate Extremal Graph Problems, here we shall concentrate on the non-degenerate cases, where the extremal structures have positive density. In this non-degenerate case I will select five topics:

(a) New results attained with the help of the Szemerédi Regularity Lemma [349] (for the older one see, e.g., [249]). There are very many new developments in this area, which will be touched on only very briefly, in Section 6.2. Here I mention only its connection to Property Testing [16] [14],... and to graph limits, where I refer the reader to some papers of Christian Borgs, Jennifer Chayes, László Lovász, Vera Sós, Kati Vesztergombi, e.g., [68, 69, 70], to the homepage of Lovász, where many of these can easily be found, and to the very new book of Lovász [263];
(b) Ramsey-Turán type results, where for the older results see the survey of Vera Sós and mine [335], and for the many new interesting developments, see among others Balogh and Lenz [39].
(c) and also the Andrásfai-Erdős-Sós type theorems [24], Erdős-Simonovits [139], Łuczak [268], Thomassen [355],...
(d) Applications in multicolor Ramsey problems, e.g., results of Łuczak [269], Gyárfás, Ruszinkó, Sárközy, and Szemerédi [194], Kohayakawa, Simonovits, Skokan [231], and many others.
(e) Typical Structures: Erdős-Kleitman-Rothschild type theorems, [131], Erdős, Frankl and Rödl [125], and Balogh, Bollobás, and Simonovits, e.g., [34],...

Again, there is no way to be complete here. Rather I chose to indicate the main lines of some of these theories... It is also very useful and informative to read the corresponding problem-posing papers of P. Erdős [113] [119] [120], [123]. I should also mention the book of Chung and Graham on Erdős problems [93].

In Section 16 I will discuss the theory of Random Matrices, but only shortly: a relatively new and excellent survey of Van Vu [370] describes this area in details. There is also another reason: Subsection 16.2 on determinants is connected to Turán the most, while in the next two parts on the probability of being singular and on the distribution of eigenvalues of random matrices is where many new interesting results were proved after Turán's death. Yet, they are connected to Turán in a slightly weaker way.[7]

Overlapping with my older surveys is inevitable. Yet I will try to "overemphasize" those parts that had to be left out from [180] and [331]. Some further related surveys and pseudo-survey papers are Füredi [167], Sidorenko [318] Simonovits [330], Simonovits and Sós [335], Kohayakawa and Rödl [229], Rödl and Schacht [303], and many others.

2. Introduction

Today one of the most developed and fastest developing areas of Graph Theory is Extremal Graph Theory and the parts of Graph Theory connected to it. There are several reasons for this. One of them is that this is a real theory with many important, highly non-trivial subfields and many related larger fields of combinatorics. I have already mentioned some some of them. Further ones are

(a) Although Extremal Hypergraph theory is still an extremely hard field to achieve new results in, several very interesting new theorems were proved for hypergraphs in the last decade.

(b) New tools were created, above all, Hypergraph Regularity Lemmas, and, connected to them, Removal Lemmas and Counting Lemmas, and Graph Limit Theory.

(c) Computers were used to solve several extremal graph and hypergraph problems, mostly using a new theory, the Razborov Flag Algebras [293, 296].

(d) Some parts of Theoretical Computer Science are connected to the above fields. I mention here four such topics:

   (i) Graph Property Testing, very strongly connected to applying Szemerédi Regularity Lemma, (see e.g. papers of Alon and Shapira) [21], [16].

   (ii) Applications of graph results, e.g., Degenerate Extremal Graph Theorems in Computer Science.

   (iii) Theory of quasi-random graphs (initiated in some sense by Thomason, [353], then by Chung, Graham and Wilson [94]...

[7] Yet I decided to include a short part on them, too.

(iv) Application of random graph methods and expanders – that are strongly connected to extremal graph theory – in Computer Science,[8]

(e) As to the tools used in Extremal Graph Theory, it is connected to the theory of Random Graphs:

(i) it uses random graphs to get lower bounds,

(ii) investigates extremal subgraphs of random graphs,

(iii) and motivates the description of typical structures,

(f) it is connected among others, to Finite Geometry (also used for constructions providing lower bounds in our problems), to Commutative Algebra, also used to get lower bounds,…(Vera Sós wrote one of the first surveys on the connections to Finite Geometries [339].)

Reading this "list" the reader immediately sees that describing the new developments in this area is much more than what such a survey paper can cover, even if in many cases it only refers to other papers or surveys. So we shall try to provide a "random tour" in this huge area.

Also, I plan to post on my homepage a slightly longer version of this survey, providing more details.

2.1. Structure of the paper

(a) We shall start with the Theory of Extremal Graphs. We shall describe the huge development of the Theory of Extremal Graphs, primarily areas neglected in [332] and [180].

(b) Section 5 describes the theory of supersaturated graphs.

(c) In Section 13 I shall describe those *applications of extremal graph results* which were initiated by Paul Turán, in the last years of his life. Also we shall describe other applications of Turán's theorem.

(d) These applications led also to the Ramsey-Turán Theory, described in more details in the survey paper of Vera Sós and myself [335]. There are quite a few new developments in this field. I shall describe some of them in Section 10.

(e) There are several connections between Ramsey Theory and the theory of Turán type problems. Section 12 contains some results on this.

(f) There is one more, very important area not to be forgotten: Erdős and Turán greatly influenced our day's mathematics just by asking the density version of Van der Waerden's theorem. This is well described, at least its early period, in the book of Graham, Rothschild and Spencer [190]. Many important details can be learned from the paper of Vera Sós [340], papers of Gowers, Green, Tao,…I also will include a very short section on this topic.

[8] For two "mini-surveys" see e.g. Spencer [341] and Alon [10].

(g) Section 16 discusses a paper of Szekeres and Turán on the average of the square of the determinants of random ± 1 matrices.

3. Turán type graph problems

Paul Turán's graph theoretical and combinatorial results can roughly be classified as follows:

(a) His classical extremal graph theorem [358], [359] and the analogous results of Kővári, V.T. Sós and Turán [252] on the extremal number of $K_2(a, b)$.

(b) His results on applications of his graph theorem, see [363, 364, 365, 366], and also the papers of Erdős, Meir, V.T. Sós and Turán [132, 133, 134][9].

(c) Results on random ± 1-matrices, estimating the average of the k^{th} power of their determinants [347, 357, 360, 362].

(d) Beside this, it was him who asked the first general question in connection with the crossing numbers (see e.g., one of his last papers [367], or Beineke and Wilson [46]).

3.1. Turán's graph theorem

In 1935 Erdős and Szekeres proved [149] that

Theorem 3.1. *For every k there exists an n_k such that if we fix n_k points in the plane arbitrarily (but in general position), then there are always k of them spanning a convex k-gon.*

To prove this, they applied Ramsey's theorem. Actually they did not know it, but rediscovered it. Motivated by Ramsey Theorem, Turán proved his famous theorem. Before formulating it we introduce some notations.

Notation. Given a graph, hypergraph, the first subscript will almost always denote the number of vertices: G_n, S_n, H_n will mostly denote graphs (digraphs, hypergraphs) of n vertices[10]. Mostly we shall restrict our considerations to ordinary graphs (without loops and multiple edges). Given a graph (digraph, hypergraph) G, $v(G)$ and $e(G)$ denote the number of vertices and edges respectively, and $\chi(G)$ is G's chromatic number. K_p denotes the complete graph on p vertices, C_ℓ and P_ℓ, are the cycle and path of ℓ vertices, respectively. $K_p(n_1, \dots, n_p)$ is the complete p-partite graph with n_i vertices in its i^{th} class, and $T_{n,p}$ is the Turán graph of n vertices and p classes, that is, $T_{n,p} = K_p(n_1, \dots, n_p)$ where $\sum n_i = n$ and $|n_i - \frac{n}{p}| < 1$.

Given two graphs G and H, denote $G \otimes H$ the graph obtained from vertex-disjoint copies of G and H by joining each vertex of G to each one of H. (Occasionally we denote their disjoint union by $G + H$, and the disjoint union of k copies of H by kH.)

[9] and a corrigendum to [134] (misprints).

[10] Very rarely we shall consider some "excluded" graphs and the subscript will just enumerate them.

Turán's problem. *Given p and n, how large can $e(G_n)$ be if G_n does not contain a K_{p+1} ?*

Clearly, $T_{n,p}$ does not contain K_{p+1}. Turán's theorem asserts that $T_{n,p}$ is extremal in the following sense:

Turán's Theorem. ([358] (1940)). *For given n and p any graph having more edges than $T_{n,p}$ or having exactly as many edges as $T_{n,p}$ but being different from it must contain a K_{p+1}, as a subgraph.*

As Turán remarks, from this form one can easily verify that the maximum number of edges a graph G_n can have without containing a K_{p+1} is

$$\frac{1}{2}\left(1-\frac{1}{p}\right)(n^2-r^2)+\binom{r}{2}, \quad \text{if } n \equiv r \pmod{p} \text{ and } 0 \le r < p. \tag{3.1}$$

In this sense Turán's theorem yields a complete solution of the posed question.[11]

How did Turán arrive at this theorem? In Ramsey's theorem we ask (in some sense): Assume we know that G_n contains no k independent vertices. For how large p can we ensure the existence of a K_{p+1} in G_n. Turán replaced the condition that G_n had no k independent vertices by a simpler condition that the graph had many edges. He asked:

> Given a graph G_n of e edges, how large K_{p+1} must occur in G_n? Or, in other words, given n and p, how large e does ensure the occurrence of a K_{p+1} in G_n?

The "complementary" form. A lesser known but equally useful form of Turán's theorem can be obtained by switching to the complementary graph $\overline{G_n} = H_n$. If H_n has no $p+1$ independent vertices, then $e(H_n) \ge e(\overline{T_{n,p}})$ and the equality implies that $H_n = \overline{T_{n,p}}$. (This is Theorem III in his original paper [358].)

On the history of Turán's theorem. As Turán remarks in the "Added in Proof of [358], he has learnt from J. Kraus that W. Mantel has already proved his theorem in the special case $p = 3$, [273]. It is interesting to realize that this theorem could have been found by Mantel back in 1907, but he missed it. It is even more surprising that P. Erdős missed to find this theorem in 1938. As a matter of fact, Erdős and E. Klein have proved an analog result in [106]. Here Erdős investigated a number theoretical question and arrived at the following graph theoretical result:

Theorem 3.2. *If G_n contains no C_4, then $e(G_n) = O(n^{3/2})$.*

[11] Letters: Mostly we shall exclude $p+1$-chromatic graphs but there will be cases when we shift the indices and exclude p-chromatic graphs.

At the same time, E. Klein gave a "finite geometric" construction showing that there exist graphs G_n with $e(G_n) > cn^{3/2}$ edges and without containing 4-cycles. Turán, proving his theorem, immediately posed several other analog problems (such as the problem of excluded path P_k, excluded loops, the problem when L is the graph determined by the vertices and edges of a regular polyhedron). This started a new line of investigations. Erdős, (as he stated many times), felt, it was a kind of blindness on his side not to notice these nice problems.

In 1949 Zykov [375] rediscovered Turán's theorem, giving a completely different proof. He used an operation which could be called *symmetrization* and which was later successfully used to prove many analog results. Since that many further proofs of Turán's theorem have been found. Some of them are similar to each other, some others are completely different. Thus e.g. proofs of Andrásfai [23] G. Dirac [100] and the proofs of Katona, Nemetz and Simonovits [215] are somewhat similar, the proof of Motzkin and Straus [277] seems to be completely new, though it is actually strongly related to Zykov's proof [375]. Most of these proofs led to interesting new generalizations. In other cases the generalizations were formulated first and only then were they proved. This is the case of the proof of Erdős, and also with the proof of Erdős-T. Sós-Bollobás-Thomason-Bondy, see [146], [60], [63]. Before turning to the general case I state three of these results.

Dirac's theorem. *Assume that $n > p$ and $e(G_n) > e(T_{n,p})$. Then, for every $j \le p$, G_n contains not only a K_{p+1} but a K_{p+2} with an edge missing,..., a K_{p+j+1} with j edges missing, assumed that $n > p + j + 1$.*

Observe that for each j this immediately implies Turán's theorem, since a $K_{p+j+1} - (j$ edges) contains a K_{p+1}.

Erdős theorem ([118]). *If G_n contains no K_{p+1} then there exists a p -chromatic graph H_n such that if $d_1 \le d_2 \le d_3 \le \cdots \le d_n$ and $d_1^* \le d_2^* \le d_3^* \le \cdots \le d_n^*$ are the degree sequences of G_n and H_n respectively, then $d_i^* \ge d_i$, $(i = 1, 2, \dots, n)$.*

This again immediately implies Turán's theorem, by

$$2e(G_n) = \sum d_i \le \sum d_i^* = 2e(H_n) \le 2e(T_{n,p}).$$

Denote by $N(x)$ the neighborhood of x.

Erdős-T. Sós-Bollobás-Thomason theorem [60, 146]. *If G_n is a graph with $e(G_n) > e(T_{n,p})$, then G_n has a vertex x of, say, degree d, for which for $G_{n-d} := G_n - N(x)$, we have $e(G_{n-d}) > e(T_{n-d,p-1})$*

This theorem was slightly improved by Bondy [63]. This result implies Turán's theorem if we apply induction on p: G_{n-d} contains a K_p yielding together with x a K_{p+1} in G_n. (Above I deliberately forgot the case $e(G_n) = e(T_{n,p})$, for the sake of simplicity.)

3.2. General Problem

Since 1941 a wide theory has developed around Turán's theorem.

> Let $\mathcal{L}$ be a finite or infinite family of graphs and let $\mathbf{ex}(n,\mathcal{L})$ denote the maximum number of edges a graph G_n (without loops and multiple edges) can have without containing any $L \in \mathcal{L}$ as a subgraph. Further, let $\mathbf{EX}(n,\mathcal{L})$ denote the family of graphs attaining this maximum. Given a family $\mathcal{L}$, determine $\mathbf{ex}(n,\mathcal{L})$ and $\mathbf{EX}(n,\mathcal{L})$.

When $\mathcal{L} = \{L\}$, we shall replace $\mathbf{ex}(n,\{L\})$ by $\mathbf{ex}(n,L)$. The general asymptotics on $\mathbf{ex}(n,\mathcal{L})$ was given by

Theorem 3.3 (Erdős and Simonovits [136], Erdős [114], [115] and Simonovits [321]). *For any family $\mathcal{L}$ of excluded graphs, if*

$$p(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(L) - 1, \tag{3.2}$$

then

$$\mathbf{ex}(n,\mathcal{L}) = \left(1 - \frac{1}{p(\mathcal{L})}\right)\binom{n}{2} + o(n^2) \qquad \textit{as} \qquad n \to \infty. \tag{3.3}$$

Further, if S_n is any extremal graph for $\mathcal{L}$, then it can be obtained from $T_{n,p}$ by changing $o(n^2)$ edges.

(The weaker result of Erdős and Simonovits, namely (3.3), is an easy consequence of the Erdős-Stone theorem [148]. The most important conclusion of these theorems is that the maximum number of edges and the structure of the extremal graphs depend only very weakly on the actual family $\mathcal{L}$, it is asymptotically determined by the minimum chromatic number. A further interesting conclusion is that for any $\mathcal{L}$ we can find a single $L \in \mathcal{L}$ such that $\mathbf{ex}(n,\mathcal{L}) - \mathbf{ex}(n,\{L\}) = o(n^2)$. This is a *compactness* type phenomenon asserting that there is not much difference between excluding many graphs or just one appropriate member of the family.)

Remark 3.4. Several authors call the result according to which (3.2) implies (3.3) as Erdős-Stone theorem, in my opinion, incorrectly. This "theorem" did not exist before our first joint paper with Erdős, [136]. It changed the whole approach to this field. Finally, Erdős always considered it as an Erdős-Simonovits result.

3.3. Degenerate extremal graph problems

If $\mathcal{L}$ contains at least one bipartite L, then $\mathbf{ex}(n,\mathcal{L}) = o(n^2)$, otherwise

$$\mathbf{ex}(n,\mathcal{L}) \ge e(T_{n,2}) = \left[\frac{n^2}{4}\right].$$

This is why we shall call the case $p(\mathcal{L}) = 1$ *degenerate*.

Here we arrive at the second – and again very important – graph paper of Turán. In 1954 Kővári, V.T. Sós and Turán proved the following result.

Kővári-T. Sós-Turán theorem [252].

$$\mathbf{ex}(n, K_2(p,q)) \le \frac{1}{2}\sqrt[p]{q-1}\, n^{2-(1/p)} + O(n). \tag{3.4}$$

We should remark that an important footnote on the first page of [252] states:

> "As we learned, after giving the manuscript to the Reduction, from a letter of P. Erdős, he has found independently most of the results of this paper."

This theorem can be regarded as a sharpening of the Erdős-Stone theorem [148] asserting that

$$\mathbf{ex}(n, K_d(m,\dots,m)) = \left(1 - \frac{1}{d-1}\right)\binom{n}{2} + o(n^2)$$

and yielding that $\mathbf{ex}(n, K_2(m,m)) = o(n^2)$. Both these theorems were motivated by some topological problems. (3.4) is probably sharp for every $p \le q$, apart from the value of the multiplicative constant, however this is not known in general. As a construction of Erdős, Rényi and T. Sós [135] and of W.G. Brown [76] shows, (3.4) is sharp for $p = 1, 2$, and 3. For $p = q = 2$ even the value of the multiplicative constant is sharp. A construction of H. Cavallius-Hylten [204] shows that it is also sharp for $p = 2$, $q = 3$. Further, the Mörs construction [278] on the analog matrix-problem, and the Füredi construction [171] show that (3.4) is sharp for $p = 2$ and all $q \ge 2$. We shall return to this question (that is, to the corresponding matrix problem) below.

Remark 3.5. It was a great surprise when it turned out that $\mathbf{ex}(n, K(3,3)) \approx \frac{1}{2}n^{5/3}$: by the lower bound given by Brown [76] we knew that the exponent $5/3$ in (3.4) is sharp, however, when Füredi [169] improved the upper bound, that showed that the multiplicative constant $\frac{1}{2}$ of the Brown construction is the right one.

Another interesting degenerate problem is the problem when a path P_k is excluded. As I learnt from Gallai, this was one of those problems asked by Turán (in a letter written to Erdős) which started the new development in this field. The answer was given much later by the

Erdős-Gallai theorem [126]. $\mathbf{ex}(n, P_k) \le \frac{k-2}{2}n.$

Clearly, if n is divisible by $k-1$, the disjoint union of $n/(k-1)$ K_{k-1}'s shows that the theorem is sharp. If n is not divisible, this construction yields only $\mathbf{ex}(n, P_k) \ge \frac{k-2}{2}n - O(k^2)$. The exact value of $\mathbf{ex}(n, P_k)$ was found by Faudree and Schelp, who used it to prove some generalized Ramsey theorems [153]. Erdős and Gallai also proved [126] that if $\mathcal{L}_k$ is the family of all the cycles of at least k vertices, then $\mathbf{ex}(n, \mathcal{L}_k) = \frac{1}{2}(k-1)n + O(k^2)$, and in some cases the extremal graphs are exactly those graphs whose doubly connected components (blocks) are K_{k-1}'s. Kopylov [250] considered the problem of connected graphs without P_k, and his results implied the earlier ones. Balister, Győri, Lehel and Schelp [31] also have results sharpening Kopylov's theorems. The reader can find further information in [180].

It is worth mentioning that Erdős and T. Sós conjectured [113] that for every tree T_k, $\mathbf{ex}(n, T_k) \le \frac{1}{2}(k-2)n$. Ajtai, Komlós, Simonovits and Szemerédi proved (under publication) this for all sufficiently large k:

Theorem 3.6 (Ajtai, Komlós, Simonovits and Szemerédi [2], [3],[4]). *There exists a k_0 such that for $k > k_0$ and $n \ge k$*

$$\mathbf{ex}(n, T_k) \le \frac{1}{2}(k-2)n.$$

We close this part with the following

Theorem 3.7 (G. Dirac, [98]). *If $P_\ell \subseteq G$, and G is (at least) 2-connected, then G also contains a C_m with $m \ge \sqrt{2\ell}$.*

3.4. Even Cycles

An unpublished result of Erdős states that

$$\mathbf{ex}(n, C_{2t}) = O(n^{1+(1/t)}). \tag{3.5}$$

Two different generalizations of this result were given by Bondy and Simonovits [66], and by Faudree and Simonovits [155]. I skip this area since it is fairly well described in [180]. Let me discuss the Cube-theorem. Turán asked that if L denotes the graph defined by the vertices and edges of a regular polyhedron, how large $\mathbf{ex}(n, L)$ is. Erdős and Simonovits [138] proved that if Q_8 denotes the cube graph, then

Theorem 3.8 (Cube theorem). $\mathbf{ex}(n, Q_8) \le C_Q \cdot n^{8/5}$.

Actually if $\tilde{Q}_8$ is obtained from Q_8 by joining two opposite vertices, then $\mathbf{ex}(n, \tilde{Q}_8) = O(n^{8/5})$, too. One intriguing open question is whether there exists a $c > 0$ such that $\mathbf{ex}(n, Q_8) > c \cdot n^{8/5}$, or at least, $\mathbf{ex}(n, \tilde{Q}_8) > c \cdot n^{8/5}$.

Remark 3.9. As I mentioned above, this topic is also discussed in much more details in the recent survey of Füredi and Simonovits [180]. The same applies to large part of the next subsection.

3.5. Finite geometric constructions

If the extremal graph problem for $\mathcal{L}$ in consideration is non-degenerate, and p is defined by (3.2) then $T_{n,p}$ yields an asymptotically extremal sequence in the sense that $T_{n,p}$ contains no $L \in \mathcal{L}$ and has asymptotically maximum number of edges. The extremal graph is often (but not always, see [329], [325]) obtained from $T_{n,p}$ by

(a) first slightly changing the sizes of the classes, that is, replacing $T_{n,p}$ by a $K_p(n_1, \dots, n_p)$, where $n_i = \frac{n}{p} + o(n)$;

(b) then adding $o(n^2)$ edges to this $K_p(n_1, \dots, n_p)$.

(c) The assertion that this is not always the case means that sometimes we need a third step too, namely, to delete $o(n^2)$ edges in a suitable way, see [329].

In this sense the non-degenerate case is relatively easy: $(T_{n,p})$ is an asymptotically extremal sequence of graphs. The extremal structures in the degenerate cases seem to be much more complicated in the sense that in most cases we do not have lower and upper bounds differing only in a constant multiplicative factor. Thus for example we do not know whether the upper bound in the cube theorem is sharp, or that the upper bound given by the Kővári-T. Sós-Turán theorem is sharp for any $p, q \geq 4$. We do not even know the existence of a positive constant c such that

$$\frac{\mathbf{ex}(n, K_2(4,4))}{n^{2-(1/3)+c}} \to \infty.$$

Still, whenever we know that our upper bound for a bipartite L is sharp, we always use either explicitly or in an equivalent form some finite geometric construction, or some algebraic construction very near to it. I have already mentioned some of these constructions, namely that of E. Klein in [106], of Erdős, Rényi and T. Sós [135] for graphs without C_4, and that of Hylten-Cavallius for graphs not containing $K_2(2,3)$. Two further very important constructions are the Brown construction [76] for graphs not containing $K_2(3,3)$ and the Benson [48] construction (see also the Singleton construction [336]) of graphs not containing C_3, C_4, C_5, C_6 and C_7, and of graphs not containing $C_3, \dots, C_{11}$. These constructions of Benson show that (3.5) is sharp for $t = 3$ and $t = 5$, while W. G. Brown's construction shows that the Kővári-T. Sós-Turán theorem is sharp for $p = q = 3$ (and therefore for all $p = 3, q \geq 3$), apart from the value of the multiplicative constants.

Remark 3.10. Since [180] is a much more detailed survey, however mostly restricted on the Degenerate Extremal Graph Problems, and since these finite geometric problems mostly refer to degenerate cases, we suggest to the interested reader to read the corresponding parts from [180]. Here we mention only that several constructions using finite geometries or related methods were found since Turán died. Perhaps Mörs [278], Füredi [171], Ball and Peppe [32] Wenger [371], should be mentioned here, and several slightly different constructions of Lazebnik, Ustimenko, and their school (see e.g., [256, 257, 258]) and also the breakthrough results of Kollár, Rónyai, and Tibor Szabó, [235] and Alon, Rónyai and Szabó [18], (see also [9] and [180]).

3.6. A digression: the extremal matrix problems

If G_n is a graph, the condition that G_n does not contain any $L \in \mathcal{L}$ implies that if we consider the adjacency matrix A of G_n and a $v(L) \times v(L)$ symmetrical submatrix of A,[12] then this submatrix cannot be the adjacency matrix of L. If for every $L \in \mathcal{L}$ we add to $\mathcal{L}$ all those graphs which are obtained from L by addition of edges, and denote by $\widehat{\mathcal{L}}$ the resulting family of forbidden graphs, then the extremal graph problems for $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are the same, further the exclusion of every $L \in \widehat{\mathcal{L}}$ is equivalent to the exclusion of their adjacency matrices as a symmetrical submatrices of A.

The number of edges of G_n is half of the 1's in the adjacency matrix, thus each extremal graph problem generates an equivalent problem for 0-1 matrices, where the number of 1's is to be maximized. Sometimes this approach is very useful, e.g., enables us to find continuous versions of graph theorems. However, in our case there is an even better matrix theoretical approach. Assume that G_n is a bipartite graph with n vertices in its first class and m vertices in the second one. Then we often represent G by an $n \times m$ 0-1 matrix, and e.g. the exclusion of $K_2(p,q)$ in G is equivalent to the condition that taking arbitrary p rows and q columns of A, at least one of the corresponding $p \times q$ entries of the matrix will be 0, further, taking arbitrary q rows and p columns the same holds.

Now, as one can read on the first page of the Kővári-T. Sós-Turán paper, K. Zarankiewicz has raised the following interesting question: given a 0-1 matrix A, of n rows and n columns, and an integer j, how large the number of 1's should be to guarantee that A contains a minor of order j consisting merely of 1's? If the solution of this problem is denoted by $k_j(n)$, then one main result of the Kővári-T. Sós-Turán paper asserts in a somewhat more complicated but sharper form that

$$k_j(n) = O(n^{2-(1/j)}). \tag{3.6}$$

Further, they show that $\lim_{n\to\infty} k_2(n)/n^{3/2} = 1$. Then they point out that their matrix results imply

$$\mathbf{ex}(n, K_2(p,p)) \le \frac{1}{2}\sqrt[p]{p-1}\cdot n^{2-\frac{1}{p}} + O(n). \tag{3.7}$$

Some historical remarks. (a) The authors of [252] mention the general of excluding a $p \times q$ submatrix of 1's and that they restrict the discussion to the Zarankiewicz problem, where $a = b$.

(b) Kővári, T. Sós and Turán used a finite geometric construction to prove that $k_2(n) \ge n^{3/2} - o(n^{3/2})$. However, they did not use finite geometric language. Neither did Erdős, describing E. Klein's construction [106].

(c) Here again we should make a historical remark. According to [252]

[12] where symmetric submatrix means that if we take some j^{th} row of A then we also take the corresponding j^{th} column and vice versa.

"S. Hartman, J. Mycielski and C. Ryll-Nardzewski have proved that

$$c_1 n^{4/3} \leq k_2(n) \leq c_2 n^{3/2} \tag{1.2}$$

with numerical c_1 and c_2".

Of course the Erdős-Klein result from 1938 was sharper, though it was formulated for graphs, and therefore formally it did not imply the Hartman-Mycielski-Ryll-Nardzewski result.

Two more historical notes should be made. Above we made a sharp distinction between degenerate and non-degenerate extremal graph problems. The germ of this distinction can be found in [252]. In Section 3 the authors write: "Let us call attention to a rather surprising fact". And this fact is that $\mathbf{ex}(n, K_2(p,p)) = O(n^{2-(1/p)})$, while to ensure a fairly similar graph, namely K_{p+1}, we need $\approx \frac{1}{2}\left(1-\frac{1}{p}\right)n^2$ edges, which is much more. Further, in Section 6 the authors formulate the conjecture that $k_j(n) \geq c_j n^{2-(1/j)}$, which is equivalent with the conjecture that (3.4) is sharp.

The reader more interested in this topic is referred to the survey of R.K. Guy [193] and to the paper of Mörs [278] completely solving the case of the Zarankiewicz problem when a $2 \times p$ submatrix of an $n \times m$ 0-1 matrix is excluded.

4. Some non-degenerate extremal problems

Let R_k denote the graph determined by the vertices and edges of a regular polyhedron.[13] Clearly, $R_4 = K_4$ is the tetrahedron graph, $R_6 = K_3(2,2,2)$ is the octahedron graph, $R_8 = Q_8$ is the cube graph and R_{12}, $D_{20} = R_{20}$ are the icosahedron graphs and the dodecahedron graphs. As we have mentioned, Turán raised the question: how many edges can G_n have without containing R_k as a subgraph? For K_4 Turán's theorem yields the answer. For the cube Q_8 Theorem 3.8 describes the situation. For the dodecahedron and the icosahedron Simonovits [325, 324] gave a sharp answer. (It is strange that the simplest polyhedron, namely the cube, creates the most trouble.) To formulate some results, we need a definition.

Definition 4.1. $H(n,p,s) := T_{n-s+1,p} \otimes K_{s-1}$: we join each vertex of K_{s-1} to each vertex of $T_{n-s+1,p}$.

It turns out that in very many cases this graph is the (only?) extremal graph. Below first I will give some examples, and then, in Section 4.1 a very general theorem on the symmetric extremal graph sequences, and finally, in Section 4.2, a few further examples.

Why is $H(n,d,s)$ a good candidate to be extremal? The simpler, shorter answer is that $H(n,p,s)$ is a simple generalization of $T_{n,p}$. But then comes the question:

[13] Here $k = 4, 6, 8, 12, 20$ is the number of vertices.

why is $(T_{n,p})$ a good candidate to be the extremal graph sequence for various extremal problems? The answer is

Theorem 4.2 (Simonovits, Critical edge, [321]). *If $p(\mathcal{L})$ is defined by (3.2), and some $L_0 \in \mathcal{L}$ has an edge e for which*

$$\chi(L_0 - e) = p, \tag{4.1}$$

then there exists an n_0, such that for $n > n_0$ $T_{n,p}$ is extremal for $\mathcal{L}$, moreover, it is the only extremal graph (for each fixed $n > n_0$).

On the other hand, if (3.2) holds and for infinitely many n $T_{n,p}$ is extremal for $\mathcal{L}$, then there is an $L \in \mathcal{L}$ and an edge e in L for which $\chi(L - e) = p$.

Remarks 4.3. (a) Erdős had some results from which he could have easily deduced the above result for $p = 2$.

(b) The above theorem has the corollary that if $T_{n,p} \in \mathbf{EX}(n, \mathcal{L})$ for infinitely many n, then for $n > n_0$ there are no other extremal graphs.

(c) Those days I have formulated the meta-theorem

"Meta-Theorem" 4.4. If we can prove some results for $L = K_{p+1}$, then most probably we can extend them to any L with critical edges.

This can be seen in the Kolaitis-Prömel-Rothschild paper [234], which extends the main results of Erdős, Kleitman and Rothschild [131], and in many-many other cases of which we list only Mubayi [279], Babai–Simonovits–Spencer [28], Prömel and Steger, [291], Balogh and Butterfield [37]...

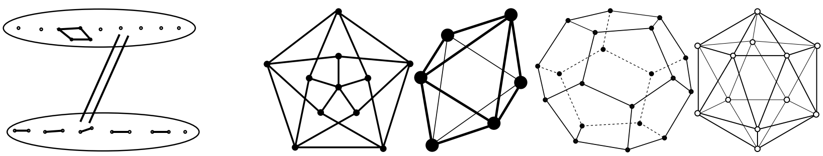
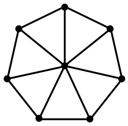


Figure 1. O_6-extremal, Grötzsch, Octahedron, Dodecahedron, Icosahedron

One interesting immediate corollary of Theorem 4.2 is the following.

Theorem 4.5. *$T_{n,2}$ is (the only) extremal graph for $L = C_{2k+1}$ for $n > n_0(k)$.*

The value of $\mathbf{ex}(n, C_{2k+1})$ can be read out from the works of Bondy [62], Woodall [373], and Bollobás [55] (pp. 147–156) concerning (weakly) pancyclic graphs for all n and k. It implies that the bound for $n_0(k)$ is $4k$ in Theorem 4.5. Füredi and Gunderson [172] gave a new streamlined proof based on works of Kopylov [250] and Brandt [71] and completely described the extremal graphs. They are unique for $n \notin \{3k-1, 3k, 4k-2, 4k-1\}$ (for $2k + 1 \geq 5$).



Another related result is that of Tomasz Dzido [103]. According to this, if we consider the even wheel $W_{2k} := K_1 \otimes C_{2k-1}$ – where we know by Theorem 4.2 that for sufficiently large n $T_{n,3}$ is the only extremal graph, Dzido also proves that

Theorem 4.6 (Dzido, even wheels [103]). *For all* $n > 6k - 10$, $\mathbf{ex}(n, W_{2k}) = \mathbf{ex}(n, K_4)$.

Theorem 4.2 immediately yields the extremal number for the 4-color-critical graphs, among others for the Grötzsch-graph seen on Fig. 1.

Theorem 4.7 (Grötzsch-extremal [321, 325, 330]). *Let* Γ_{11} *be the Grötzsch-graph on Figure 1. For* $n > n_0$, $T_{n,3}$ *is the only extremal graph.*

Theorem 4.8 (Dodecahedron theorem [325]). *For* $n > n_0$, $H(n, 2, 6)$ *is the only extremal graph for the dodecahedron graph* $D_{20} = R_{20}$.

Theorem 4.9 (Icosahedron theorem [324]). *For* $n > n_0$ $H(n, 3, 3)$ *is the only extremal graph for the icosahedron graph* R_{12}.

Let us return to the questions:
(α) "When is $H(n, p, s)$ extremal for $\mathcal{L}$?", and
(β) "When is $H(n, p, s)$ the only extremal graph for $\mathcal{L}$, for $n > n_{\mathcal{L}}$?"

In [330] I asked if there are cases when $H(n, p, s)$ is extremal graph but there are infinitely many other extremal graphs as well. Now I know that YES, there are. (We skip the details). The next question is: why is $H(n, p, s)$ extremal graph in many cases? In particular, why is $H(n, 2, 6)$ extremal for D_{20}? Of course, for such questions there are no clear cut answers, yet I try to answer this later, see Remark 4.22.

The octahedron graph problem was solved (or, at least reduced to the sufficiently well described problem of $\mathbf{ex}(n, C_4)$) by Erdős and Simonovits.

Theorem 4.10 (Octahedron theorem [137]). *If* S_n *is extremal for* R_6, *then one can find an extremal graph* A_m *for* C_4 *and an extremal graph* B_{n-m} *for* P_3 *of* $\frac{1}{2}n + O(\sqrt{n})$ *vertices each, such that* $S_n = A_m \otimes B_{n-m}$.

Clearly, B_{n-m} is either a set of $(n-m)/2$ independent edges or a set of $\frac{1}{2}(n-m-1)$ independent edges and an isolated vertex.

Some very similar theorems can be found in Griggs, Simonovits and Thomas [192], see Section 15.1, and some general results on $L = K_p(a, b, c, \ldots, c)$ in [137].

In the late 60's and early 70's some basic techniques were found, mainly by Erdős and Simonovits, to prove non-degenerate extremal graph theorems. Often sharp solutions are given in terms of the solution of some degenerate problems. This is the case in the Octahedron theorem (which is the simplest case of some more general theorems [137]). The reason of this phenomenon is discussed in details in [326], [327] and

[329]. Further, many particular extremal graph results can mechanically be deduced from a fairly general theorem of Simonovits [325]. This is the case e.g. with Moon's theorem, [275] or with the dodecahedron theorem. In some other cases, e.g, in the case of the icosahedron, this deduction is possible but not too easy.

Questions related to this will be discussed in the next subsection.

4.1. How to solve non-degenerate extremal problems?

Given a family $\mathcal{L}$ of forbidden subgraphs, beside the subchromatic number $p(\mathcal{L})$ defined in (3.2) the so called "Decomposition family" of $\mathcal{L}$ is the second most important factor influencing $\mathbf{ex}(n,\mathcal{L})$ and $\mathbf{EX}(n,\mathcal{L})$. So first we define it, then give a few examples and show how it influences the extremal structures.

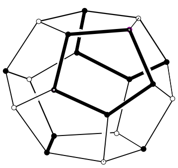
Definition 4.11 (Decomposition $\mathbb{M}$ of $\mathcal{L}$). Given a family $\mathcal{L}$ of forbidden subgraphs, with a p defined by (3.2), we collect in $\mathbb{M}$ those graphs M for which there exists an $L \in \mathcal{L}$, such that $M \otimes K_{p-1}(v(L),\dots,v(L))$ contains L. [14]

In other words, $M \in \mathbb{M}$ if putting[15] it into a class A_i of a large $T_{n,p}$, the resulting graph contains some $L \in \mathcal{L}$. The extremal graph problem of $\mathbb{M}$ is always degenerate, since $p+1$-coloring some $L_0 \in \mathcal{L}$ and taking subgraphs spanned by any two color-classes of L_0 we get (several) bipartite $M \in \mathbb{M}$.

In the general results of Erdős [114, 115] and myself [321] we proved that comparing an extremal graph for $\mathcal{L}$ and $T_{n,p}$, the error terms are determined up to some multiplicative constants, by $\mathbf{ex}(n,\mathbb{M}(\mathcal{L}))$.

EXAMPLES

(a) If $\mathcal{L} = \{K_{p+1}\}$, then $\mathbb{M}(\mathcal{L}) = \{K_2\}$. More generally, if there is an $L \in \mathcal{L}$ of minimum chromatic number: $\chi(L) = p(\mathcal{L})+1$, and there is a critical edge $e \in E(L)$, i.e., $\chi(L-e) = p$, then $\mathbb{M} = \{K_2\}$.



(b) If $\mathcal{L} = \{D_{20}\}$, the Dodecahedron graph, then $6K_2 \in \mathbb{M}(\mathcal{L})$ where $6K_2$ is the graph consisting of 6 independent edges. However, $\mathbb{M}(D_{20})$ contains also $C_5 + P_4 + K_2$, see the figure.

(c) If $\mathcal{L} = \{R_{12}\}$, the Icosahedron graph, then $P_6, 2K_3 \in \mathbb{M}(\mathcal{L})$.

(d) The decomposition class of $\mathcal{L} = \{K_3(a,b,c)$ consists of $K(a,b)$, if $a \le b \le c$.

Remark 4.12. The Decomposition family does not (always) determine the extremal graphs. Thus e.g., $K(2,2,2)$ and $K(2,2,3)$ have the same decomposition, however, by [137], their extremal numbers are different.

[14] To get finite families $\mathbb{M}$ when $\mathcal{L}$ is finite, we may also assume that M is minimal for the considered property, or at least $M \subseteq L$.

[15] "putting" means selecting $v(M)$ vertices in this class and joining them so that the resulting subgraph be isomorphic to M.

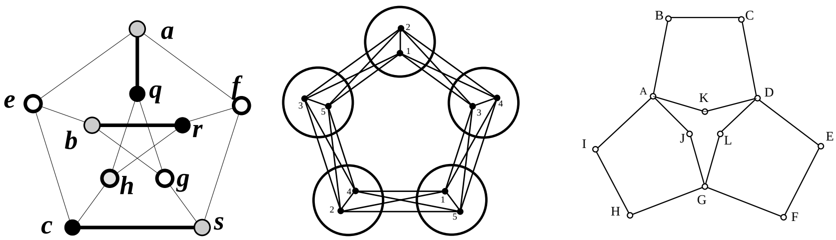
4.2. Some further examples

If the decomposition $\mathbb{M}(\mathcal{L})$ contains a tree (or forest), then the remainder terms in the general theorems become linear. A subcase of this, when $\mathbb{M}(\mathcal{L})$ contains a path (or a subgraph of a path) is described in my paper [325].

Giving a lecture in Štiřin (1997) I wanted to illustrate the general power of these results to solve extremal graph problems. So I selected one excluded graph from Łuczak's lecture, another one from Nešetřil's lecture, seen in Fig. 2. I called in [330] these graphs of Fig. 2 accordingly Łuczak and Nešetřil graphs.

Theorem 4.13 (Łuczak-extremal). *For $n > n_0$, $H(n, 4, 2)$ is the only extremal graph for the Łuczak graph L_{10}.*

Theorem 4.14 (Nešetřil-extremal). *For $n > n_0$, $H(n, 2, 2)$ is the only extremal graph for the Nešetřil-graph N_{12}.*



(a) Petersen graph (b) Łuczak Graph (b) Nesetril graph

Figure 2. Some excluded subgraphs

Theorem 4.15 ($H_{n,p,k}$-theorem). *(i) Let $L_1, \ldots, L_\lambda$ be given graphs with* $\min \chi(L_i) = p + 1$. *Assume that omitting any $k-1$ vertices of any L_i we obtain a graph of chromatic number $\geq p+1$, but L_1 can be colored in $p+1$ colors so that the subgraph of L_1 spanned by the first two colors is the union of k independent edges and (perhaps) of some isolated vertices. Then, for $n > n_0(L_1, \ldots, L_\lambda)$, $H_{n,p,k}$ is the (only) extremal graph.*

(ii) Further, there exists a constant $C > 0$ such that if G_n contains no $L_i \in \mathcal{L}$ and

$$e(G_n) > e(H_{n,p,k}) - \frac{n}{p} + C,$$

then one can delete $k-1$ vertices of G_n so that the remaining G_{n-k+1} is p–colorable.

This theorem is strongly connected with Theorem 4.2. [325] and [330] contain much more general theorems than the above ones, these are just illustrations of the general results. Without going too much into details, I define a sequence of symmetric graphs and provide a fairly general theorem.

Definition 4.16. $\mathcal{G}(n,p,r)$ is the family of graphs G_n, where $V(G_n)$ can be partitioned into $p+1$ classes $U_1,\dots,U_p$ and W with

$$\left||U_i| - \frac{n}{p}\right| < r, \qquad |W| < r$$

where $G[U_i]$ is the vertex-disjoint union of the connected, pairwise isomorphic subgraphs of G_n, the "blocks" $B_{i,j}$. Further, each $x \in W$ is joined – for each $i = 1,\dots,p$ – to each block $B_{i,j}$ in the same way: the isomorphisms $\psi_{i,j} : B_{i,1} \to B_{i,j}$ are fixed and $x \in W$ is joined to a $y \in B_{i,1}$ iff it is joined to each $\psi_{i,j}(y)$.

Theorem 4.17. *If $\mathbb{M}(\mathcal{L})$ contains a path P_τ then there exists an r such that for every sufficiently large n, $\mathcal{G}(n,p,r)$ contains an extremal graph $S_n \in \mathbf{EX}(n,\mathcal{L})$.*

This theorem helps to prove many extremal graph results. Some other results of [325] ensure the uniqueness of the extremal graphs, too. One reason why these results are easily applicable in several cases is that they apply not only to ordinary extremal graph problems but to extremal graph problems with "chromatic conditions".

Assume that instead of only excluding subgraphs from $\mathcal{L}$ we also have some additional conditions on G_n:

> Consider a graph property $\mathcal{P}$ and assume that $G_n \in \mathcal{P}$. Does this change the maximum in a Turán type problem?

Denote by $\mathbf{ex}(n,\mathcal{L},\mathcal{P})$ the maximum of $e(G_n)$ under the condition that G_n has no subgraphs from $\mathcal{L}$ and satisfies $\mathcal{P}$. Mostly we think of "chromatic properties" (see Definition 4.18).

Clearly, if no $\mathcal{L}$-extremal graph has property $\mathcal{P}$, then $\mathbf{ex}(n,\mathcal{L},P) < \mathbf{ex}(n,\mathcal{L})$. If the condition is that $\chi(G_n) > t$, for some $t > p$, that will only slightly diminish the maximum: we can take a fixed graph H_v of high chromatic number and high girth and then consider $H_v + T_{n-v,p}$.[16]

Definition 4.18 (Chromatic conditions). The chromatic property $\mathcal{C}_{s,t}$ is the family of graphs from which one can not delete s vertices of L to get a t-chromatic graph.

Theorem 4.19. *Assume that $\mathcal{L}$, s,t are given, and $\mathbf{ex}(n,\mathcal{L},\mathcal{C}_{s,t})$ is the maximum number of edges an $\mathcal{L}$-free $G_n \in \mathcal{C}_{s,t}$ can have. If $\mathbb{M}(\mathcal{L})$ contains a path P_τ then there exists an r such that for every sufficiently large n, $\mathcal{G}(n,p,r)$ contains an extremal graph $S_n \in \mathbf{EX}(n,\mathcal{L},\mathcal{C}_{s,t})$.*

Theorem 4.17 can be used to solve the extremal graph problem "algorithmically", since W and $B_{i,\ell}$ have bounded sizes. The details are omitted.

Below we describe an algorithms to solve extremal graph problems: This algorithm works if we know the appropriate information on $\mathcal{L}$.

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[16] There is an exception when $\mathcal{L}$ contains some trees.

Algorithm 4.20 (The stability method). (a) We look for a property $\mathcal{P}$ which we feel is an important feature of the conjectured extremal graphs S_n.

(b) Show that if G_n does not contain some $L \in \mathcal{L}$ and does not have the property $\mathcal{P}$, then $e(G_n)$ is significantly smaller than the conjectured extremal number.

(c) This shows that all the extremal graphs have property $\mathcal{P}$. Using this extra information we prove the conjectured structure of the extremal graphs.

Example 4.21. If the decomposition class $\mathbb{M}$ contains an M consisting of r independent edges, then we can immediately see that if any $B_{i,\ell}$ has at least two vertices (and therefore, being connected, has an edge), then the symmetric graph sequences contain some L, a contradiction. Hence the blocks $B_{i,\ell}$ reduce to vertices. Therefore any $x \in W$ is either joined to each vertex of U_i or to none of them. Now it is not too difficult to see that the extremal graphs must be (almost) the $H(n,p,k)$ graphs: The only difference which can occur is that the vertices of degree $n - O(1)$ do not necessarily form a complete subgraph.

Remark 4.22. So we have seen that if the decomposition class $\mathbb{M}(\mathcal{L})$ contains an M consisting of independent edges, then we have can apply the theorems from [325] and have a good chance to have $H(n,p,s)$ as the extremal graph.

Following this line, one can easily deduce Theorem 4.15 from Theorem 4.19. The next few results follow from these theorems.

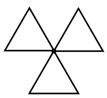
Theorem 4.23 (Petersen-extremal graphs). *For $n > n_0$, $H_{n,2,3}$ is the (only) extremal graph for the Petersen graph $\mathbb{P}_{10}$.*

(An alternative proof of this can be derived from Theorem 4.30 of the next section.)

I close this part with two cases, when Theorem 4.17 is applicable but the extremal graph is not a $H(n,p,s)$. Both results follow from Theorem 4.15.[17] Let $\mathcal{L}_{k,\ell}$ denote the graphs with k vertices and ℓ edges.

Theorem 4.24 (Simonovits [323]). *Let k be fixed and $\ell := e(T_{k,p}) + b$, for $1 \le b \le k/(2p)$. If n is sufficiently large, then*

$$\mathbf{ex}(n, \mathcal{L}_{k,\ell}) = e(T_{n,p}) + b - 1.$$



A theorem of Erdős, Füredi, Gould, and Gunderson determines $\mathbf{ex}(n, F_{2k+1})$, where $F_{2k+1} := (kK_2) \otimes K_1$: k triangles with one common vertex. Clearly, here the Decomposition class contains a kK_2, hence Theorem 4.17 is applicable. Yet the extremal graph is not a $H(n,2,s)$, since even one vertex completely joined to a $T_{2k,2}$ creates an F_{2k+1}. (For even k, the extremal graph is obtained from a $T_{n,2}$ by putting two K_k's into its first class.)

[17] They can be obtained directly, by much simpler arguments, as well.

4.3. Andrásfai-Erdős-Sós type theorems

We have seen that $\mathbf{ex}(n, \mathcal{L}) - \mathbf{ex}(n, \mathcal{L}) = O(n)$ if $\mathcal{P}$ is that $\chi(G_n)$ is high. The situation completely changes if we try to maximize $d_{\min}(G_n)$, instead of $e(G_n)$.

Theorem 4.25 (Andrásfai-Erdős-Sós [24]). *If G_n does not contain K_p, and $\chi(G_n) \geq p$, then*

$$d_{\min}(G_n) \leq \left(1 - \frac{1}{p - \frac{4}{3}}\right) n + O(1).$$

Comparing this with Turán's theorem, where $d_{\min}(T_{n,p-1}) \approx (1 - \frac{1}{p-1})n$, we see that because of the extra condition $\chi(G_n) \geq p$, the maximum of $d_{\min}(G_n)$ dropped by $c_p n$, for some $c_p \approx \frac{1}{3p^2} > 0$. Below we shall need

Definition 4.26 (Blowing up a graph). Given a graph M_v, its blown-up version $M[a_1, \dots, a_v]$ is a graph where each vertex $x_i \in V(M_v)$ is replaced by a set X_i of a_i independent vertices (and these X_i's are disjoint) and we join a $u \in X_i$ and a $w \in X_j$ if the original vertices x_i and x_j were joined in M_v. If $a_1 = \dots = a_v = a$, then we use the simpler notation $M[a]$.

To generalize Theorem 4.25, Erdős and Simonovits [139] defined

$$\psi(n, L, t) := \max\{e(G_n) \ : \ L \not\subseteq G_n \text{ and } \chi(G_n) \geq t\},$$

where L is a fixed excluded graph, t is fixed, and $n \to \infty$. Using this language and including some further results of [24], we can say that

Theorem 4.27 (Andrásfai-Erdős-Sós [24]).

$$\psi(n, K_p, p) = \left(1 - \frac{1}{p - \frac{4}{3}}\right) n + O(1). \tag{4.2}$$

For $n > n_0$, the extremal graph S_n for this problem is a product: $S_n = T_{m,p-3} \otimes C_5[a_1, a_2, \dots, a_5]$, where the parameters m and a_i should be chosen to maximize $e(S_n)$ among these structures.

The above description of S_n almost completely determines its structure: if $T_{m,p-3} = K_{p-3}(m_1, \dots, m_{p-3})$, then

$$a_i = \frac{n}{3n-4} + O(1) \qquad \text{and} \qquad m_i = \frac{3n}{3n-4} + O(1).$$

To formulate a more general and sharper result, assume that

$$L \text{ has a critical edge: an } e \text{ for which } \chi(L - e) < \chi(L). \tag{4.3}$$

Theorem 4.28 (Erdős-Simonovits [139]). *If $\chi(L)=p$ and L has a critical edge, then, for $n>n_0(L)$,*

$$\psi(n,L,p)\le\psi(n,K_p,p).$$

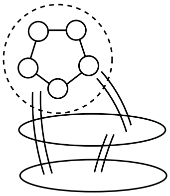


Fig. 3: Extremal structure

Actually, equality may hold only for $L=K_p$.

Theorem 4.29 (Erdős-Simonovits [139]). *Let $\chi(L)=p$ and $L\ne K_p$ satisfy (4.3). Then, for $n>n_0(L)$,*

$$\psi(n,L,p)\le\left(1-\frac{1}{p-\frac{3}{2}}\right)n+O(1). \tag{4.4}$$

Of course, this theorem does not cover the case of the Petersen graph: it has no critical edge. Figure 2 shows that one can delete 3 independent edges from $\mathbb{P}_{10}$ to get a bipartite graph. Moreover, if $T(v,p,s)$ is the graph obtained from $T_{n,p}$ by putting s independent edges into the first class of $T_{n,p}$, then Figure 2 shows that $\mathbb{P}_{10}\subseteq T_{12,2,3}$. So the "stability" of $\mathbb{P}_{10}$-extremal graphs is covered by

Theorem 4.30 (Simonovits [330]). *For every v (and $t\le v/2$) there exists a $K=K(v)$ such that if*

$$d_{\min}(G_n)>\frac{2}{5}n+K$$

and $T_{v,2,t}\not\subset G_n$, then one can delete K vertices of G_n to get a bipartite graph.

Remarks 4.31. (a) Theorem 4.30 is sharp, as shown by $C_5[\frac{1}{5}n]$. Clearly, $\delta(C_5[\frac{1}{5}n])\ge\frac{2}{5}n-2$ and $T_{v,2,t}\not\subset C_5[\frac{1}{5}n]$. Further, replacing $T_{v,2,t}$ by any graph $L\subseteq T_{v,2,t}$ we get the same sharpness if $K_3\subseteq L$, since $C_5[\frac{1}{5}n]$ contains no K_3.

(b) Moreover, Theorem 4.30 is sharp also for $\mathbb{P}_{10}$: one can relatively easily show that $\mathbb{P}_{10}$ cannot be embedded into $C_5[\frac{1}{5}n]$.

(c) The theorem is *not sharp* if $\chi(L)=3$ and $L\subseteq C_5[\mu]$ for some μ.[18]

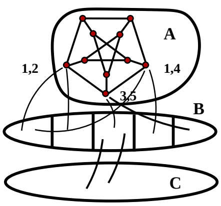


Fig. 4: Hajnal Construction

The real question was if $\psi(n,K_3,t)\le c_tn+o(n)$ for some constants $c_t\to 0$ as $t\to\infty$. In other words, is it true that if the chromatic number tends to ∞, we can push down the degree density arbitrarily?

In [24] it was conjectured that YES, however, it turned out in the Erdős-Simonovits paper [139] that NO. This follows from Construction 4.33 of A. Hajnal below.[19] For this we shall need the definition of the Kneser graph $\mathbf{KN}(2k+\ell,k)$. Its vertices are

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[18] $C_{2\mu+1}\subseteq C_5[\mu]$ for $\mu>1$.

[19] I think that this construction was found by Hajnal, but now that I reread our paper, I cannot exclude that it was found by Erdős and Hajnal.

the k-subsets of a $(2k+\ell)$-element set U and we join $X, Y \subseteq U$ if $X \cap Y = \emptyset$. It is easy to color $\mathbf{KN}(2k+\ell, k)$ with $\ell+2$ colors. The Petersen graph $\mathbb{P}_{10} = \mathbf{KN}(5,2)$ is the simplest non-trivial Kneser graph.

Theorem 4.32 (Kneser Conjecture, Lovász Theorem [262]).

$$\chi(\mathbf{KN}(2k+\ell, k)) = \ell + 2. \tag{4.5}$$

Construction 4.33 (A. Hajnal, in [139]). Let $k, \ell, h \to \infty$, $\ell = o(k)$, $k = o(n)$. Our graph H_n has $n \approx 3h$ vertices partitioned into three groups $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, where

$$H[\mathbb{A}] = \mathbf{KN}(2k+\ell, k), \qquad |\mathbb{B}| \approx 2h, \qquad |\mathbb{C}| \approx h.$$

(Case $k = 2$, $\ell = 1$ can be seen in Figure 4).

(a) Each vertex v of $\mathbf{KN}(2k+\ell, k)$ is a subset of $\{1, \dots, 2k+\ell\}$: call its elements the “names” of v. The vertices of $\mathbb{B}$ are partitioned into $2k+\ell$ subclasses B_j. $j = 1, 2, \dots, 2k+\ell$ of approximately equal sizes. We join the vertices of B_j to those vertices of $\mathbb{A}$ whose name-set contains j. Finally, join each vertex from $\mathbb{C}$ to each one of $\mathbb{B}$.

Let us verify the implicitly or explicitly stated properties of H_n. $\chi(H_n) \ge \ell + 2$, by (4.5). H_n contains no K_3, because there are no edges between $\mathbb{C}$ and $\mathbb{A}$, so all the triangles have to be in $\mathbb{A} \cup \mathbb{B}$. However, $\mathbb{A}$ does not contain K_3’s, and by the “name-rule”, if $x, y \in \mathbb{A}$ are connected, then they have no common neighbors in $\mathbb{B}$. Finally, if $k, \ell, n \to \infty$, $k = o(n)$, $\ell = o(k)$, then $d_{\min}(H_n) \ge n/3 - o(n)$, since the vertices $x \in \mathbb{A}$ have

$$d(x) \approx \frac{k}{2k+\ell} \frac{2n}{3}, \tag{4.6}$$

because of the name-rule, while for the vertices of $\mathbb{B}$ (4.6) is trivial; for an $x \in \mathbb{C}$, $d(x) = \frac{2}{3}n - o(n)$.

Remark 4.34. When we described this construction originally, the Kneser Conjecture was still unproved: we used a much weaker assertion (an unpublished argument of Szemerédi, based on a theorem of Kleitman) that $\chi(\mathbf{KN}(2k+\ell, k)) \to \infty$. Soon the Kneser conjecture was proved by Lovász [262], then an alternative proof was given by Bárány [44] and then many nice results were proved, of which we mention here just one, due to Schrijver [315], describing the color-critical subgraphs of $\mathbf{KN}(m, k)$.

There are many interesting related results in this area. We mention here only a few of them:

Theorem 4.35 (Häggkvist [197], Guoping Jin [207]).

$$\psi(n, K_3, 4) = \frac{11}{29} n + O(1).$$

The sharpness of this result follows from an "optimally" blown up version of the Grötzsch graph, where "optimally" means that n vertices are partitioned into 11 classes $U_1, \dots .U_{11}$ and the classes are joined as in the Grötzsch graph, however the proportions are chosen so that the number of edges be maximized, which happens when each degree is approximately the same. Improving earlier an result of Thomassen [355], Łuczak proved

Theorem 4.36 (Łuczak [268]). *For every $\varepsilon > 0$ there exists an $L = L(\varepsilon)$ such that if G_n is triangle-free and $d_{\min}(G_n) > (\frac{1}{3} + \varepsilon)n$, then G_n is contained in some blown up version of a triangle free H_m for some $m \le L(\varepsilon)$.*

As Erdős and myself, using the construction of Hajnal, pointed out such a result does not hold below $n/3$, more precisely, with an $\varepsilon < 0$. The results above leave open the case $\varepsilon = 0$ which was very recently answered by Brandt and Thomassé [74], who also completely described the structure of triangle free graphs G_n with $d_{\min}(G_n) > n/3$. Their results imply

Theorem 4.37. *All graphs G_n with $d_{\min}(G_n) > \frac{1}{3}n$ are 4-colorable.*

4.4. The structure of dense L-free graphs

Below we shall write $G \to H$ if H contains a homomorphic image of G, or, in other words, a blown up version $H(t)$ of H contains G. To avoid too technical arguments, we restrict ourselves to the 3-chromatic case. For a graph L we define

$$\begin{aligned} \xi(L) &= \max\left\{m : m \text{ is odd and } L \to C_m\right\} \\ &= \max\left\{m : m \text{ is odd and } L \subseteq C_m[v(L)]\right\}. \end{aligned}$$

Note that if $\chi(L) = 3$, then $\xi(L)$ cannot be larger than $girth_{\text{odd}}(L)$, the length of the shortest odd cycle contained in L. Finally, by $\beta(G)$ we denote the minimum number of edges that must be deleted from G to make it bipartite.

In this section we study the structure of L-free graphs of large minimum degree for a general 3-chromatic graph L. Our main result can be stated as follows.

Theorem 4.38 (Łuczak and Simonovits [271]). *Let L be a 3-chromatic graph. Then for every $\alpha, \eta > 0$, there exists an n_0 such that for every L-free graph G with $v(G) = n \ge n_0$ and*

$$d_{\min}(G) > \left\lceil \frac{2n}{\xi(L) + 2} \right\rceil + \eta n\,, \tag{4.7}$$

we have $\beta(G) \le \alpha n^2$.

Furthermore, for every $\alpha > 0$ there exist an $\bar{\eta} > 0$ and an $\bar{n}_0$ such that each L-free graph G with $v(G) = n \ge \bar{n}_0$ and

$$d_{\min}(G) > \left\lceil \frac{2n}{\xi(L) + 2} \right\rceil - \bar{\eta} n\,, \tag{4.8}$$

contains a subgraph G' with at least $e(G) - \alpha n^2$ edges such that $G' \to C_{\xi(L)+2}$.

Similar but sharper results were proved by Győri, Nikiforov and Schelp for the special case when L is an odd cycle.

Theorem 4.39 (Győri, Nikiforov and Schelp [196]). *If a non-bipartite graph G_n has minimum degree $d_{\min}(G_n) \geq n/(4k+2) + c_{k,m}$, where $c_{k,m}$ does not depend on n and n is sufficiently large, and if $C_{2s+1} \subset G_n$ for some $k \leq s \leq 4k+1$ then $C_{2s+2j+1} \subset G_n$ for every $j = 1, \dots, m$.*

They describe the structure of all graphs on n vertices with $d_{\min}(G_n) \geq n/(4k+2)$ not containing odd cycles longer than $2k+1$. In particular they prove that these graphs can be made bipartite by deletion of a fixed number of edges or vertices.

Further sources to read: Alon and Sudakov [22].

5. Problem of Supersaturated Graphs

5.1. Counting complete subgraphs

For the sake of simplicity we restrict ourselves to the case when $\mathcal{L}$ has only one member L. By definition, if $e(G_n) = \mathbf{ex}(n, L) + 1$, then G_n contains an L. It is rather surprising that generally $e(G_n) > \mathbf{ex}(n, L)$ ensures much more than just one L. The first result in this direction is an unpublished theorem of Rademacher (1941) according to which a graph G_n with $\left[\frac{n^2}{4}\right] + 1$ edges contains at least $\lfloor \frac{n}{2} \rfloor$ copies of K_3. This was immediately generalized by

Theorem 5.1 (Erdős [109]). *There exists a constant $c > 0$ such that if $e(G_n) = \left[\frac{n^2}{4}\right] + k$, $1 \leq k \leq cn$, then G_n contains at least $k\lfloor \frac{n}{2} \rfloor$ copies of K_3.*

$T_{n,2,k}$ shows that this result is sharp, apart from the value of c. Indeed, $e(T_{n,2,k}) = \left[\frac{n^2}{4}\right] + k$ and it has only $k\lfloor \frac{n}{2} \rfloor$ triangles. Later Erdős extended this result to K_{p+1} and graphs G_n with $e(T_{n,p}) + k$ edges [117]. Many similar results were proved by Erdős [117, 112], Moon and Moser [276], Bollobás [53], [54], Lovász and Simonovits, [264, 265].

For complete graphs Lovász and Simonovits proved a conjecture of Erdős and formulated a general conjecture in [264, 265] which they could prove only for special values of $k = e(G_n) - \mathbf{ex}(n, K_{p+1})$, namely, when $k \in [1, \varepsilon n^2]$,[20]. Later, in several steps it was solved by Fisher, [158, 159], Razborov [295], Nikiforov [286] and finally, "completely", by Reiher [297].

[20] More precisely, when for some $q \geq p$, $e(T_{n,q}) < e(G_n) < e(T_{n,q}) + \varepsilon_q n^2$.

We have already mentioned the "meta-theorem" that if one can prove a result for K_p, then one can also prove it for graphs with critical edges. One example of this is

Theorem 5.2 (D. Mubayi, [279]: critical edges). *Let L be $p+1$-chromatic with a critical edge. Let $c(n,L)$ be the minimum number of copies of L produced by the addition of an edge to $T_{n,p}$. There exist $n_0(L)$ and $\delta(L)$ such that every graph G_n of order $n > n_0$ with $e(G_n) = \mathbf{ex}(n, K_{p+1}) + k$ edges contains at least $kc(n,L)$ copies of L, provided $k \le \delta n$.*

The proof uses the graph removal lemma and the Erdős-Simonovits stability theorem.

5.2. General sample graphs

Turning to the general case we fix an arbitrary L and call a graph G_n supersaturated if $e(G_n) > \mathbf{ex}(n,L)$. The problem is, at least how many copies of L must occur in a G_n with $\mathbf{ex}(n,L)+k$ edges. Erdős and Simonovits [140] proved that

Theorem. *For every $c > 0$ there exists a $c^* > 0$ such that if $e(G_n) > \mathbf{ex}(n,L) + cn^2$ and $v = v(L)$, then G_n contains at fewest $c^* n^v$ copies of L.*

Further sources to read: The reader interested in further information is suggested to read the papers of Lovász-Simonovits on structural stability [265], Erdős–Simonovits, [140], or Brown–Simonovits [85], or my survey [328].

5.3. Razborov's method, Flag algebras

Given a graph G_n, we may count the occurrences of several possible subgraphs in it. Denote by $c(L, G_n)$ the number of occurrences of L in G_n. Inequalities for such "counting functions" were the basic tools in several cases, see e.g. [252], [276] [265]. The connection between Supersaturated Graph theorems and proofs of ordinary extremal graph problems was discussed e.g. in [328]. In the last few years Razborov has developed a new method which enables the researchers to apply computers to prove inequalities between counting functions on a graph. This method turned out to be very successful and popular. To describe it and its applications would go far beyond our scope. I just mention one of the first papers of A. Razborov [293] and his very recent survey [296] on this topic, or Keevash [218].

5.4. The general case, bipartite graphs

As we have mentioned, the theory of supersaturated graphs started with Rademacher's theorem, and the first few papers in the field counted complete subgraphs of supersaturated graphs, [117], [100]…(Perhaps one exception should be mentioned here:

counting walks in graphs, e.g. Blakley and Roy [49] that was found independently also by [282], [260]. Counting walks is important e.g., if we wish to get information on the eigenvalues of a graph.)

The theory of supersaturated graphs is completely different for (a) the case when the excluded graph, L is bipartite, and (b) when it is not. The case when it is bipartite is described in details in [180], and from other viewpoints, in my survey, [328], so I will describe the situation here only very shortly.

For $e(G_n) \le \mathbf{ex}(n, L)$, of course, it may happen that G_n contains no copies of L. As soon as we go above $\mathbf{ex}(n, L)$, we immediately have very many copies. Yet, to give a precise description is hopeless, even for one of the the simplest cases, for C_4: we do not know enough of the finite geometries to tell, how many C_4 must occur in G_n if $e(G_n) = \mathbf{ex}(n, C_4) + 1$.

Erdős and I conjectured (see [328]) that if $\chi(L) = 2$ then for every $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ such that if $e(G_n) > (1 + \varepsilon)\mathbf{ex}(n, L)$, then G_n contains at least $\eta n^{v(L)}$ copies of L. We also formulated a weaker conjecture, asserting that – for any fixed L – there exist a (small) $\eta > 0$ and a $C > 0$ such that if $e(G_n) > C\mathbf{ex}(n, L)$, then G_n contains at least $\eta n^{v(L)}$ copies of L. It is also mentioned (implicitly?) in [328] that these conjectures mean that the random graph has the fewest copies of L.[21] Sidorenko [319], [320] considered dense graph sequences, turned the corresponding inequalities into integrals, the error terms disappeared, and he formulated more explicitly that for given number of edges the Random Graph has the least copies of L..

Today this became one of the most important conjectures in this area. The simplest case when the conjecture is unknown is when L is obtained from a $K(5, 5)$ by deleting edges of a C_{10}. We could mention here several results, however basically we refer the reader to [180] and mention only Simonovits, [328], Conlon, Fox and Sudakov [95].

Remark 5.3. Earlier we always first proved an extremal graph theorem and then the corresponding supersaturated graph theorem. Today this is not quite so: For $k \ge 4$ we do not really know any reasonable upper bound on $\mathbf{ex}(n, Q_{2^k})$ (for the k-dimensional cube), while the corresponding Erdős-Simonovits-Sidorenko conjecture is proved by Hatami [199]. This may seem to be surprising, however, the Sidorenko Conjecture is about *dense* graphs.

5.5. Ramsey-supersaturated?

The general question would be (though not the most general one) that if we have a sample graph L and $n > n_0$, and we r-color K_n, at least how many monochromatic

[21] Those days quasi-random graphs were “non-existent”, today we know that from this point of view the random and the quasi-random graphs are indistinguishable.

subgraphs must occur.[22] The simplest case is to determine

$$\min\left(c(K_p, G_n) + c(K_p, \overline{G_n})\right).$$

For K_3 the answer is relatively easy, see Goodman [184]. Erdős conjectured [110] that the minimum is achieved by the Random Graph. This was disproved by Thomason [354]. (See also [205].)

6. Regularity Lemma

When the Szemerédi Regularity Lemma [349] "arrived", first it seemed something too complicated. The reason for this was that those days most graph theorists felt uneasy about having this "approximation type statements".[23]

Today we know that (a) it is not that complicated and that (b) it is one of the *most important* tools in Extremal Graph Theory. This is not the place to explain it. Surveys like Komlós-Simonovits [249], [248] describe sufficiently well the usage of the Regularity Lemma in our setting, for "dense graph sequences"[24], several excellent newer surveys are also available, like Kohayakawa and Rödl [229], Rödl and Schacht [302], Gerke and Steger [183], and many others. Yet, for the sake of completeness we formulate it.

6.1. The original regularity lemma

Definition 6.1 (ε-regular pairs). The pair of two disjoint vertex-sets, $A, B \subseteq V(G)$ is ε-regular in G, if for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, we have

$$\left|\frac{e(X,Y)}{|X||Y|} - \frac{e(A,B)}{|A||B|}\right| < \varepsilon. \tag{6.1}$$

Theorem 6.2 (Szemerédi Regularity Lemma). *For every $\kappa > 0$ and $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon, \kappa)$ such that for each graph G_n, $V(G_n)$ can be partitioned into $k \in (\kappa, k_0)$ vertex-sets $(U_1, \ldots, U_k)$, of $\le \lceil n/k \rceil$ vertices (each), so that for all but $\varepsilon\binom{k}{2}$ pairs (U_i, U_j) ($1 \le i < j \le k$) the subgraph $G[U_i, U_j]$ induced by U_i, U_j is ε-regular.*

The meaning of this "lemma" is that any graph can be approximated by a "generalized random graph". Its applicability comes from the fact that embedding certain structures into randomlike graphs is much easier than into arbitrary graphs. This approximation helps us to prove (instead of statements on "embedding into arbitrary graphs") the simpler assertions on "embedding into generalized random graphs".

---

[22] A related question is, how many monochromatic forbidden subgraphs appear near the Ramsey bound, see e.g., Rosta and Surányi, [307], Károlyi and Rosta [212],. . .

[23] Harary, e.g., did not like assertions containing statements like "for $n > n_0$". . .

[24] where $e(G_n) > cn^2$ for some constant $c > 0$ as $n \to \infty$.

The Regularity Lemma completely changed that part of graph theory we are considering here. There are many excellent introductions to its applications. One of the first ones was that of Komlós and myself [249], or its extension, [248].

Remarks 6.3. (a) The Regularity Lemma can be applied primarily when a graph sequence (G_n) is given with positive edge density: $e(G_n) > cn^2$, for some fixed $c > 0$.

(b) For ordinary graphs it has several weaker or stronger versions, and one could assert that if one knows the statement, the proofs are not that difficult: the breakthroughs came from finding the right Regularity Lemma versions.

(c) For hypergraphs the situation completely changes: the regularity lemmas are much more complicated to formulate and often their proofs are also very painful (?). For a related survey see the PNAS paper of Rödl, Nagle Skokan, Schacht and Kohayakawa [298] and the "attached" Solymosi paper [337], and Gowers, [188], and Tao [351].

(d) Regularity Lemmas are connected with "removal lemmas", and "counting lemmas" however, for ordinary graphs they are easy, while for hypergraphs they are much deeper.

(e) Regularity Lemmas can be applied to sparse graph sequences (G_n) as well, [225, 228] assumed that the graphs G_n satisfy some technical assumptions, according to which they do not have too dense subgraphs. Subgraphs of random graphs satisfy this condition, therefore Sparse Regularity Lemmas were applicable in several cases for non-random subgraphs of sparse random graphs.

(f) Regularity Lemmas were "invented" to ensure small subgraphs of given properties of a graph G_n. Later Komlós, G.N. Sárközy, and Szemerédi started using it to ensure spanning subgraphs. This is for what the "Blow Up Lemmas" were invented, see Komlós, [245], Komlós, Sárközy, Szemerédi, [240]. Later they worked out algorithmic versions of the Blow-Up lemma too [242], (see also Rödl and Ruciński [300]) and hypergraph versions (Keevash, [217]) were established. We return to this topic in Subsection 6.6.

(g) There are many cases where Regularity Lemmas are used to give a first proof for some theorems, but later it turns out that the "regularity lemma" can be eliminated.

(h) Regularity Lemmas play crucial role in the theory of quasi-randomness, in "property testing", and in the theory of graph limits.

6.2. Some newer regularity lemmas

In [249] we tried to give an easy introduction to the applications of the Regularity Lemma. We have described the earliest applications, the Alon-Duke-Lefmann-Rödl-Yuster paper [13] about the algorithmic aspects of the Regularity Lemma, which helps to turn existence theorems using the Regularity Lemma into algorithms, the Frieze-Kannan version [164] which helps to make algorithms faster, since it uses a weaker Regularity Lemma, however, with much fewer classes. Beside [164], see also [**?**]. The weak Regularity Lemma in my opinion also connects the combinatorial approach to the Mathematical Statistics, above all, to Principal Component Analysis.

There are also continuous versions of Regularity Lemmas. Here we refer the Reader to the paper of Lovász and B. Szegedy [266] and to the book of Lovász [263]. Many further remarks and references could be added here but we have to cut it short.

6.3. Regularity Lemma for sparse graphs

The Kohayakawa-Rödl version of the Szemerédi Regularity Lemma uses a "technical" assumption that the considered G_n does not contain subgraphs G_m of much higher density than G_n. Very recently Alex Scott proved a new version of the Regularity Lemma, for Sparse graphs [316]. Yet this have not solved all the problems. As Scott points out, it may happen in the applications of the Scott Lemma that most of the edges are in the "wrong place". We skip the details. On the connection of Random graph models and Regularity Lemmas, we mention Bollobás and Riordan [59].

6.4. Regularity Lemma and Quasi-randomness

Quasi-randomness informally means that

(Q) We consider graph sequences (G_n) and look for "properties" $\mathcal{P}_i$ that are obvious for the usual random graphs (say, from the binomial distribution $\mathcal{R}_{n,p}$) and equivalent to each other.

Here there are two notions relatively near to each other: the pseudo-random and the quasi-random graphs. The investigations in this area were initiated by Andrew Thomason (see e.g. his survey [353]) and were motivated (partly?) by Ramsey problems. Chung, Graham and Wilson [94] showed that if we weaken the error terms, then there are six properties satisfying (Q). Vera Sós and I proved that there is another property $\mathcal{P}_R$ being equivalent to quasi-randomness:

Theorem 6.4 (Simonovits–Sós [333]). *A graph sequence (G_n) is p-quasi-random in the Chung-Graham-Wilson sense iff for every κ and $\varepsilon > 0$ there exist two integers $k(\varepsilon,\kappa)$ and $n_0(\varepsilon,\kappa)$ such that for $n > n_0$ $V(G_n)$ has a (Szemerédi) partition into k classes $U_1,\dots,U_k$ (where $|U_i - n/k| \le 1$, $\kappa < k < k(\varepsilon,\kappa)$) where all but at most εk^2 pairs $1 \le i < j \le k$ are ε–regular with densities $d(U_i,U_j)$ satisfying*

$$|d(U_i,U_j) - p| < \varepsilon.$$

Several extensions exist for sparse graph sequences and hypergraph sequences, however, we do not discuss them in details. For the sparse case see, e.g., Kohayakawa and Rödl [229]. For hypergraph extensions (which are much more technical) see, e.g., Keevash [217].

6.5. Regularity lemma and property testing

Property testing is among the important "Computer Science motivated" areas. It is perhaps two steps away from Turán's results, yet I write very shortly about it. Assume that we have a graph property $\mathcal{P}$. We would like to decide if a graph $G_n \in \mathcal{P}$ or not. However, we may ask only a few questions about pairs xy if they are edges of G_n or not? For example, we would like to decide if G_n contains a given L or not. Obviously, we cannot decide this for sure – using only a few questions – unless we allow some errors in the answer: if we can change a few edges in G_n to get a $\tilde{G_n} \in \mathcal{P}$ then we accept a YES. Some of the earliest questions of this type were coming from Paul Erdős, though in somewhat different form. In the papers of Alon and Shapira it turned out that – in the reasonable cases – one can decide the question if one can decide it by applying the regularity lemma to G_n and then considering the densities between the partition classes.

6.6. Blow up lemma

In many cases we embed a small graph L into a large one, G_n. There are some exceptions, when we wish to find in G_n a Hamiltonian cycle, or a spanning tree of given structure, ... In these cases mostly (a) we have to assume some sparseness condition on L, say a bound on $d_{\max}(L)$. (b) Even if we can embed L into G_n, if $v(L) = n$, then we have to struggle with finding places for the last few vertices.

To solve this problem Komlós, G. Sárközy and Szemerédi [240] established a special "extension" of the Regularity Lemma, called the *Blow-Up Lemma*. Komlós has a survey [245] on early successes of the Blow Up lemma. This survey very nicely describes the classification of embedding problems [25] and lists several conjectures solved with the help of the Blow-Up Lemma.

We call a pair (X, Y) of vertex-sets in G_n (ε, τ)-super-regular if $|X| \approx |Y|$, it is ε-regular, $d(X, Y) \geq \tau$ and the minimum degree of $G(X, Y)$ is also at least $(d(X, Y) - \varepsilon)|X|$.[26]

Theorem 6.5 (Blow Up Lemma, short form). *For every $\delta, \Delta > 0$ there exists an $\varepsilon_0 > 0$ such that the following holds. Given a graph H_ν, and a positive integer m, and G_n and U_n are obtained by replacing every vertex of H_ν by m or $m-1$ vertices, and replacing the edges of H_ν with (ε, δ)-super-regular pairs and by complete bipartite graphs, respectively. If $L_n \subseteq U_n$ and $d_{\max}(L_n) \leq \Delta$, then $L_n \subseteq G_n$.*

The meaning of this is that if we do not have large degrees in L_n and small degrees in G_n and we apply the Regularity Lemma to G_n, and replace each of the ε-regular τ-dense pairs by complete bipartite graphs, then, if we can embed L_n into the so obtained U_n, then we can embed L_n into the original, much sparser G_n as well.

[25] fixed size L, $o(n)$ size L, $v(L) = cn$, $v(L) = n$

[26] We could define this basic notion also slightly differently.

The basic idea was (i) first to use a randomized greedy embedding algorithm for most of the vertices of the graph to be embedded and (ii) then take care of the remaining ones by applying a König-Hall type argument [240].

The Blow Up Lemma successfully solved several open problems, see e.g., Komlós, Sárközy, and Szemerédi, proving the Pósa-Seymour conjecture, [246], the Alon-Yuster conjecture [243], ... Here the Pósa-Seymour conjecture asks for ensuring the k^{th} power of a Hamiltonian cycle, (meaning that we have a Hamiltonian cycle, where all the vertices are joined whose distance on this H is at most k).

The randomization was later eliminated by Komlós, Sárközy and Szemerédi and the embedding became an algorithmic one [242]. An alternative "derandomized" proof was also given by Rödl and Ruciński [300]. This approach turned out to be extremely successful. The Blow-up lemma was also extended to hypergraphs, see Keevash [217].

When using the Regularity Lemma, or the Blow Up Lemma, we often apply some "classical" result to the Cluster Graphs. Here we often need the famous

Theorem 6.6 (Hajnal–Szemerédi [244]). *If n is divisible by p and*

$$d_{\min}(G_n) \geq \left(1 - \frac{1}{p}\right) n,$$

then $V(G_n)$ can be covered by vertex-disjoint copies of K_p.

When Hajnal and Szemerédi proved this conjecture of Erdős, that was an enormous technical achievement, but I do not think that most people in the surrounding new that this will be also an important "tool".

Further sources to read: Several related results discuss, how can one get rid of applying the Blow Up lemma (or variants of the Regularity Lemma, see, e.g. Levitt, Sárközy and Szemerédi [247]). Kühn and Osthus have a related survey [255], and Rödl and Ruciński another one [301]. See also Alon-Rödl-Ruciński [19], B, Csaba, [96].

7. Arithmetic structures and combinatorics

This will be the shortest section of this survey. Clearly, writing of the influence of Turán in Discrete Mathematics one cannot avoid the Erdős-Turán conjecture, nowadays Szemerédi's $r_k(n)$-theorem. This asserts that

Theorem 7.1 (Szemerédi [348]). *For any fixed k, if a sequence A of integers does not contain k-term arithmetic progressions, then it has only $o(n)$ elements in $[1, n]$.*

This theorem was very strongly connected to combinatorics. Szemerédi's proved and used an earlier, weaker version of his Regularity Lemma, to prove Theorem 7.1.

Vera Sós has a paper describing the origins of this conjecture [340], (based on the letters exchanged by Erdős and Turán, during the war).

Remarks 7.2. (a) Szemerédi's theorem is one of the roots of many results that connect Combinatorics (Graph Theory?) and Combinatorial Number Theory. Beside this it also connects Ergodic Theory and Combinatorial Number Theory, since Fürstenberg [181] gave an ergodic theoretic proof of it, then Fürstenberg, Katznelson [182] and others gave several generalizations, using ergodic theoretic methods. The reader is recommended to read e.g. the corresponding chapter of the book of Graham, Rothschild and Spencer [190], The same time, there are fascinating approaches to this field using deep analysis, due to Gowers, and others,[27] see recent papers of Gowers [186], or an even newer paper of Gowers [189] on these types of problems, on arithmetic progressions.

(b) Historically it may be interesting to read the first, fairly weak results of Erdős and Turán in this topic, in [150]. They start with proving that $r_3(n) < \frac{1}{2}n$. Then they prove a slight improvement, and formulate a conjecture of Szekeres which turned out to be false.

One of the most famous conjectures of Erdős was

Conjecture 7.3. If $A = (a_1, \dots, a_n, \dots)$ is a sequence of integers with

$$\sum \frac{1}{a_i} = \infty,$$

then, for any k, A contains a k-term arithmetic progression.

One motivation of this conjecture is that it would imply

Theorem 7.4 (Green-Tao [191]). *For arbitrary k there exist k-term arithmetic progressions in the set of primes.*

Further sources to read: Elek and Szegedy on the nonstandard methods in this area, [104, 105]

8. Multigraph and digraph extremal problems

Here I formulate only the digraph problem, which includes the multigraph case. Let r be fixed and consider digraphs in which for any two vertices at most r arcs of the same orientation can join them. (Hence the number of arcs joining two vertices is at most $2r$.) The problem is obvious:

[27] This approach originates from Roth.

> For a given family $\overrightarrow{\mathcal{L}}$ of digraphs what is the maximum number of arcs a digraph $\overrightarrow{D}_n$ can possess without containing any $\overrightarrow{L} \in \overrightarrow{\mathcal{L}}$?

The concepts of $\mathbf{ex}(n, \overrightarrow{\mathcal{L}})$ and $\mathbf{EX}(n, \overrightarrow{\mathcal{L}})$ are defined in the obvious way. Brown and Harary [84] started investigating multigraph extremal problems. Several general theorems were proved by W. G. Brown, P. Erdős and M. Simonovits [78], [79], [80], [81]. Some results concerning directed multihypergraphs can also be found in a paper of Brown and Simonovits [85]. For the Erdős conference in 1999 we wrote a longer survey on the topic [86]. The case $r = 1$, at least, the asymptotics of $\mathbf{ex}(n, \overrightarrow{\mathcal{L}})$ in this case, is sufficiently well described. Below we formulate only one theorem, indicating that the whole theory of digraph extremal problems is strongly connected to the theory of matrices with non-negative integer entries.

Brown-Erdős-Simonovits theorem [78]. *Let us consider digraphs where any two vertices are joined by at most one arc in each direction. Let $\overrightarrow{\mathcal{L}}$ be a given family of forbidden digraphs. Then there exists a* 0-1 *matrix A (of say t rows and columns) such that*

(a) if we partition n vertices into t classes $U_1, \dots, U_t$, and for $i \neq j$ join each vertex of U_i to each vertex of U_j, by an arc oriented from U_i, to U_j, iff $a_{i,j} = 1$, and put transitive tournaments into the classes U_i iff $a_{i,i} = 1$ (otherwise these are independent vertices) then the resulting digraph does not contain subdigraphs from $\overrightarrow{\mathcal{L}}$.

(b) One can partition n vertices into t classes $U_1, \dots, U_t$ in such a way that the resulting digraphs $\overrightarrow{D}_n$ form an almost extremal sequence: $e(\overrightarrow{D}_n)/\mathbf{ex}(n, \overrightarrow{\mathcal{L}}) \to 1$ (and $\overrightarrow{D}_n$ contains no forbidden subdigraphs).

The meaning of this theorem is that for $r = 1$ we can always find an almost extremal graph sequence of fairly simple structure, where the structure itself exclude the containment of forbidden subgraphs.

Example 8.1. (a) Let $r = 1$. Let L_3 be the following digraph: a is joined to b and c by two arcs of opposite directions and b is joined to c by one arc. The extremal structure is a $\overrightarrow{G_n}$ obtained from $\mathcal{T}_{n,2}$ replacing each edge by two arcs of opposite direction. Any tournament $\overrightarrow{T}_n$ is also an almost-extremal graph, and there are many other extremal graphs, see [86].

(b) There are digraph families for which the structure on Fig 5(a) is extremal, and for some other family $\overrightarrow{\mathcal{L}}$ the structures on 5(b)-(e) forms an extremal sequence, respectively.

Brown, Erdős, and myself had conjectures asserting that most of the results for $r = 1$ can be generalized to any fixed r, however, most of our conjectures were "killed" by some counterexamples of Sidorenko [317] and then of Rödl and Sidorenko [304].

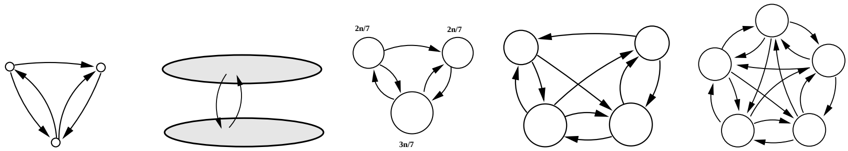

Figure 5. (a) Excluded (b), (c), (d) and (e) extremal structures for some $\overline{\mathcal{L}}$

9. Hypergraph extremal problems

Just to emphasize that we are speaking of hypergraphs, hyperedges, $\dots$, we shall use script letters, and occasionally an upper index indicates the r-ity: $\mathcal{H}_n^{(r)}$ denotes an r-uniform hypergraph on n vertices.

Given two positive integers h and r, we may consider h-uniform r-multihypergraphs, that is, h-uniform hypergraphs, where the edges may have some multiplicities $\le r$. Obviously, given a family of such multihypergraphs, $\mathbf{ex}(n, \mathcal{L})$ is defined as the maximum number of h-tuples (counted with multiplicity) such a multihypergraph on n vertices can have without containing some members of $\mathcal{L}$ as submultihypergraphs. Some results on such general extremal graph problems were obtained by W.G. Brown and M. Simonovits [85], but for the sake of simplicity we shall confine our considerations to $r = 1$, that is, to ordinary h-uniform hypergraphs. Even for $h = 3$ most of the problems we meet prove to be hopeless or at least extremely hard. Therefore we shall mostly restrict our considerations to 3-uniform hypergraphs.

9.1. Degenerate hypergraph problems

Let $\mathcal{K}_h^{(h)}(m)$ be the following h-uniform hypergraph: it has hm vertices partitioned into disjoint m-tuples $U_1, \dots, U_h$, and the edges are those h-tuples which have exactly one vertex from each U_i.

Theorem 9.1 (Erdős' theorem [111]). *There exist two constants $c = c_h > 0$ and $A = A_h$ such that*

$$n^{h-cm^{-(h-1)}} < \mathbf{ex}(n, \mathcal{K}_h^{(h)}(m)) < An^{h-m^{-(h-1)}}.$$

Clearly, $\mathcal{K}_2^{(2)}(m) = K_2(m, m)$, and the above theorem is a generalization of the Kővári-T. Sós-Turán theorem. For the sake of simplicity, Theorem 9.1 was given only for the case when the sizes of classes of the excluded h-uniform h-partite graph were equal. One annoying feature of this theorem is that we do not have matching lower and upper bounds for the exponents even in the simplest hypergraph case $h = 3$ and $m = 2$.[28] At this point, it is worth defining two different chromatic numbers of hypergraphs.

[28] This is the octahedron hypergraph, defined by the triangles of an octahedron.

Definition 9.2 (Strong-Weak chromatic number). A hypergraph $\mathcal{H}$ is strongly t-colorable, if $V(\mathcal{H})$ can be t-colored so that each hyperedge uses each color at most once; the strong chromatic number $\chi_s(\mathcal{H})$ is the smallest such t.

A hypergraph $\mathcal{H}$ is weak t-colorable if we can t-color its vertices so that each of them gets at least 2 colors; $\chi(\mathcal{H})$ is the smallest such t.

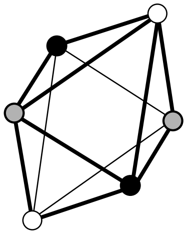

Fig. 6: Octahedron hypergraph

This way we see, by Theorem 9.1, that for r-uniform hypergraphs $\mathbf{ex}(n, \mathcal{L}^{(r)}) = o(n^r)$ if and only if there is an $\mathcal{H}^{(r)} \in \mathcal{L}^{(r)}$ that is strongly r-colorable. This extends from $r = 2$ to $r > 2$ what we already knew from Section 3.3.

Let $\mathcal{L}_{k,t}$ denote the family of 3-uniform hypergraphs of k vertices and t edges. Brown, Erdős and T. Sós [82] started investigating the function $f(n, k, t) = \mathbf{ex}(n, \mathcal{L}_{k,t})$.[29] The problem of finding good estimates of $f(n, k, t)$ is sometimes relatively simple, for some other values of k and t it seems to be extremely hard. One case which they could not settle was if $f(n, 6, 3) = o(n^2)$. Ruzsa and Szemerédi [311] proved the following surprising result.

Ruzsa-Szemerédi theorem. *Let $r_k(n)$ denote the maximum number of integers one can choose in $[1, n]$ so that no k of them form an arithmetic progression.*[30] *Then there exists a constant $c > 0$ such that*

$$cnr_3(n) < f(n, 6, 3) = o(n^2).$$

It is known that

Theorem 9.3 (Behrend [45], and Roth [309]).

$$n^{1-\frac{c}{\sqrt{\log n}}} < r_3(n) < c^* \frac{n}{\log \log n}.$$

The upper bound was recently improved by Tom Sanders [312] to

$$r_3(n) < c^{**} n \frac{(\log \log n)^5}{\log n}.$$

So, among others, the Ruzsa-Szemerédi theorem is surprising, since it shows the nonexistence of an $\alpha \in (1, 2)$ such that $C_1 n^\alpha < f(n, 6, 3) < C_2 n^\alpha$. Another surprising feature is that it implies that $r_3(n) = o(n)$, which was considered a beautiful result of K. F. Roth [308, 309], though superseded by the famous result of Szemerédi:

[29] The same question was investigated in some sense by Dirac [100] and in several papers of Erdős, and of Simonovits, see also Griggs, Simonovits and Thomas [192].

[30] We have already considered this problem in Section 7.

Theorem 9.4 (Szemerédi on arithmetic progressions [348]). *For every fixed k, as $n \to \infty$, $r_k(n) = o(n)$.*

For some related generalizations, see Alon and Shapira [20].

9.2. The "simplest" hypergraph extremal problem?

Next we turn to a hypergraph extremal problem which has a very simple extremal structure. G.O.H. Katona conjectured and Bollobás proved that

Theorem 9.5 (Bollobás [52]). *If $\mathcal{H}^{(3)}_{3n}$ is a 3-uniform hypergraph with $n^3 + 1$ triples, then it contains three triples where one contains the symmetric difference of the other two.*

This can be viewed as a possible generalization of Turán's theorem: K_3 has three pairs and the symmetric difference of two of them is contained in the third one. To understand a statement like Theorem 9.5, one always has to consider the conjectured extremal structure. Now this is the complete 3-partite 3-uniform hypergraph with (almost) equal class sizes. For us it is much more interesting that such a simple nice-looking extremal problem exists for hypergraphs.

9.3. Turán's hypergraph conjecture

We finish this part with the famous unsolved problem of P. Turán [361]:

Given a p, we define the complete h-uniform p-graph $\mathcal{K}^{(h)}_p$ as the h-uniform hypergraph on p vertices and with all the $\binom{p}{h}$ hyperedges. What is the maximum number of hyperedges in an h-uniform hypergraph $\mathcal{H}^{(h)}_n$ if it does not contain $\mathcal{K}^{(h)}_p$ as a subhypergraph?

For $h = 3$ Turán formulated some plausible conjectures. The conjectured extremal hypergraphs differed in structure for the cases if p was even or odd. For the sake of simplicity we formulate them only for $p = 4$ and $p = 5$.

(a) For $p = 4$ let us consider the 3-uniform hypergraph obtained by partitioning n vertices into 3 classes U_1, U_2 and U_3 as equally as possible and then taking all the triples of form (x, y, z) where x, y, and z belong to different classes; further, take all the triplets (x, y, z) where x and y belong to the i^{th} class and z to the $(i+1)^{\text{th}}$, $i = 1, 2, 3$, and $U_4 := U_1$.

(b) For $p = 5$ Turán had a construction with 4 classes and another one with 2 classes. The one with 2 classes is simple: we take all the triples having two vertices in one class and the third vertex in the other class. V.T. Sós observed that the construction with 2 classes can be obtained from the construction with 4 classes by moving some triples in some simple way. Later J. Surányi found a construction, showing that Turán's conjecture for $p = 5$ is false for $n = 9$. As far as I know Kostochka has found a

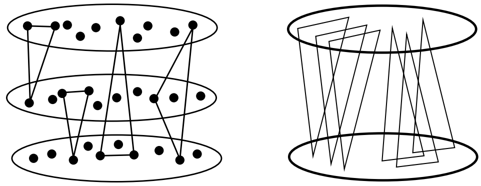

Figure 7. The conjectured extremal hypergraphs for $\mathcal{K}_4^{(3)}$ and $\mathcal{K}_5^{(3)}$

generalization of Surányi's construction: counterexamples for every $n = 4k + 1$. Still Turán's conjecture may be asymptotically sharp.

(c) Let us return to the case of $L = \mathcal{K}_4^{(3)}$. Even in this simple case Turán's conjecture seems to be very hard, even if we look only for asymptotics, that is, for $\lim \mathbf{ex}(n, \mathcal{K}_4^{(3)})/n^3$. There are no counterexamples to the conjecture, however, first Katona, Nemetz and Simonovits [215] have found some other constructions, slightly different from Turán's one, and only for $n = 3k+1$ and $n = 3k+2$. Later W. G. Brown [77] gave another construction without $\mathcal{K}_4^{(3)}$ and with the same number of triples, having 6 classes, depending on one parameter and containing Turán's construction as a special case. Finally Kostochka [251] has found a construction with t parameters, $3t$ classes, for arbitrary t, and having the same number of triples as Turán's one, without containing $\mathcal{K}_4^{(3)}$. His construction was a generalization of Brown's one. In these new constructions $n = 3k$, which seems to be the most interesting case. Next Fon der Flaass [160] gave a characterization of all of Kostochka's (3,4)-graphs, "explaining" why do the Kostochka constructions work. Recently Andrew Frohmader [166] found some new constructions. As to numerical estimates, see e.g. Chung and Lu [92].

Some people include intersection results into extremal hypergraph theory. I prefer to distinguish between them. Yet, I will include here a very famous problem of Erdős and Rado.

Problem 1 (Delta-systems, [130], [124].). Let us call a system of sets, $A_1, \dots, A_k$ a strong Δ-system, if the intersection of any two of them is the same. Is it true that if $\mathcal{A}$ is a system of r-tuples on an n-element set, without a k-Delta-system, then $|\mathcal{A}| < C_r^n$, for some constant $C_r > 0$.

9.4. Do Hypergraphs jump?

Definition 9.6 (Jumping constants). The number $\alpha \in [0, 1)$ is a jump for r if for any $\varepsilon > 0$ and integer $m \ge r$, any r-uniform hypergraph $\mathcal{H}_n^{(r)}$ with $n > n_o(\varepsilon, m)$ vertices and at least $(\alpha + \varepsilon)\binom{n}{r}$ edges contains a subhypergraph $\mathcal{H}_m^{(r)}$ with at least $(\alpha + c)\binom{m}{r}$ edges, where $c = c(\alpha)$ does not depend on ε and m.

By the Erdős-Stone-Simonovits theorem, for ordinary graphs (i.e. $r = 2$) every α

is a jump. Erdős asked [111] whether the same is true for $r \geq 3$. For the sake of simplicity we restrict ourselves to 3-uniform hypergraphs. For such a hypergraph $\mathcal{H}_n^{(3)}$ define the triple density as

$$\zeta(\mathcal{H}_n^{(3)}) = \frac{e(\mathcal{H}_n^{(3)})}{\binom{n}{3}}.$$

Theorem 9.1 of Erdős shows that if for a three-uniform hypergraph sequence $(\mathcal{H}_n^{(3)})$ the triple-density $\zeta(\mathcal{H}_n^{(3)}) > \alpha > 0$,[31] then there exist some subgraphs $\mathcal{H}_{m(n)}^{(3)} \subset \mathcal{H}_n^{(3)}$ with $m(n) \to \infty$, for which

$$\zeta(\mathcal{H}_{m(n)}^{(3)}) \geq \frac{6}{27} \quad \text{as } n \to \infty.$$

This means that – in this sense – the density jumps from $\alpha = 0$ to $\alpha' = 2/9$. It seems to me that Erdős wanted to know if this minimum density, $2/9$ (i.e. the density of $K_3^{(3)}(m)$) is a jumping constant. However, he formulated his question in a more general form and that was disproved (by a "random graph construction"), by Frankl and Rödl:

Theorem 9.7 (Frankl and Rödl [163]). *Suppose that* $r \geq 3$ *and* $\ell > 2r$. *Then* $1 - \frac{1}{\ell^{r-1}}$ *is not a jumping constant.*

Theorem 9.8 (Baber–Talbot [30]). *If* $\alpha \in [02299, 02316]$, *then* α *is a jump for* $r = 3$.

These are the first non-trivial jumping constants. The proof uses Razborov's flag algebra method. Theorem 9.8 follows from that for an appropriately chosen family $\mathcal{F}$ of 3-uniform hypergraphs $\mathbf{ex}(n, \mathcal{F}) < 0.2299\binom{n}{3} + o(n^3)$.

Remark 9.9. The jumping constant problem came up slightly differently, (perhaps earlier) in the digraph extremal problems, in the following form: "prove that the extremal densities form a well ordered set under the ordinary relation '$<$' ". Actually, a YES answer implies that the corresponding digraph extremal problems can algorithmically be solved. For the details we refer the reader to [81, 86]. The answer was YES for $r = 1$ and NO for large values of r, see Sidorenko [317], and Rödl–Sidorenko [304].

9.5. The story of the Fano problem

Consider the 3-uniform hypergraph defined by the "lines" of the Fano geometry (see Fig 8(a)). This hypergraph has 7 vertices and 7 triples and any two (distinct) of them intersect in exactly 1 vertex. This is the smallest finite geometry. As a hypergraph, it will be denoted by $\mathcal{F}_7$.

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[31] We may define the density dividing by n^r and by $\binom{n}{r}$.

Figure 8. (a) Fano hypergraph (b) Fano Extremal graph

Vera Sós asked what is the extremal graph for $\mathcal{F}_7$, and conjectured [339] that it is the complete bipartite 3-uniform graph shown in Fig. 8(b). Why is this conjecture natural? [32]

(i) Because $\mathcal{F}_7$ is 3-chromatic, by Definition 9.2,

(ii) however, deleting any triple of $\mathcal{F}_7$ we get a 2-chromatic hypergraph;

(iii) $\mathcal{F}_7$ is relatively sparse.

Theorem 9.10 (de Caen and Füredi[90]).

$$\mathrm{ex}_3(n, \mathcal{F}_7) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

Theorem 9.11 (Füredi–Simonovits [179], Keevash–Sudakov [219]). *For $n > n_0(\mathcal{F}_7)$ the complete bipartite 3-uniform hypergraph is the only extremal hypergraph for $\mathcal{F}_7$.*

Actually, in [179] a stronger, stability result was proved, easily implying Theorem 9.11. Observe that the degrees of the conjectured extremal graph are around $\frac{3}{4}\binom{n}{2}$.

Theorem 9.12. *There exist a $\gamma_2 > 0$ and an n_2 such that the following holds. If $\mathcal{H}$ is a triple system on $n > n_2$ vertices not containing the Fano configuration $\mathcal{F}_7$ and*

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2}$$

holds for every $x \in V(\mathcal{H})$, then $\mathcal{H}$ is bipartite, $\mathcal{H} \subseteq \mathcal{H}(X, \overline{X})$ for some $X \subseteq V(\mathcal{H})$.

This result is a distant relative of Theorem 4.25 (of Andrásfai, Erdős and T. Sós).

Remark 9.13 (Tools). These proofs heavily use some multigraph extremal results of Füredi and Kündgen [174]: the basic approach is that one finds first a $\mathcal{K}_4^{(3)} \subset \mathcal{H}_n^{(3)}$. If its vertices are a, b, c, d, then one considers the four link graphs of these vertices, where the link-graph of an x in a 3-uniform hypergraph is the pairs uv forming a 3-edge with x. [33] These link-graphs define a (colored) multigraph on $V(\mathcal{H}_n^{(3)}) - \{a, b, c, d\}$. We

[32] We used the complete k-chromatic graph for Theorem 9.1 in a slightly different way. Actually, there we considered the strong chromatic number, here the weak one.

[33] Actually, we use only the three largest ones of them.

apply a multigraph extremal theorem of [174] to get an $\mathcal{F}_7 \subset \mathcal{H}_n^{(3)}$. The boundedness of multiplicities is trivial.

There are a few further cases where we have sharp results on hypergraph extremal problems. I mention here e.g. Füredi, Pikhurko and Simonovits [176, 177, 178], where the last one refers to 4-hypergraphs. Other sharp results can be found on 4-hypergraph cases in Füredi-Mubayi-Pikhurko [175].

9.6. Co-degree problems

For hypergraphs we have several options to define degrees. Below we restrict our considerations again to the 3-uniform case and instead of degrees we consider co-degrees: the co-degree of two vertices x and y is the number of triples of $\mathcal{H}_n^{(3)}$ containing both of them.

Theorem 9.14 (Mubayi [280]). *For every $\varepsilon > 0$ there exists an n_0 such that for $n > n_0$, if for any pair of vertices $x, y \in V(\mathcal{H}_n^{(3)})$ their co-degree is at least $(\frac{1}{2} + \varepsilon)n$ then $\mathcal{F}_7 \subset \mathcal{H}_n^{(3)}$.*

Mubayi conjectured that $\varepsilon = 0$ would be sufficient to ensure a Fano subgraph. Mubayi and Zhao remark in [281] that for co-degree problems many questions have answers different from that of the ordinary hypergraph extremal problems. One such case is the problem of jumping constants (see Section 9.4). The co-degree densities are defined in the obvious way, thus the jumping constants are defined almost the same way as for hyper-edge densities.

Theorem 9.15 (Mubayi-Zhao [281]). *For co-degree problems every $c \in (0, 1)$ is a non-jumping constant.*

Further sources to read: We close this section mentioning some references on hypergraph extremal theorems: Balogh, Bohman, Bollobás, and Yi Zhao: [33], Frankl and Füredi [162], Keevash and Sudakov [220].

10. Ramsey-Turán theory

Vera Sós [338] and then Erdős and Vera Sós [143] initiated a whole new research field, the Ramsey-Turán theory. We shall concentrate primarily on the most recent results, since a longer survey of Vera Sós and myself [335] covers the earlier results well.

The extremal configuration in Turán's original theorem is too regular. This is why one could feel that perhaps better estimates could be achieved by replacing Turán's original theorem by some version of it, where the too regular configurations are somehow excluded. One way to exclude regular patterns is to assume that G does not contain too many independent vertices – Turán's extremal graph does. This means that we exclude large complete graphs in the complementary graphs. This is, how we arrive at problems which, as a matter of fact, are combinations of Ramsey and Turán type problems. Very soon after the first results of Erdős and Vera T. Sós [143, 144, 145] were published, many others joined to this research.

As we mentioned, Turán's original theorem was motivated by Ramsey's theorem. It would have been quite natural to ask sooner or later, whether the two results could be combined. The questions thus arising would have been interesting on their own, too. However, only much later, in connection with the applications discussed in Section 13 did the Ramsey-Turán problems emerge.

We denote by $RT(n, L, m)$ the maximum number of edges a graph G_n can have if $L \not\subseteq G_n$ and $\alpha(G_n) \le m$. Setting $m = n$ we arrive at Turán's extremal theorem. On the other hand, if m is too small then, by Ramsey theorem, there are no graphs in the considered class. The first problems and results in this field can be found in Sós [338], generalized by Burr, Erdős and Lovász [87].

As we shall see in Section 13, if we wish to apply Turán's theorem to find lower bounds on "geometric sums" of type (13.1), then we use many different graphs on the same vertex set, simultaneously. We know that the first one contains no complete p_1-graph, the second one contains no complete p_2-graph, and so on. We would like to find some estimate on some weighted sum of the number of their edges. The simplest case is, when these weights are equal. This is how Vera T. Sós arrived in [338] at the following question:

> Partition the edges of a K_n into k sets, thus obtaining the graphs $G_1, \dots, G_k$ on $V(K_n)$. We know that for $i = 1, \dots, k$, G_i contains no complete p_i-graph. What is the maximum of $e(G_1) + \cdots + e(G_{k-1})$?

Of course, if k and $p_1, \dots, p_k$ are fixed and $|V|$ is too large, then such graphs simply do not exist. This is just Ramsey's theorem. However, in the cases interesting for us $p_1, \dots, p_{k-1}$ are fixed and p_k tends to infinity. We assume only that $p_k = o(n)$, or more generally, that $p_k = o(f(n))$. Thus we could use the notation

$$RT(n, L_1, \dots, L_{k-1}; o(f(n)) \le cn^2$$

or $RT(\dots) = o(f(n))$ where the left hand size means that we consider a graph sequence (G_n) with $\alpha(G_n) = o(f(n))$.

Surprisingly enough, such questions sometimes prove to be extremely difficult. The simplest tractable case was when we had two graphs, G_n and its complementary graph H_n and wanted to maximize $e(G_n)$ under the assumption that G_n contains no K_{p+1} and the largest complete graph in H_n is of size $o(n)$. The first real breakthrough was

Theorem 10.1 (Erdős and Sós [143]).

$$RT(n, K_{2p+1}, o(n)) = e(T_{n,p}) + o(n^2). \tag{10.1}$$

So the estimate of $RT(n, K_m, o(n))$ was solved by Erdős and V.T. Sós [143] for the case when m is odd. The case of even p's was much more difficult. Thus e.g. it was a longstanding problem whether for $p = 4$ $e(G_n) = o(n^2)$ or not. Finally Szemerédi proved that

Theorem 10.2 ([350]). $RT(n, K_4, o(n)) < \frac{1}{8}n^2 + o(n^2)$.

Later Bollobás and Erdős [58] constructed graphs, showing that Szemerédi's estimate is sharp.

Theorem 10.3 ([58]). $RT(n, K_4, o(n)) = \frac{1}{8}n^2 + o(n^2)$.

The next breakthrough was when Erdős, Hajnal, V.T. Sós and Szemerédi, [129], determined (among others) the limit of $RT(n, K_{2p}, o(n))/n^2$, (thus generalizing Theorem 10.3). Ramsey–Turán theory is one of the areas of Extremal Graph Theory where many new results were proved lately. In [127] Erdős, Hajnal, Simonovits, Sós, and Szemerédi asked:

Problem 2. Does there exist a $c > 0$ for which $RT(n, K_4, \frac{n}{\log n}) < (\frac{1}{8} - c)n^2$?

One step to answer Problem 2 was

Theorem 10.4 (Sudakov [342]). *If* $\omega(n) \to \infty$, *and* $f(n) = n/e^{\omega(n)\sqrt{\log n}}$, *then* $RT(n, K_4, f(n)) = o(n^2)$.

Then Problem 2 was answered in the negative by

Theorem 10.5 (Fox, Loh and Zhao [161]). *For* $\sqrt{\frac{\log\log^3 n}{\log n}} \cdot n < m < \frac{1}{3}n$,

$$RT(n, K_4, m) \geq \frac{1}{8}n^2 + \left(\frac{1}{3} - o(1)\right) mn.$$

On the other hand,

Theorem 10.6 (Fox, Loh and Zhao [161]). *There is an absolute constant* $c > 0$, *such that for every* n, *if* $e(G_n) > \frac{1}{8}n^2$, *and* $K_4 \not\subseteq G_n$, *then*[34]

$$\alpha(G_n) > c\frac{n}{\log n} \log\log n.$$

In other words, if $\tilde{c} > 0$ *is small enough, then*

$$RT\left(n, K_4, \tilde{c}\frac{n \log\log n}{\log n}\right) \leq \frac{1}{8}n^2.$$

[34] Let us use binary log here, but assume that $\log n > 1$.

In addition, they proved that

Theorem 10.7 (Fox, Loh and Zhao [161]).

$$RT(n, K_4, \alpha) \leq \frac{1}{8}n^2 + 10^{10}\alpha n.$$

J. Balogh, Ping Hu, and M. Simonovits [40] proved (among many other results) the following phase transition phenomenon.

Theorem 10.8. $RT(n, K_5, o(\sqrt{n \log n})) = o(n^2)$.

One difficulty in this area is that there are no known Erdős-Stone-Simonovits type results (though there are some related conjectures in [129]). Thus, e.g. if $L(t)$ is a blown up version of L, $RT(n, L, o(n))$ and $RT(n, L(t), o(n))$ may behave completely differently, even for $L = K_3$. We close this part with a related construction of V. Rödl. Erdős asked if

$$RT(n, K(2,2,2), o(n)) = o(n^2). \tag{10.2}$$

Rödl modified the Bollobás-Erdős construction [58]; his version still did not decide if (10.2) holds, however, it answered another question of Erdős:

Theorem 10.9 (Rödl [299]). *There exist graphs G_n with $e(G_n) > \frac{1}{8}n^2 - o(n^2)$ edges and with $\alpha(G_n) = o(n)$, however, not containing K_4, nor $K(3,3,3)$.*

Further sources to read: Erdős and Sós [143, 144].

10.1. Sparse Ramsey-Turán problems

Starting out from completely different problems, Ajtai, Komlós and Szemerédi also arrived at Ramsey-Turán type problems. To solve some number theoretical and geometry problems, they arrived at the following Ramsey-Turán theorem:

Theorem 10.10 ([5, 1, 6]). *If the average degree of G_n is d and $K_3 \not\subseteq G_n$ then*

$$\alpha(G_n) > c\frac{\log d}{d} \tag{10.3}$$

This means a $\log d$ improvement over the ordinary Turán theorem. Another interpretation of this is that excluding a triangle in the complementary graph makes G_n random-looking. These and similar results, e.g. [1] were used to improve earlier estimates in some problems in Geometry [239] [238] Combinatorial Number Theory [6] and Ramsey Theory [5]. We skip the details.

10.2. α_p-independence problems

We close this very short part with two relatively new results of Balogh and Lenz [39]. Hypergraph Ramsey-Turán problems motivate the following problem:

> Given two sample graphs H and L, and two integers n, and m. How many edges can a graph G_n have if any induced $G_m \subseteq G_n$ contains an H and G_n does not contain L.

For $H = K_2$ we get back the ordinary $RT(n, L, m)$, while for $H = K_p$ we call the maximum m in the condition α_p-independence and denote it by $\alpha_p(G_n)$. Several related results can be found in [127, 128], and for newer results see Balogh and Lenz [39]. We mention here just one of them:

Theorem 10.11 (Balogh–Lenz). *For $t \geq 2$ and $2 \leq \ell \leq t$, let $u = \lceil t/2 \rceil$. Then $RT_t(n; K_{t+\ell}, o(n)) \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2} n^2$.*

This is a breakthrough result, answering our earlier questions, where we [128] wanted to decide, for which ℓ is $RT_t(n; K_{t+\ell}, o(n)) \geq c(\ell, t)n^2$ for some constant $c(\ell, t) > 0$. Balogh and Lenz found important "generalizations" of the Bollobás-Erdős construction [58].

Further sources to read: Balogh and Lenz [38].

11. Anti-Ramsey theorems

Anti-Ramsey problems[35] (in the simplest case) have the following form: Given an arbitrary coloring of a graph, we call a subgraph H Totally Multi-colored (TMC) or Rainbow if all its edges have distinct colors. [36]

Problem 3. We have a "sample graph" H. Let $\mathbf{AR}(n, H)$ be the maximum number of colors K_n can be colored with without containing a TMC H.

The problem of determining $\mathbf{AR}(n, H)$ is connected not so much to Ramsey-theory but to Turán-type problems. For a given family $\mathcal{H}$ of finite graphs, the general result corresponding to Theorem 3.3 is

Theorem 11.1 (Erdős-Simonovits-Sós [141]). *Let*

$$d + 1 := \min_{e \in E(H)} \{\chi(H - e) \ : \ e \in E(H)\}. \tag{11.1}$$

[35] I heard this expression "anti-Ramsey" first from Richard Rado and it is also the title of his paper [292], on sequences. There the topic is analogous but not really connected to our problems.

[36] Originally we called it TMC, later Erdős and Tuza started calling such an H "rainbow"-colored, and some people would call it heterochromatic.

Then

$$\mathbf{AR}(n,H) = e(T_{n,d}) + o(n^2), \qquad \textit{if} \qquad n \to \infty. \tag{11.2}$$

The reason for this Transfer Principle: Assume that $H - e$ has the minimum chromatic number in (11.1). Consider an edge-coloring of K_n and choose one edge from each color. This way we get a TMC graph G_n. Now, $e(G_n) > \mathbf{ex}(n, H-e)+\varepsilon n^2$ would guarantee $cn^{v(H)}$ copies of $H - e$. Hence some pair uv would be contained in $c'n^{v(H)-2}$ copies of $H - e$, yielding with uv this many copies of H. We could choose two of them having no common vertices but u and v. Since all the colors in this union are distinct, whichever way we color uv, we get a TMC copy of H. □

11.1. Path, cycles and further related results

The above approach gives a good asymptotics if $d > 1$ in (11.1). On the other hand, for $d = 1$ new problems have to be overcome. The Anti-Ramsey problem of P_ℓ was solved by Simonovits and Sós [334]. The question of C_ℓ was much more complicated.

Problem 4 (Erdős–Simonovits–Sós [141]). How many colors ensure a totally multicolored (Rainbow) C_ℓ with some $\ell > k$.

One immediately sees that this problem is an analog of the Erdős-Gallai problem on cycles. One of the important open problems in this area was the problem of Rainbow cycles.

Conjecture 11.2 (Erdős, Simonovits and Sós [141]). Fix a cycle length ℓ. Consider the following edge-coloring of K_n. First we cover the vertices by complete subgraphs of $\ell - 1$ vertices each and a remainder smaller one, K_r (they form an extremal graph for P_ℓ.) Give a "private color" to these edges. Enumerate the complete subgraphs as $H_1, \dots, H_m, \dots$ and color the edges between H_i and H_j by the new color c_i if $i < j$. One can easily see that this coloring of K_n has no totally multicolored (rainbow) C_ℓ. Show that this is the maximum number of colors one can use:

$$\mathbf{AR}(n, C_\ell) = \frac{1}{2}(\ell - 2)n + \frac{n}{\ell - 1} + O(1).$$

The conjecture is easy for K_3, was proved for C_4 by Noga Alon [8], then for $\ell = 5, 6$ independently by Schiermeyer [313] and by Jiang Tao and Doug West [206], and finally the problem was completely settled by Montellano-Ballesteros and Neumann-Lara [274].

11.2. Other types of Anti-Ramsey graph problems

In the results of the previous section typically some colors are used very many times but the others only once. To eliminate this, Erdős and Tuza counted the "color-degrees":

Theorem 11.3 (Erdős and Tuza [151]). *Consider an arbitrary coloring of K_n. Denote by $k(i)$ the number of colors at the i^{th} vertex. If K_n does not contain TMC (rainbow) triangles, then $\sum 2^{-k(i)} > 1$.*

They consider the cases when the color distribution is forced to be uniform in some sense and list several problems and provide further theorems.

Theorem 11.4 (Frieze–Reed [165]). *If $c > 0$ is a sufficiently small constant, n is large, and the edges of K_n are colored so that no color appears more than $k = c\frac{n}{\log n}$ times, then K_n has a TMC Hamilton cycle.*

We close this part with mentioning results stating that there are very sparse graphs having the Anti-Ramsey property. In the next two theorems – instead of assuming that the number of colors used is large, – we assume that they form a proper coloring.

Theorem 11.5 (Rödl and Tuza [305]). *There exist graphs G with arbitrarily high girth such that every proper edge coloring of G contains a cycle all of whose edges have different colors.*

The proof of the above results was probabilistic. Haxell and Kohayakawa proved that the Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak [267] also have this property.

Theorem 11.6 ([200]). *For every positive integer t, every real δ such that $0 < \delta < 1/(2t+1)$, and every n sufficiently large with respect to t and δ, there is a graph G_n such that (i)* $\text{girth}(G) = t + 2$*, and*

(ii) for any proper edge-coloring of G_n there is a rainbow $C_\ell \subset G_n$ for all $2t+2 \le \ell \le n^\delta$.

Further sources to read: Babai and Sós [29], Babai [27], Alon, Lefmann and Rödl [17], Hahn and Thomassen [198], Axenovich and Kündgen [26], Burr, Erdős, Graham, Sós, Frankl [89, 88]…

12. Turán-like Ramsey theorems

Considering Ramsey theorems for ordinary graphs we may observe the following “dichotomy”:

(a) Pseudo-random graphs: In many cases the Ramsey extremal graphs look as if they were random graphs. [37]

[37] A famous conjecture of V.T. Sós suggests that (at least for complete graphs) these are quasi-random graphs.

(b) Canonical structures: In other cases the Ramsey extremal structures look (almost?) *Canonical Graph Sequences*: n vertices are partitioned into q classes $U_1, U_2, \ldots, U_q$ and the graphs $G[U_i]$ are monochromatic cliques, the bipartite graphs $G[U_i, U_j]$ are also monochromatic complete bipartite graphs, and the sizes of these classes may vary. (However, in our cases it may happen that Canonical Sequences are Ramsey-extremal, but there are also some other almost-canonical graph sequences that are Ramsey-extremal: we can change the colors of a negligible number of edges without creating monochromatic forbidden subgraphs.)

Denote by $R_k(L_1, L_2, \ldots, L_k)$ the Ramsey number corresponding to $L_1, L_2, \ldots, L_k$: the minimum N for which, if we k-edge-color K_N, then for some i the i^{th} color will contain an L_i.

Conjecture 12.1 (Bondy-Erdős). If n is odd, then

$$R_k(C_n) := R_k(C_n, C_n, \ldots, C_n) = 2^{k-1}(n-1) + 1. \tag{12.1}$$

The background of this conjecture is that for two colors, according to the Bondy–Erdős theorem [65], or the Faudree-Schelp [154] or Rosta theorems [306] the conjecture is true. The sharpness can be seen if we take two complete BLUE K_{n-1}'s and join them completely by RED edges.

Now, if we have a construction on $N = 2^{k-1}(n-1)$ vertices, k-colored, without monochromatic C_n, then we may take two copies of this construction and a new color k and join the two copies completely by this new color. This provides the lower bound in (12.1).

For $k \geq 3$, the conjecture seemed to be harder to prove. Łuczak [269] proved that if n is odd, then $R_3(C_n) = 4n + o(n)$, as $n \to \infty$. Later, Kohayakawa, Simonovits and Skokan (adding some fairly involved stability arguments to Łuczak's original one) showed that

Theorem 12.2 (Kohayakawa, Simonovits and Skokan, [231], [232]). *There exists an n_0 for which for $n > n_0$,*

$$R_3(C_n, C_n, C_n) = 4n - 3. \tag{12.2}$$

The special case $n = 7$ of (12.2) was proved in [152]. Conjecture 12.1 is still open for $k > 3$. Bondy and Erdős [65] remarked that they could prove $R_k(C_n) \leq (k+2)!n$ for n odd. The next result improves this:

Theorem 12.3 (Luczak-Simonovits-Skokan [272]). *For every odd $k \geq 4$,*

$$R_k(C_n) \leq k3^{k-1}n + o(n), \qquad \textit{as} \qquad n \to \infty.$$

The following conjecture is unknown even for $k = 4$:

Conjecture 12.4 (Kohayakawa, Simonovits, Skokan). If $n_1, n_2, \dots, n_k$ are fixed, then there are asymptotically Ramsey-extremal graphs U_N for the corresponding Ramsey problem of finding $R_k(C_{n_1}, C_{n_2}, \dots, C_{n_k})$, where $V(U_N)$ can be partitioned into a bounded number $O_k(1)$ of classes and – apart from $O_k(N)$ edges – the color of each edge depends only on the classes it joins.

The case of even cycles has a slightly different answer, since the construction described above contains long monochromatic even cycles. Related results can be found in e.g. Łuczak [270], Figaj and Łuczak, Benevides and Skokan [47], . For further related results see the 3-color-Path results of Gyárfás, Ruszinkó, Sárközy, and Szemerédi [194], [195].

Slightly different, yet related questions are discussed in the paper of Faudree and Simonovits [156].

13. Applications of Turán's graph theorem

13.1. Distance distribution

Here we shall discuss very shortly some applications of Turán's graph theorem to the distribution of distances in metric spaces. Perhaps Erdős noticed first that Turán's theorem can be applied to distance distributions.

Theorem 13.1 (Erdős [107]). *If we have a set X of n points in the plane, $X = \{P_1, \dots, P_n\}$ and the diameter of X is at most 1, then at least*

$$\binom{n}{2} - \mathbf{ex}(n, K_4) \approx \frac{1}{3}\binom{n}{2}$$

of the distances P_iP_j is at most $1/\sqrt{2}$.

To prove this, observe that for any 4 points – by an easy argument – at least one of the 6 distances is $\le 1/\sqrt{2}$. So the graph G_n defined by the distances $> 1/\sqrt{2}$ contains no K_4. Hence $e(G_n) \le \mathbf{ex}(n, K_4)$. □

Obviously, this result is sharp: if we fix an equilateral triangle of diameter 1 and put $n/3$ points into each of its vertices, then roughly 1/3 of the $\binom{n}{2}$ distances will be 0 and all the others are equal to 1.

14 years later Turán pointed out that a slight generalization of this simple observation may yield far-fetching and interesting results (estimates) in geometry, analysis and some other fields, too. Turán's basic observation was as follows: Instead of $d = 1/\sqrt{2}$, we can apply the same idea simultaneously to several distances. We define the corresponding *Packing Constants*:

Definition 13.2. Given a metric space $\mathbb{M}$ with the metrics $\rho(x, y)$ and an integer k, let

$$d_k := \max_{\mathbf{diam}\{P_1,\dots,P_k\} \le 1} \min_{i \ne j} \rho(P_i, P_j).$$

(If $|\mathbb{M}| = \infty$, it may happen that we have to replace the min by inf.)

Now, the above argument shows that if the ρ-diameter of an n-element set is at most 1, then it contains at least $\binom{n}{2} - \mathbf{ex}(n, K_k)$ distances $\rho(P_i, P_j) \le d_k$. Using Abel summation, we may obtain good estimates on sums of the form

$$\sum f(\rho(P_i, P_j)). \tag{13.1}$$

This way, through distance-distribution results, Turán [363], V.T. Sós [338], and later Erdős, Meir, V.T. Sós and Turán [132, 133, 134] could give estimates on certain integrals, potentials, certain parameters from functional analysis, and other geometric sums. In [132] the authors write:

> "In what follows, we are going to discuss systematic applications of graph theory – among others – to geometry, potential theory and to the theory of function spaces…These applications show that suitably devised graph theorems act as flexible logical tools (essentially as generalizations of the pigeon hole principle)…We believe that the applications given in this sequence of papers do not exhaust all possibilities of applications of graph theory to other branches of mathematics. Scattered applications of graph theory, (mostly via Ramsey theorem) existed already in the papers of Erdős and Szekeres [149] and Erdős [106], [116]."

Remarks 13.3. These lines are 40 years old, however, the development of Discrete Mathematics really shows that Discrete Mathematics became a very applicable theory in very many areas of mathematics. Strangely enough, or perhaps because Turán died too soon, not too many results were published on application of extremal graph results to distance distribution, after Turán's death.

However, two further areas were strongly connected to this approach. The first one was the application of Turán type graph results in estimating distributions in Probability Theory. This area was pioneered by G.O.H. Katona. He was able to prove some inequalities concerning the distribution of certain random variables [213]-[216]. Next several important results of the field were proved by A. Sidorenko. This volume has a separate article on this topic, by Katona [214]. I would risk the opinion that among the several steps that led to the theory of graph limits one important step was this: introducing integrals in areas related to extremal graph theory.

The other one is *Ramsey-Turán theory* discussed in Section 10.

13.2. Application to Geometry

Given n points in the space (or in any bounded metric space), for every $c > 0$ we can define a graph $G^{(c)}$ by joining the points P and Q iff $PQ > c$. By establishing some appropriate geometric facts, we may ensure that $G^{(c)}$ contains no complete $p = p(c)$-graph. Hence we know (by Turán's theorem) that the number of pairs (P, Q) with $PQ > c$ is at most $\mathbf{ex}(n, K_{p(c)})$.

Assume that we apply this method with several constants $c_1 > c_2 > \cdots > c_k > 0$. If $f(x)$ is a monotone decreasing function in (13.1), then we may obtain lower bounds on this expression by replacing all the distances between c_i and c_{i+i} by c_i. The 'only' problems to be solved are:

How to choose the constants $c_1 > c_2, > \cdots > c_k > \cdots > 0$?

How to choose the integers p_k for the constants c_k, to get good results?

This was the point, where the packing constants (depending largely on the geometric situation) came in. Their investigation goes back at least to a dispute between Newton and Gregory, see Turán [364]. It was also somewhat surprising that not all packing constants count in our application. It is enough to regard those ones, where $c_k > c_{k+1}$. It is not worth giving a detailed description of the results obtained this way, since the Introduction of [134] does it. We make only one critical remark on a side-issue:

In [364] Turán remarks that perhaps his method, implemented on a good computer would help to decide problems such as the one in the Newton-Gregory dispute. Namely, it could decide whether $c_t = c_{t+1}$ or not.

This is not quite so. First of all, such an algorithm can never give a positive answer. Further, even if the answer is in the negative, and that could be proved by the method suggested by Turán, then probably that could be decided also without using Turán's method.

13.3. Other applications

An old unsolved problem is that if we have n points in the k-dimensional Euclidean space, how many unit distances can occur. For the plane Erdős observed that the graph given by the unit distances cannot contain a $K_2(2, 3)$. Hence – by the Kővári-T. Sós-Turán theorem – the number of unit distances is $O(n^{3/2})$. A similar argument works in $\mathbb{R}^3$: the 3-space, but for higher dimension the situation changes. Unfortunately, the application of Turán type theorems is not enough to get the conjectured bounds: to prove that the number of unit distances is at most $O(n^{1+\varepsilon})$.

(b) Some other type of applications of hypergraph extremal problems are found in the works of Simonovits [322] and Lovász [261] yielding sharp bounds on some questions related to color-critical graphs. For more details see either the original papers or the Füredi-Simonovits survey [180].

Further sources to read: Erdős [116], Erdős and Simonovits [142], . . .

14. Extremal subgraphs of random graphs

What happens if, instead of considering all the $\mathcal{L}$-free graphs G_n, we consider only $\mathcal{L}$-free subgraphs G_n of some host-graphs R_n and maximize their number of edges. One of the most investigated subcases of this problem is when R_n is a random graph with some given distribution. The maximum is $\mathbf{ex}(R_n, \mathcal{L})$, however this is a random number, depending on the random graph R_n. So we can state only that certain events will hold with high probability.

Rödl and Schacht wrote very recently an excellent survey [303] on this topic, so we shall give only a very short introduction to this area.

Assume that R_n is a random graph of binomial distribution, with given edge-probability: $R_n \in \mathcal{G}_{n,p}$. The phenomena to be discussed are

> If L is a sample graph, $k = \chi(L)-1$, and we take a random graph $R_n \in \mathcal{G}_{n,p}$ with edge probability $p > 0$,
>
> (a) is the subgraph $F_n \subseteq R_n \in \mathcal{G}_{n,p}$ not containing L and having the maximum number of edges k-chromatic with probability $1 - o(1)$?
>
> (b) if (a) does not hold, is it true that at least we can delete $o(e(R_n))$ edges from R_n to get a k-chromatic graph, almost surely?

An early result in this area was

Theorem 14.1 (Babai-Simonovits-Spencer [28]). *There exists a $p_0 < \frac{1}{2}$ for which in a random $R_n \in \mathcal{G}_{n,p}$, almost surely, the maximum size K_3-free subgraph, $F_n \subseteq R_n$ is bipartite.*

Several generalizations of this were proved in [28], however, those days no "Sparse Regularity Lemma" was known, and the proofs of Babai, Simonovits and Spencer were using the (ordinary) Szemerédi Regularity Lemma [349] and the stability method. Hence [28] could cover only the case when the edge probability was $p > p_0 > 0$. As soon as the Kohayakawa-Rödl version of the Regularity Lemma was proved and became known, the possibility to generalize the results of [28] became possible. First Brightwell, Panagiotou and Steger [75] proved that Theorem 14.1 holds under the much weaker condition that $p > n^{-1/250}$ and very recently B. De Marco and Jeff Kahn [97] proved that

Theorem 14.2. *There exists a $C > 0$ such that if the edge probability is $p > C\sqrt{\log n/n}$, then every maximum triangle-free subgraph of $G_{n,p}$ is bipartite, with probability tending to 1, as $n \to \infty$.*

This is best possible.

Let

$$d_2(H) = \max\left\{\frac{e(H')}{v(H')} \ : \ H' \subseteq H, \text{ and } v(H') \geq 3\right\}.$$

Conjecture 14.3 (Kohayakawa-Rödl-Schacht [230]). Let $v(H) \geq 3$ and $e(H) > 0$. Let $G = G_{n,p}$ be a random graph with edge probability $p = p_n$ where $p_n n^{1/d_2(H)} \to \infty$. Then

(i) almost surely (as $n \to \infty$),

$$\mathbf{ex}(G, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) e(G) + o(e(G)).$$

(ii) Further, for $\chi(H) \geq 3$, a stability phenomenon also holds: almost surely, deleting $o(e(G_{n,p}))$ edges, one can make $G_{n,p}$ $(\chi(H) - 1)$-colorable.

The above conjecture is proved for several cases. Thus, e.g., for cycles it was proved by Haxell, Kohayakawa and Łuczak [201] and [202], while the paper of Kohayakawa, Łuczak and Rödl [227] contains a proof of (i) for $H = K_4$.

15. Typical structure of L-free graphs

Here we consider the following problem:

> What is the typical structure of L-free graphs? Or, more generally, we have a Universe (graphs, hypergraphs, multigraphs, permutations, ordered sets, ...) and a property $\mathcal{P}$, can we say something informative about the typical structures in $\mathcal{P}$?

This question has basically two subcases: the exclusion of some L as a not necessarily induced subgraph and the exclusion of some induced subgraphs.

15.1. Starting in the middle

In this part excluding $L \subset G_n$ we do not assume that (only) the induced subgraphs are excluded. The difference can be seen already for C_4: If we define a complete graph on A and an independent set on B and join them arbitrarily, the resulting G_n contains many C_4's but no induced C_4. So first we consider the case of not necessarily induced subgraphs.

First we assume that the forbidden graphs are non-bipartite, and return to the degenerate case in the next, very short subsection. Denote by $\mathcal{P}(n, \mathcal{L})$ the family of n-vertex graphs without subgraphs from $\mathcal{L}$. Since all the subgraphs of any $S_n \in \mathbf{EX}(n, \mathcal{L})$ belong to $\mathcal{P}(n, \mathcal{L})$, therefore

$$|\mathcal{P}(n, \mathcal{L})| \geq 2^{\mathbf{ex}(n, \mathcal{L})}. \tag{15.1}$$

This motivated

Conjecture 15.1 (Erdős).

$$|\mathcal{P}(n, \mathcal{L})| = 2^{\mathbf{ex}(n, \mathcal{L}) + o(n^2)}. \tag{15.2}$$

Of course, the meaning of this is that $\mathcal{P}(n, \mathcal{L})$ cannot be much larger than the right hand side of (15.1). This was confirmed first for K_{p+1}. The result for K_3 was much sharper than for the general case.

Theorem 15.2 (Erdős–Kleitman–Rothschild [131]). *(i) Almost all triangle-free graphs G_n are bipartite.*

(ii) In general,

$$|\mathcal{P}(n, K_{p+1})| \le 2^{\left(1-\frac{1}{p}\right)n+o(n^2)}.$$

Later Erdős, Frankl, and Rödl proved the original Erdős conjecture.

Theorem 15.3 (Erdős, Frankl, and Rödl [125]).

$$|\mathcal{P}(n, \mathcal{L})| \le 2^{\mathbf{ex}(n,\mathcal{L})+o(n^2)}.$$

As we have already pointed out, the finer structure in the extremal graph problems depends on the "Decomposition family" $\mathbb{M}$ of $\mathcal{L}$. So Balogh, Bollobás and myself improved Theorem 15.3 in several steps. First, in [34] we improved the error term $o(n^2)$ of Theorem 15.3 to $O(n^{2-c})$.

Theorem 15.4. *For every $\mathcal{L}$, if $\mathbb{M}$ is the decomposition family of $\mathcal{L}$ and $\mathbb{M}$ is finite, then*

$$|\mathcal{P}(n, \mathcal{L})| \le n^{\mathbf{ex}(n,\mathbb{M})+c_{\mathcal{L}}\cdot n} \cdot 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2}, \tag{15.3}$$

for some sufficiently large constant $c_{\mathcal{L}} > 0$.

This was an improvement, indeed: if $L \in \mathcal{L}$ and $v = v(L)$ is of minimum chromatic number, then we can choose a bipartite $M \subseteq L$ from $\mathbb{M}$. Hence $\mathbf{ex}(n, \mathbb{M}) < c \cdot n^{2-\frac{2}{v}}$, yielding a better error term in the exponent in (15.3).

Our next result yields also structural information.

Theorem 15.5 (Balogh, Bollobás, Simonovits [35]). *Let $\mathcal{L}$ be an arbitrary finite family of graphs. Then there exists a constant $h_{\mathcal{L}}$ such that for almost all $\mathcal{L}$-free graphs G_n we can delete $h_{\mathcal{L}}$ vertices of G_n and partition the remaining vertices into p classes, $U_1, \dots, U_p$, so that each $G[U_i]$ is $\mathbb{M}$-free.*

For some particular cases we can provide even more precise structural information. A good test-case is when the Octahedron graph is excluded. In our main result below we describe the structure of almost all octahedron-free graphs. We say that a graph G has property $\mathcal{Q} = \mathcal{Q}(C_4, P_3)$ if its vertices can be partitioned into two sets, U_1 and U_2, so that $C_4 \not\subseteq G[U_1]$ and $P_3 \not\subseteq G[U_2]$. If $G \in \mathcal{Q}$ then G does not contain O_6. It was proved by Erdős and Simonovits [137] that for n sufficiently large every O_6-extremal G_n has property $\mathcal{Q}$. The typical structure of O_6-free graphs is described by

Theorem 15.6 (Balogh, Bollobás, Simonovits [36]). *The vertices of almost every O_6-free graph can be partitioned into two classes, U_1 and U_2, so that U_1 spans a C_4-free graph and U_2 spans a P_3-free graph.*

A similar, slightly simpler, result is the following. Denote $\mathcal{P}(n;a,b)$ the family of graphs G_n for which no a vertices of G_n span at least b edges. In some sense, G. Dirac started investigating such problems [100]. Several results of Erdős and Simonovits are related to this topic, and they became very important for hypergraphs, see e.g., Brown, Erdős and T. Sós [82], or Ruzsa and Szemerédi [311]. Much later, Griggs, Simonovits and Thomas [192] proved that, for n sufficiently large, the vertex set of any $\mathcal{P}(n,6,12)$-extremal graph G_n can be partitioned into U_1 and U_2 so that the induced subgraphs, $G[U_1]$ is $\{C_3,C_4\}$-free and $G[U_2]$ is an independent set. Note that if G_1 is $\{C_3,C_4\}$-free and $e(G_2)=0$ then $G_1 \otimes G_2$ is $(6,12)$-free.

Theorem 15.7 (Balogh, Bollobás, Simonovits [36]). *The vertex set of almost every graph in $\mathcal{P}(n;6,12)$ can be partitioned into two classes, U_1 and U_2, so that U_1 spans a $\{C_3,C_4\}$-free graph and U_2 is an independent set.*

To avoid technicalities, we formulated only this special case. Another line is the problem of critical edges.

Theorem 15.8 (Prömel and Steger [291]). *For every L having a critical edge, almost all L-free graphs have chromatic number $\chi(L)-1$.*

This is sharp, since no graph with chromatic number $\chi(L)-1$ contains L as a subgraph, (see also Hundack, Prömel, and Steger [203].) To demonstrate the power of our methods we proved a generalization of their result. Denote by sH the vertex-disjoint union of s copies of H. Let the excluded graph be $L=sH$, where H has a critical edge, and $\chi(H)=p+1\geq 3$. Simonovits [321] proved that for n sufficiently large, the unique L-extremal graph is $H(n,p,s)$, see Theorem 4.15. Observe that if one can delete $s-1$ vertices of a graph G_n to obtain a p-partite graph, then G_n is L-free.

Theorem 15.9 (Balogh, Bollobás, Simonovits [36]). *Let p and s be positive integers and H be a $p+1$-chromatic graph with a critical edge. Then almost every sH-free graph G_n has a set S of $s-1$ vertices for which $\chi(G_n-S)=p$.*

15.2. Degenerate cases

One could think that if L is bipartite but not a tree, then (15.2) remains valid:

$$|\mathcal{P}(n,L)| < 2^{\mathbf{ex}(n,L)(1+o(1))}. \tag{15.4}$$

Yet, this is not known even in the simplest case, for $L=C_4$. The first important result in this area was

Theorem 15.10 (Kleitman–Winston [224]).

$$2^{(\frac{1}{2}-o(1))n\sqrt{n}} \le |\mathcal{P}(n, C_4)| < 2^{cn\sqrt{n}} \qquad \textit{with} \qquad c = 1.082.$$

The result itself is highly non-trivial. The next result in this direction was

Theorem 15.11 (Kleitman–Wilson [372]).

$$|\mathcal{P}(n, C_6)| < 2^{cn\sqrt[3]{n}}.$$

The corresponding results for C_{2k} for $k \ge 4$ are still open. Balogh and Samotij also have analogous results for $K_{t,t}$, and – more generally, – for $K_{s,t}$.

Theorem 15.12 (Balogh and Samotij [42, 43]). *For $L = K_{s,t}$, there exist a constant $c = c_L$ for which*

$$|\mathcal{P}(n, L)| \le 2^{c\mathbf{ex}(n,L)}.$$

Their method also implies that

Theorem 15.13 (Balogh and Samotij [42, 43]). *For $L = K_{2,t}$, there exists a constant $\tilde{c} = \tilde{c}_L$ for which for almost all L-free G_n, we have*

$$\frac{1}{12}\mathbf{ex}(n, L) \le e(G_n) \le (1 - c)\mathbf{ex}(n, L).$$

Several of the related papers contain a "mini-survey" of the situation, so we stop here.

15.3. Typical hypergraph structures

As we have mentioned, for many years there were only a few hypergraph extremal results. In the last few years this dramatically changed. As we have seen in Section 9, several interesting extremal hypergraph theorems were proved lately. Also some corresponding "typical structure results" were obtained, e.g. [41]. Here we give only a few examples. The first one is connected to the Fano-results [179] and [219].

Theorem 15.14 (Person and Schacht [287]). *Almost all $\mathcal{F}_7$-free 3-uniform hypergraphs are 2-chromatic.*

Call the following three edges a triangle: (u, v, w), (u, v, x), (x, y, w). The following result extends the sharper version of Theorem 15.2, at least for triangles.

Theorem 15.15 (Balogh and Mubayi [41]). *Almost all triangle free 3-uniform hypergraphs are tripartite.*

The following result attacks already the general case, extends the Erdős-Frankl-Rödl Theorem to 3-uniform hypergraphs.

Theorem 15.16 (Nagle and Rödl [283]). *For any fixed 3-uniform hypergraph L,*

$$|\mathcal{P}(n,L)| < 2^{\mathbf{ex}(n,L)+o(n^3)}.$$

This was extended to k-uniform graphs by Nagle, Rödl and Schacht [284].

Other structures. There are some other structures where analogous results were proved fairly early, showing that some specific structures dominate (in number) the others. Here we mention some results of Kleitman and Rothschild [222] on the number of partially ordered sets on n elements.

Consider $\mathcal{Q}(n)$, the family of partial orders of the following structures: n vertices are distributed in three classes L_1, L_2, and L_3, where $|L_1| = n/4 + o(n)$, $|L_2| = n/2 + o(n)$, $|L_3| = n/4 + o(n)$. Define a partial order by its Hasse diagram. Define the partial order Q as follows: the arcs go from L_i to L_{i+1}, $i = 1, 2$, and if we forget about the orientations, we get a $\frac{1}{2}$-quasi-random graph between L_i and L_{i+1}. Kleitman and Rothschild proved that [222]]

Theorem 15.17 (Kleitman and Rothscild [222], see also [221]).

$$|\mathcal{P}_n| = \left(1 + O\left(\frac{1}{n}\right)\right)|\mathcal{Q}_n|.$$

Thus

$$|\mathcal{P}_n| = 2^{n^2/4+o(n^2)}.$$

See also Kleitman, Rothschild and Spencer [223].

15.4. Induced subgraphs?

If instead of excluding some not necessarily induced subgraphs, we exclude induced subgraphs, the situation completely changes. The first results in this direction were proved by Prömel and Steger [289] [290]... Several extensions were proved by Alekseev, Bollobás and Thomason, and others.

Definition 15.18. The sub-coloring number $p_c(\mathcal{P})$ of a hereditary graph property $\mathcal{P}$ is the maximum integer p for which if we put complete graphs into some classes of a $T_{n,p}$ (somehow), and delete some original edges, the resulting graph cannot have property $\mathcal{P}$.

Example 15.19. Let the property $\mathcal{P}$ be that G_n contains an induced C_4. Consider a complete graph K_ℓ and a set I_m of independent vertices (with disjoint vertex sets) and join them arbitrarily. The resulting graph will not contain induced C_4's. It is easy to see that here $p_c(\mathcal{P}) = 2$.

Theorem 15.20 (Alekseev [7], Bollobás-Thomason [61]). *If $\mathcal{P}$ is a hereditary property of graphs, and $\mathcal{P}(n, \mathcal{L})$ denotes the family of n-vertex graphs of property $\mathcal{P}$, and $p := p_c(L)$ then*

$$|\mathcal{P}(n, \mathcal{L})| = 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2+o(n^2)}.$$

This was improved in [12].

Definition 15.21. Given an integer k, the universal graph $U(k)$ is the bipartite graph with parts $A = \{0,1\}^k$ and $B = \{1, \dots, k\}$, where $j \in B$ is joined to a k-tuple X if $j \in X$, (i.e., the j^{th} coordinate of X is 1).

Theorem 15.22 (Alon, Balogh, Bollobás, Morris [12]). *Let $\mathcal{P}$ be a hereditary property of graphs, with coloring number $\chi_c(\mathcal{P}) = p$. Then there exist constants $k = k(\mathcal{P}) \in \mathbb{N}$ and $\varepsilon = \varepsilon(\mathcal{P}) > 0$ such that the following holds. For almost all graphs $G_n \in \mathcal{P}$, there exists a partition $(A, S_1, \dots, S_p)$ of $V(G_n)$, such that:*
(a) $|A| < n^{1-\varepsilon}$,
(b) $G[S_j]$ *is* $U(k)$*-free for every* $j \in [p]$.
Moreover, if $\mathcal{P}_n$ is the family of n-vertex graphs of $\mathcal{P}$, then

$$2^{(1-1/p)\binom{n}{2}} < |\mathcal{P}_n| \le 2^{(1-1/p)\binom{n}{2}+n^{2-\varepsilon}}$$

for every sufficiently large $n \in \mathbb{N}$.

There are several further interesting results in [12], but we stop here.

Further sources to read: Bollobás [56].

15.5. Counting the colorings

Some of the above results are strongly connected to estimating

$$c_{r,F}(\mathcal{H}) := \#\{r - \text{colorings of } \mathcal{H} \text{ without monochromatic copies of } F\}$$

Estimating $c_{r,F}$ is strongly connected to the extremal problem of F, i.e. determining $\mathbf{ex}(n, F)$ and also with Erdős-Frankl-Rödl type theorems, first of all, with Theorems 15.2 and 15.3. Erdős and Rothschild conjectured that

Conjecture 15.23.

$$c_{2,K_\ell}(G_n) \le 2^{\mathbf{ex}(n,K_\ell)}.$$

For triangles this was proved by Yuster [374]. This was extended to arbitrary complete graphs by Alon, Balogh, Keevash and Sudakov [11]. A similar coloring-counting theorem was proved by Lefmann, Person, Rödl and Schacht [259], also explaining the connection of these results to each other. We skip the details.

16. "Random matrices"

This part is devoted to random ± 1 matrices, where the questions are:

(i) How large is the determinant of a random matrix,
(ii) what is the probability that a random matrix is singular,
(iii) what can be said about the eigenvalues of a random matrix.

Recently very many new results were obtained in this field. Below I shall mention some of them and provide some references, and also refer the reader to the excellent survey paper of Van Vu [370].

Szekeres and Turán [347] were primarily interested in (i), more precisely, in the average of the absolute value of the determinant of a ± 1 matrix. Later Turán continued this line, Szekeres went into another direction.

16.1. Hadamard Matrices

According to the famous theorem of Hadamard, given a matrix $A = (a_{ij})$, $|\det(A)|$ can be estimated from above by the product of the lengths of the row vectors. Equality holds iff the row vectors are pairwise orthogonal. If the entries of the matrix are 1's and -1's, then Hadamard's result yields that

$$|\det(A)| \le n^{n/2}. \tag{16.1}$$

It is natural to ask whether the equality in (16.1) can be achieved for ± 1 entries. In other words, are there orthogonal $n \times n$ matrices with ± 1 entries? Such matrices are called Hadamard matrices. The smallest ones are (1) and $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. One can easily prove that if for some $n > 2$ such a matrix does exist, then n is divisible by 4. It is a very famous, old and widely investigated but still open conjecture that

Conjecture 16.1. Hadamard matrices exist for every n divisible by 4.

One can easily construct Hadamard matrices for $n = 2^k$ and it is not too difficult to construct them for $n = 4k$ if $n - 1$ is a prime.

16.2. Szekeres-Turán approach

In connection with the Hadamard problem, Gy. Szekeres and P. Turán arrived at the following question [347]:

Problem 5. Consider all the ± 1 matrices A of n rows and columns. How large is the average of $|\det(A)|^k$, as a function of n?

They proved in [347] that

Theorem 16.2. *The average of* $|\det(A)|^2$ *for the* $n \times n$ ± 1 A*, is* $n!$.

They simply calculated the sum of the squares of the determinants of all the $n \times n$ ± 1 matrices. Their proof was very simple and elegant. They have also calculated the sum of the fourth powers of these determinants, proving that this is $(n!)^2 \cdot \varphi(n)$, where $\varphi(n)$ is a function defined by the recursion

$$\varphi(1) = 1, \qquad \varphi(2) = 2, \qquad \varphi(n) = \varphi(n-1) + \frac{2}{n}\varphi(n-2). \tag{16.2}$$

Remark 16.3. For every $c > 0$, $\varphi(n)$ is between n^{2-c} and n^2, if n is sufficiently large. This means that the average of the squares and fourth powers of these determinants are (in some weak sense) fairly near to the desired maximum. Geometrically, if we take n ± 1-vectors independently, at random, they will be roughly orthogonal to each other.

Remark 16.4. Superficially we could think that the main goal of the Szekeres-Turán paper was to prove the existence of a good approximation of Hadamard matrices, using Random Matrix methods. Maybe, originally this was their purpose. However, as they remarked, Erdős had pointed out[38] that the following direct construction provides a much better result on the maximum value of the determinant:

Find a prime $p = 4k - 1 < n$ sufficiently near to n and then build an Hadamard matrix for this $\tilde{n} = 4k$. Using the monotonicity of the maximum, one gets a much better estimate than by the Szekeres-Turán argument.

Is this result more than merely answering an important and interesting mathematical problem in an elegant way? YES, in the following sense:

Here we can see one of the first applications of stochastic methods instead of giving constructions for some optimization problem in Discrete Mathematics. Later this method was applied many times and proved to be one of our most powerful methods. (In combinatorics and graph theory it was Paul Erdős who started applying probabilistic methods *systematically*.) From this point of view the Szekeres-Turán paper was definitely among the pioneering ones.

16.3. Turán's and Szekeres' continuation

Later both Turán [357, 360, 362] and Szekeres [344, 345] returned to these questions. They generalized their original results in various ways. However, they did not really succeed in estimating the average of the $2k^{\text{th}}$ power of the considered determinants.[39] (The average of the odd powers is, by symmetry, 0!) Turán seemed to be more interested in finding analytically various averages of ± 1 determinants. Szekeres went basically into two directions:

[38] This was remarked in the paper of Turán and Szekeres and also, e.g., in the "problem collection paper" of Erdős [108].

[39] As I see, they could not estimate the average of the 6^{th} powers.

(a) He considered the so called *skew Hadamard matrices*, restricted the averaging to these matrices i.e., where for $i \neq j$ $a_{i,j} = -a_{j,i}$. For them the averaging method [344] gave higher average.

(b) Also, Szekeres invented new combinatorial/algebraic constructions of Hadamard Matrices, Skew Hadamard Matrices [345]. He also used computer search to find "small" examples. e.g. for $n = 52, 92$.

16.4. Expected or typical value?

The paper of Szekeres and Turán determines the average and the square average of $\det(A)^2$. In many cases the typical values of some random variable ξ are very near to its expected values. This is e.g. the case in Turán's "Hardy-Ramanujan" paper [356]. In case of the ± 1 determinants the situation is different.

A Correction/Historical Remark. Here I have to make a "Correction": Writing my notes for Turán' Collected Papers [368] I "overstated" Theorem 16.2. I wrote that Szekeres and Turán proved that the determinant of almost all A in Theorem 16.2 is near to the average $\sqrt{n!}$. This holds only in some fairly weak logarithmic sense. In ordinary sense, not only they did not state this, but – as it turns out below, – this is not even true.

Of course, Szekeres and Turán did not speak of "probability". The point is that they did not use Chebishev inequality, and they did not calculate the standard deviation. (Slightly earlier, Turán, in his proof of the Hardy-Ramanujan theorem, without speaking of probabilities, calculated the mean and the standard deviation of the number of prime divisors and then applied Chebishev inequality.) Theorem 16.6 below implies that for a positive percentage of the considered random matrices the determinant is above $(1 + c)\sqrt{n!}$, for some fixed $c > 0$.

This question, when ξ is noticeably above $\mathbb{E}(\xi)$ (where $\mathbb{E}$ denotes the expected value) is discussed in e.g. in

Theorem 16.5 (Schlage-Puchta [314]). *Let ξ be a non-negative real random variable, and suppose that $\mathbb{E}(\xi) = 1$ and $\mathbb{E}(\xi^2) = a > 1$. Then the probability $P(\xi \geq a)$ is positive, and for every $b < a$ we have $\int_{|\xi|>b} \xi^2 \geq a - b$.*

The paper remarks that this theorem is nearly a triviality, but it has several interesting corollaries. One of them is a lower estimate for $|\det(A)|$ in the Szekeres-Turán problem. Since the 4$^{\text{th}}$ moment is much larger than the 2$^{\text{nd}}$, (by (16.2)), Theorem 16.5 is applicable here.

16.5. The Hadamard "goodness" of Random Matrices

Denote the (Euclidean) norm of $\mathbf{a}$ by $||\mathbf{a}||$. Let A be an $n \times n$ matrix with column vectors $\mathbf{a}_i$, $(i = 1, \dots, n)$. Define its "Hadamard goodness" as

$$h(A) = \frac{\det(A)}{\prod ||\mathbf{a}_i||},$$

if the denominator does not vanish, otherwise define $h(A) = 0$.

John Dixon [101] wrote a nice and interesting paper on the above discussed question, primarily on the typical goodness of the random method in the "Hadamard approach". He wrote that for him a paper of Cabay and Lam suggested that (logarithmically, in some natural settings) the values of the determinants of random matrices are close to their maximum. He proved that this is not so: the logarithmic distance is typically what is suggested in the Szekeres-Turán theorem: $\det(A)^{1/n} \approx (\sqrt{n!})^{1/(2n)} \approx \sqrt{n/e}$.

The question investigated by Dixon [101] is, how large the expected value of $h(A)$ is if A is a random matrix, where the distribution of entries obey some weak smoothness conditions. The conclusion of Dixon's results is that typically $h(A)^{1/n} \approx 1/\sqrt{e}$.

Condition (D1) If $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the columns of A, then the density of the distribution at A depends only on the values of $||\mathbf{a}_1||, \dots, ||\mathbf{a}_n||$.

Condition (D2) The probability that $\det(A) \neq 0$ is 1.

Theorem 16.6 (Dixon [101]). *Let A be a random matrix whose distribution satisfies (D1) and (D2). Denote by μ_n and σ_n^2 the mean and variance of the random variable* $\log h(A)$. *Then*

(i) $\mu_n = -\frac{1}{2}n - \frac{1}{4}\log n + O(1)$, *and* $\sigma_n^2 = \frac{1}{2}\log n + O(1)$, *as* $n \to \infty$*;*

(ii) For each $\varepsilon > 0$, *the probability of that*

$$n^{-\frac{1}{4}-\varepsilon}e^{-\frac{1}{2}n} < h(A) < n^{-\frac{1}{4}+\varepsilon}e^{-\frac{1}{2}n}$$

tends to 1 as $n \to \infty$.

16.6. Probability of being singular

In this section we are discussing the upper bounds for the probability that $\det(A) = 0$. One interested in more details is suggested to read some of the following sources: Komlós [237], Kahn, Komlós, and Szemerédi [211], or some more recent papers of Van Vu [370], Terry Tao and Van Vu [352].

Obviously, for continuous distributions this probability is 0. One can easily see that this probability must be the largest for ± 1 matrices, where both values are taken with equal probabilities.

Theorem 16.7 (Komlós, [237]). *Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries are random independent variables, taking values ± 1 with probability $\frac{1}{2}$. Then* $\det(A) \neq 0$ *with probability $p_n \to 1$ as $n \to \infty$.*

A more general result is

Theorem 16.8 (Komlós, [237]). *Let $A = (\xi_{ij})$ be an $n \times n$ matrix whose entries are random independent variables, with common, non-degenerate distribution*[40]. *Then* $\det(A) \neq 0$ *with probability $p_n \to 1$ as $n \to \infty$.*

Conjecture 16.9. Let P_n be the probability that a random $n \times n$ matrix with elements ± 1 is singular. Then $P_n = (1 + o(1))n^2 2^{1-n}$.

The first breakthrough was

Theorem 16.10 (Kahn, Komlós and Szemerédi [211]). *There is a positive constant ε for which $P_n < (1 - \varepsilon)^n$.*

This is a considerable improvement on the best previous bound, $P_n = O(1/\sqrt{n})$ given by Komlós in 1977.

16.7. Eigenvalues of Random Matrices

This field is again a very wide one, with many interesting results. The beginnings of this part heavily relies on the Füredi-Komlós paper [173].

Investigating the distribution of the eigenvalues of matrices goes back to E. P. Wigner (1955), who was motivated by quantum mechanics. The following generalization is due to L. Arnold [25].

Theorem 16.11 (Wigner, Semicircle law.). *Assume that A is a random symmetric matrix with random independent entries a_{ij} for $i \geq j$. Let the distribution of these entries be F for $i \neq j$ and G for $i = j$. Assume that $\int |x|^k \, dF < \infty$, $\int |x|^k \, dG < \infty$ for $k = 1, 2, \dots$ and set $D^2 a_{ij} = \mathbf{Var}\, a_{ij} = \sigma^2$. $W_n(x)$ be the empirical distribution of the number of eigenvalues of A not exceeding xn. Let*

$$W(x) = \begin{cases} \frac{2}{\pi}\sqrt{1 - x^2} & \textit{for } |x| \leq 1, \\ 0 & \textit{for } |x| > 1. \end{cases}$$

Then

$$\lim_{n \to \infty} W_n(2\sigma\sqrt{n} \cdot x) = W(x).$$

[40] A distribution is degenerate if with probability 1, its outcome is the same.

This implies that for $c > 2\sigma$ with probability $1-o(1)$, all but $o(n)$ of the eigenvalues belong to $[-c\sqrt{n}, c\sqrt{n}]$. Yet, this does not give information on the largest eigenvalues. Ferenc Juhász [209] gave some weak estimates on this and those were improved to much better ones by the Füredi-Komlós theorems which basically assert that

Theorem 16.12 (Füredi, Komlós [173]). *Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ symmetric matrix where a_{ij} are independent, (not necessarily identically distributed) random real variables bounded with a common bound K, for $i \geq j$. Assume that, for $i > j$, a_{ij} have a common expectation μ and variance σ^2. Further, assume that $\mathbb{E}(a_{ii}) = \nu$. (Here $a_{ij} = a_{ji}$.) The numbers K, μ, σ^2, ν will be kept fixed as $n \to \infty$.*

If $\mu > 0$ then the distribution of the largest eigenvalue of $A = (a_{ij})$ can be approximated in order $1/\sqrt{n}$ by a normal distribution of expectation

$$(n-1)\mu + \nu + \sigma^2/\mu \tag{16.3}$$

and variance $2\sigma^2$. Further, with probability tending to 1,

$$\max_{i\geq 2} |\lambda_i(A)| < 2\sigma\sqrt{n} + O(\sqrt{n}\log n), \tag{16.4}$$

where λ_i is the i^{th} eigenvalue of A.[41]

Remark 16.13. The semi-circle law implies that $\max_{i\geq 2} |\lambda_i(A)|$ cannot be much smaller than $2\sigma\sqrt{n}$.

16.8. Singularity over finite fields

One could ask what happens if we take the entries of a random $n \times n$ matrix from a finite field $\mathcal{F}$.

Theorem 16.14 (Jeff Kahn, J. Komlós [210]). *The probability that a random square matrix of order n, with entries drawn independently from a finite field $F(q)$ according to some distribution, is nonsingular is asymptotically (as $n \to \infty$) the same as for the uniform distribution (excepting certain pathological cases, see below):*

$$\Pr(M_n \textit{ is nonsingular }) \to \prod_{i\geq 1}\left(1 - \frac{1}{q^i}\right) \qquad \textit{as} \qquad n \to \infty. \tag{16.5}$$

What is pathological? Kahn and Komlós write that if the entries of the random matrix M_n are chosen independently and uniformly from $\mathcal{F}$, that is enough to ensure (16.5) and this was fairly widely known. Among others in [91] (see also [253, 254]) it is proved that

[41] $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Theorem 16.15. *Let M_n be a random $n \times n$ $\mathcal{F}$-matrix with entries chosen according to some fixed non-degenerate probability distribution μ on $\mathcal{F}$. Then (16.5) holds if and only if the support of μ is not contained in any proper affine field of $\mathcal{F}$.*

We skip the details here, again.

Acknowledgments. I would like to thank the help of several of my colleagues for carefully reading various versions of this manuscript. I would mention above all, János Pintz, Balázs Patkós, Dániel Korándi, and Cory Palmer. For the errors, misprints, only I am responsible.

Bibliography

[1] M. Ajtai, P. Erdős, J. Komlós and E. Szemerédi: *On Turán's theorem for sparse graphs,* Combinatorica, **1**(4) (1981), 313–317.

[2] M. Ajtai, J. Komlós, M. Simonovits and E. Szemerédi: *On the approximative solution of the Erdős-Sós conjecture on trees,* Manuscript.

[3] M. Ajtai, J. Komlós, M. Simonovits and E. Szemerédi: *The solution of the Erdős-Sós conjecture for large trees.* Manuscript.

[4] M. Ajtai, J. Komlós, M. Simonovits and E. Szemerédi: *The Erdős-Sós conjecture and graphs with dense blocks.* Manuscript.

[5] M. Ajtai, J. Komlós, and E. Szemerédi: *A note on Ramsey numbers,* J. Combin. Theory Ser. A **29**(3) (1980), 354–360.

[6] M. Ajtai, J. Komlós, E. Szemerédi: *A dense infinite Sidon sequence.* European J. Combin. **2**(1) (1981), 1–11.

[7] V. E. Alekseev: *Range of values of entropy of hereditary classes of graphs,* (Russian) Diskret. Mat. **4**(2) (1992), no. 2, 148–157; translation in Discrete Math. Appl. **3**(2) (1993), 191–199.

[8] N. Alon: *On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems,* J. Graph Theory **7** (1983), 91–94.

[9] N. Alon: *Tools from Higher Algebra,* Chapter 32 of Handbook of Combinatorics (ed. Graham, Lovász, Grötschel), 1995, 1749–1783, North-Holland, Amsterdam.

[10] N. Alon: *Paul Erdős and Probabilistic Reasoning,* In: Erdős Centennial Volume, Bolyai Soc. Math. Stud. 25 (L. Lovász, I. Ruzsa and V. Sós, eds.), Springer, 2013, 11–33.

[11] N. Alon, J. Balogh, P. Keevash and B. Sudakov: *The number of edge colourings with no monochromatic cliques,* J. London Math. Soc., **70**(2) (2004), 273–288.

[12] N. Alon, J. Balogh, B. Bollobás, and R. Morris: *The structure of almost all graphs in a hereditary property,* J. Combin. Theory Ser. B 101 (2) (2011), 85–110.

[13] N. Alon, R.A. Duke, H. Lefmann; V. Rödl, and R. Yuster: *The algorithmic aspects of the regularity lemma,* J. Algorithms 16 (1994), no. 1, 80–109. (see also the extended abstract, in Proceedings of the 33rd Annual Symposium on Foundations of Computer Science, Pittsburgh, PA, 1992, IEEE Computer Society Press, Los Alamitos, CA, 1992, 473–481.)

[14] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy: *Efficient testing of large graphs* (extended abstract), in Proceedings of the 40th Annual Symposium on

Foundations of Computer Science, New York, NY, 1999, IEEE Computer Society Press, Los Alamitos, CA, (1999), 656–666.

[15] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy: *Efficient testing of large graphs,* Combinatorica, **20** (2000), 451–476.

[16] N. Alon, E. Fischer, I. Newman, and A. Shapira: *A combinatorial characterization of the testable graph properties: it's all about regularity,* SIAM J. Comput. **39**(1) (2009), 143–167.

[17] N. Alon, H. Lefmann, and V. Rödl: *On an anti-Ramsey type result,* In Sets, graphs and numbers (Budapest, 1991), Vol. 60 of Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, (1992), 9–22.

[18] N. Alon, L. Rónyai and T. Szabó: *Norm-graphs: variations and applications,* J. Combinatorial Theory, Ser. B 76 (1999), 280–290.

[19] N. Alon, V. Rödl, and A. Ruciński: *Perfect matchings in ε-regular graphs,* Electronic Journal of Combinatorics, 5, 1998, R13.

[20] N. Alon and A. Shapira: *On an extremal hypergraph problem of Brown, Erdős and Sós,* Combinatorica **26** (2006), no. 6, 627–645.

[21] N. Alon and A. Shapira: *Testing subgraphs in directed graphs,* J. Comput. System Sci., 69 (2004), 353–382. (see also Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, 700–709 (electronic), ACM, New York, 2003.)

[22] N. Alon and B. Sudakov: H-free graphs of large minimum degree, // Electron. J. Combin. 13 (2006), no. 1, Research Paper 19, 9 pp.

[23] B. Andrásfai: *Neuer Beweis eines graphentheoretishen Satzes von P. Turán,* Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 95-107.

[24] B. Andrásfai, P. Erdős and V.T. Sós: *On the connection between chromatic number, maximal clique and minimal degree of a graph,* Discrete Math. **8** (1974), 205–218.

[25] L. Arnold: *On the asymptotic distribution of the eigenvalues of random matrices,* J. Math. Analysis and Appl. **20** (1967), 262–268.

[26] M. Axenovich and A. Kündgen: *On a generalized anti-Ramsey problem,* Combinatorica **21** (2001), 335–349.

[27] L. Babai: *An anti-Ramsey theorem,* Graphs Combin. **1**(1) (1985), 23–28.

[28] L. Babai, M. Simonovits, and J. Spencer: *Extremal subgraphs of random graphs,* J. Graph Theory **14**(5) (1990), 599–622.

[29] L. Babai and V.T. Sós: *Sidon sets in groups and induced subgraphs of Cayley graphs,* Europ. J. Combin. **6** (1985), 101–114.

[30] R. Baber and J. Talbot: *Hypergraphs do jump,* Combin. Probab. Comput., **20** (2010), 161–171.

[31] P.N. Balister, E. Győri; J. Lehel, and R.H. Schelp: *Connected graphs without long paths,* Discrete Math. 308 (2008), no. 19, 4487–4494.

[32] S. Ball and V. Pepe: *Asymptotic improvements to the lower bound of certain bipartite Turán numbers,* Combin. Probab. Comput. 21 (2012), no. 3, 323–329.

[33] J. Balogh, T. Bohman, B. Bollobás, and Yi Zhao: *Turán densities of some hypergraphs related to K_{k+1}^k,* SIAM J. Discrete Math. **26**(4) (2012), 1609–1617.

[34] J. Balogh, B. Bollobás, and M. Simonovits: *The number of graphs without forbidden subgraphs,* J. Combin. Theory Ser. B **91** (2004), 1–24.

[35] J. Balogh, B. Bollobás, and M. Simonovits: *The typical structure of graphs without given excluded subgraphs,* Random Structures Algorithms **34** (2009), 305–318.

[36] J. Balogh, B. Bollobás and M. Simonovits: *The fine structure of octahedron-free graphs,* Journal of Combinatorial Theory, Series B. **101**(2) (2011), 67–84.

[37] J. Balogh and J. Butterfield: *Excluding induced subgraphs: critical graphs,* Random Structures Algorithms, **38** (2011), 1–2, 100–120.

[38] J. Balogh and J. Lenz: *Some exact Ramsey-Turán numbers,* Bull. Lond. Math. Soc. 44 (2012), no. 6, 1251âĂŞ1258.

[39] J. Balogh and J. Lenz: *On the Ramsey-Turán numbers of graphs and hypergraphs,* Israel J. Math. **194** (2013), no. 1, 45–68.

[40] J. Balogh, Ping Hu, and M. Simonovits: *Phase transitions in the Ramsey-Turán theory,* submitted and on the Arxiv

[41] J. Balogh and D. Mubayi: *Almost all triangle-free triple systems are tripartite,* Combinatorica **32**(2) (2012), 143–169.

[42] J. Balogh and W. Samotij: *The number of $K_{s,t}$-free graphs,* J. Lond. Math. Soc. (2) **83**(2) (2011), 368–388.

[43] J. Balogh and W. Samotij: *The number of $K_{m,m}$-free graphs,* Combinatorica, **31** (2011), 131–150.

[44] I. Bárány: *A short proof of Kneser's conjecture,* J. Combin. Theory Ser. A **25**(3) (1978), 325–326.

[45] F. Behrend: *On sets of integers which contain no three terms in arithmetical progression,* Proc. National Acad. Sci. USA **32** (1946), 331-332.

[46] L. Beineke and R. Wilson, R.: *The early history of the brick factory problem,* Math. Intelligencer **32**(2) (2010), 41–48.

[47] F. S. Benevides and J. Skokan: *The 3-colored Ramsey number of even cycles,* Journal of Combinatorial Theory, (B) **99**(4) (2009), 690–708.

[48] C. Benson: *Minimal regular graphs of girth eight and twelve,* Canad. J. Math. **18** (1966), 1091-1094.

[49] Blakley, G., Roy, P.: *A Hölder type inequality for symmetric matrices with nonnegative entries,* Proc. Amer. Math. Soc. **16** (1965), 1244–1245.

[50] Blatt, D. and G. Szekeres: *A skew matrix of order 52,* Canad. J. Math. **21** (1969), 1319–1322.

[51] B. Bollobás: *Extremal graph theory,* In L. Lovász, R. Graham, M. Grötschel, editors, Handbook of Combinatorics, 1995, 1231–1292, North-Holland, Elsevier, Amsterdam, MIT Press.

[52] B. Bollobás: *Three-graphs without two triples whose symmetric difference is contained in a third,* Discrete Math. **8** (1974), 21–24.

[53] B. Bollobás: *Relations between sets of complete subgraphs,* Proceedings of the Fifth British Combinatorial Conference (C. St.J. A. Nash-Williams and J. Sheehan, editors), Utilitas Mathematica Publishing, 1976, 79–84.

[54] B. Bollobás: *On complete subgraphs of different orders,* Math. Proc. Cambridge Philos. Soc **79** (1976), 19–24.

[55] B. Bollobás: *Extremal Graph Theory,* Academic Press, London, 1978.

[56] B. Bollobás: *Hereditary properties of graphs: asymptotic enumeration, global structure, and colouring,* Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998). Doc. Math. 1998, Extra Vol. III, 333–342 (electronic).

[57] B. Bollobás: *Random Graphs (2nd edition),* Cambridge Studies in Advanced Mathematics 73, Cambridge Univ. Press (2001).

[58] B. Bollobás and P. Erdős: *On a Ramsey-Turán type problem,* J. Combinatorial

Theory Ser. B 21 (1976), no. 2, 166–168.

[59] BOLLOBÁS, B. AND O. RIORDAN: *Metrics for sparse graphs,* In Surveys in Combinatorics 2009, Vol. 365 of London Mathematical Society Lecture Notes, Cambridge University Press, 211–287.

[60] B. BOLLOBÁS AND A. THOMASON: *Dense neighbourhoods in Turán's theorem,* J. Combin. Theory (B) **31** (1981), 111-114.

[61] B. BOLLOBÁS, AND A. THOMASON: *Hereditary and monotone properties of graphs,* in: R.L. Graham, J. Nešetřil (Eds.), The Mathematics of Paul Erdős, II, in: Algorithms Combin., vol. 14, Springer-Verlag, New York, Berlin, 1997, 70–78.

[62] J.A. BONDY: *Large cycles in graphs,* Discrete Math. **1** (1971), 121–132.

[63] J.A. BONDY: *Large dense neighbourhoods and Turán's theorem,* J. Combinatorial Theory (B) 34/1 (1983), 99-103; Erratum in JCT(B) **35** p80.

[64] J.A. BONDY: *Basic Graph Theory: Paths and Circuits,* Handbook of Combinatorics (eds Graham, Grötschel, Lovász) (1995), I 3–110. North-Holland, Elsevier, Amsterdam, MIT Press.

[65] J. A. BONDY AND P. ERDŐS: Ramsey numbers for cycles in graphs, J. Combinatorial Theory Ser. B **14** (1973), 46–54.

[66] J.A. BONDY AND M. SIMONOVITS: *Cycles of even length in graphs,* J. Combinatorial Theory (B) 16 (1974), 97–105.

[67] J.A. BONDY AND Z. TUZA: *A weighted generalization of Turán's theorem,* J. Graph Theory 25 (1997), 267–275.

[68] C. BORGS, J. CHAYES, L. LOVÁSZ, V.T. SÓS AND K. VESZTERGOMBI: *Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing,* Adv. Math. 219 (2008), no. 6, 1801–1851. (also available at http://arxiv.org/abs/math/0702004.

[69] C. BORGS, J. CHAYES, L. LOVÁSZ, V.T. SÓS, AND K. VESZTERGOMBI: *Counting Graph Homomorphisms,* in Topics in discrete mathematics (Eds. M. Klazar, J. Kratochvil, M. Loebl, J. Matoušek, R. Thomas, and P. Valtr), Algorithms and Combinatorics, **26**. Springer-Verlag, Berlin, 2006, 315–371.

[70] C. BORGS, J. CHAYES, L. LOVÁSZ, V.T. SÓS, B. SZEGEDY, AND K. VESZTERGOMBI: *Graph limits and parameter testing,* in: STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, ACM, New York, 2006, 261–270.

[71] S. BRANDT: *A sufficient condition for all short cycles,* 4th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1995), Discrete Appl. Math. **79** (1997), 63–66.

[72] S. BRANDT: *Triangle-free graphs and forbidden subgraphs,* Sixth Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1999). Discrete Appl. Math. **120**(1–3) (2002), 25–33.

[73] S. BRANDT: *A 4-colour problem for dense triangle-free graphs,* Cycles and colourings (Stará Lesná, 1999), Discrete Math. **251**(1-3) (2002), 33–46.

[74] S. BRANDT AND S. THOMASSÉ: *Dense triangle-free graphs are four-colorable: a solution to the Erdős-Simonovits problem,* J. Combin. Theory B (in press).

[75] G. BRIGHTWELL, K. PANAGIOTOU, AND A. STEGER: *Extremal subgraphs of random graphs,* Random Structures Algorithms **41**(2) (2012), 147–178.

[76] W. G. BROWN: *On graphs that do not contain a Thomsen graph,* Canad. Math. Bull. 9 (1966), 281–285.

[77] W. G. BROWN: *On an open problem of P. Turán concerning 3-graphs,* Studies in Pure

Mathematics (dedicated to Paul Turán), Birkhäuser Verlag, 1983, 91–93.

[78] W. G. Brown, P. Erdős, and M. Simonovits: *Extremal problems for directed graphs,* J. Combin Theory Ser B 15 (1973), 77–93.

[79] W. G. Brown, P. Erdős and M. Simonovits: *On multigraph extremal problems,* Problemes Combinatoires et Theorie des Graphes (ed. J. C. Bermond et al.), CRNS Paris, (1978), 63–66.

[80] W. G. Brown, P. Erdős and M. Simonovits: *Inverse extremal digraph problems,* Finite and Infinite Sets, Proc. Coll. Math. Soc. J. Bolyai **37** (1981), 119–156.

[81] W. G. Brown, P. Erdős, and M. Simonovits: *Algorithmic solution of extremal digraph problems,* Trans Amer Math Soc **292** (1985), 421–449.

[82] W. G. Brown, P. Erdős and V.T. Sós: *On the existence of triangulated spheres in 3-graphs and related problems,* Periodica Math. Hungar. **3** (1973), 221–228.

[83] W. G. Brown, P. Erdős, and V.T. Sós: *Some extremal problems on r-graphs,* New directions in the theory of graphs (Proc Third Ann Arbor Conf, Univ Michigan, Ann Arbor, Mich, 1971), 53–63, Academic Press, New York 1973.

[84] W. G. Brown and F. Harary: *Extremal digraphs,* Combinatorial theory and its applications, Colloq. Math. Soc. J. Bolyai, **4** (1970), I. 135–198;

[85] W. G. Brown and M. Simonovits: *Digraph extremal problems, hypergraph extremal problems, and the densities of graph structures,* Discrete Mathematics, **48** (1984), 147–162.

[86] W. G. Brown and M. Simonovits: *Multigraph extremal problems,* In Paul Erdős and his Mathematics, Springer, (2002), 1–46.

[87] S.A. Burr, P. Erdős, and L. Lovász: *On graphs of Ramsey type,* Ars Combinatoria **1**(1) (1976), 167–190.

[88] S.A. Burr, P. Erdős, V.T. Sós, P. Frankl, and R.L. Graham: *Further results on maximal anti-Ramsey graphs,* Graph theory, combinatorics, and applications, Vol. 1 (Kalamazoo, MI, 1988), 193–206, Wiley-Intersci. Publ., Wiley, New York, 1991.

[89] S.A. Burr, P. Erdős, R.L. Graham, V.T. Sós: *Maximal anti-Ramsey graphs and the strong chromatic number,* J. Graph Theory **13**(3) (1989), 263–282.

[90] D. de Caen and Z. Füredi: *The maximum size of 3-uniform hypergraphs not containing a Fano plane,* J. Combin Theory Ser B **78** (2000), 274–276.

[91] Charlap, L. S., Rees, H. D. and Robbins, D. P.: *The asymptotic probability that a random biased matrix is invertible,* Discrete Math. **82** (1990), 153–163.

[92] F. Chung and L. Lu: *An upper bound for the Turán number t3(n,4),* J. Combin. Theory Ser. A **87** (1999), 381–389.

[93] Fan Chung and R. L. Graham: *Erdős on graphs, His legacy of unsolved problems,* A. K. Peters, Ltd., Wellesley, MA, 1998. xiv+142 pp. ISBN: 1-56881-079-2; 1-56881-111-X

[94] F.R.K. Chung, R.L. Graham, and R.M. Wilson: *Quasi-random graphs,* Combinatorica 9 (1989), **4**, 345–362.

[95] D. Conlon, J. Fox, Jacob, and B. Sudakov: *An approximate version of Sidorenko's conjecture,* Geom. Funct. Anal. **20**(6) (2010), 1354–1366.

[96] B. Csaba: *On the Bollobás - Eldridge conjecture for bipartite graphs,* Combin Probab Comput. 16 (2007), 661–691.

[97] B. DeMarco and Jeff Kahn: *Mantel theorem for random graphs,* submitted, arXiv:1206.1016.

[98] G.A. Dirac: *Some theorems on abstract graphs,* Proc. London. Math. Soc. 2 (1952),

69–81.
[99] G. A. DIRAC: *On the maximal number of independent triangles in graphs*, Abh. Math. Seminar Univ. Hamburg, 26 (1963), 78-82.
[100] G. DIRAC: *Extensions of Turán's theorem on graphs,* Acta Math Acad Sci Hungar 14 (1963), 417–422.
[101] J.D. DIXON: *How good is Hadamard's inequality for determinants?* Canad. Math. Bull. **27** (1984), 260–264.
[102] R. DOTSON, AND B. NAGLE: *Hereditary properties of hypergraphs,* J. Combin. Theory Ser. B **99** (2009), 460–473.
[103] T. DZIDO: *A note on Turan numbers for even wheels, manuscript,* submitted
[104] G. ELEK AND B. SZEGEDY: *Limits of hypergraphs, removal and regularity lemmas: A Non-standard Approach,*
arXiv:0705.2179, 2007.
[105] G. ELEK AND B. SZEGEDY: *A measure-theoretic approach to the theory of dense hypergraphs.* Adv. Math. **231** (2012), no. 3-4, 1731–1772.
[106] P. ERDŐS: *On sequences of integers no one of which divides the product of two others and some related problems,...* Mat. i Mech. Tomsk, **2** (1938), 74–82.
[107] P. ERDŐS: Elemente der Math. 10 (1955), 114.
[108] P. ERDŐS: *Some unsolved problems,* Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 221–254.
[109] P. ERDŐS: *On a theorem of Rademacher-Turán,* Illinois J. Math. **6** (1962), 122-127.
[110] P. ERDŐS: *On the number of complete subgraphs contained in certain graphs,* Publ. Math. Inst. Hung. Acad. Sci., VII, Ser. A **3** (1962), 459–464.
[111] P. ERDŐS: *On extremal problems of graphs and generalised graphs,* Israel J. Math **2** (1964), 183–190.
[112] P. ERDŐS: *On the number of triangles contained in certain graphs,* Canad. Math. Bull. **7**(1) (1964), 53–56.
[113] P. ERDŐS: *Extremal problems in graph theory,* Theory of graphs and its Applications, Proc. Coll. Smolenice (Czechoslovakia, 1963), (1964), 29–36.
[114] P. ERDŐS: *Some recent results on extremal problems in graph theory,* Theory of Graphs, Intern. Symp. Rome (1966), 118–123.
[115] P. ERDŐS: *On some new inequalities concerning extremal properties of graphs,* Theory of graphs, Tihany (Hungary, 1966), 77–81.
[116] P. ERDŐS: *On some applications of graph theory to geometry,* Canad. J. Math. **19** (1967), 968–971.
[117] P. ERDŐS: *On the number of complete subgraphs and circuits contained in graphs,* Casopis Pest. Mat. **94** (1969), 290–296.
[118] P. ERDŐS: *On the graph theorem of Turán,* (in Hungarian), Mat. Lapok **21** (1970), 249–251,
[119] P. ERDŐS: *Some unsolved problems in graph theory and combinatorial analysis,* Combinatorial Math. and its Applications, (Proc. Conf. Oxford, 1969) Acad. Press, 1971, 97–109.
[120] P. ERDŐS: *Problems and results in graphs theory and combinatorial analysis,* Proc. Fifth British Combin. Conference, 1975 (Aberdeen), 169–192.
[121] P. ERDŐS: *Paul Turán, 1910-1976: his work in graph theory,* J. Graph Theory **1**(2) (1977), 97-101.
[122] P. ERDŐS: *The art of counting: Selected writings,* Edited by Joel Spencer and with

a dedication by Richard Rado. Mathematicians of Our Time, Vol. 5. The MIT Press, Cambridge, Mass.-London, 1973. xxiii+742 pp.

[123] P. Erdős: *On the combinatorial problems which I would most like to see solved,* Combinatorica **1** (1981), 25–42.

[124] P. Erdős: *Some recent problems and results in graph theory,* The Second Krakow Conference on Graph Theory (Zgorzelisko, 1994), Discrete Math. **164**(1-3) (1997), 81–85.

[125] P. Erdős, P. Frankl, and V. Rödl: *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent,* Graphs and Combinatorics, **2** (1986), 113–121.

[126] P. Erdős and T. Gallai: *On maximal path. and circuits of graphs,* Acta Math. Acad. Sci. Hungar. **10** (1959), 337–356.

[127] P. Erdős, A. Hajnal, M. Simonovits, V.T. Sós, and E. Szemerédi: *Turán-Ramsey theorems and simple asymptotically extremal structures,* Combinatorica **13**(1) (1993), 31–56.

[128] P. Erdős, Hajnal, A., M. Simonovits, V.T. Sós, E. Szemerédi: *Turán-Ramsey theorems and K_p-independence numbers,* Combin. Probab. Comput. **3**(3) (1994), 297–325. (Reprinted by Cambridge Univ. Press, 1997.)

[129] P. Erdős, A. Hajnal, V.T. Sós and E. Szemerédi: *More results on Ramsey-Turán type problems,* Combinatorica **3**(1) (1983), 69–81.

[130] P. Erdős and R. Rado: *Intersection theorems for systems of sets, I and II,* J. London Math . Soc ., 35 (1960), 85-90 a

[131] P. Erdős, D.J. Kleitman and R. L. Rothschild: *Asymptotic enumeration of K_n-free graphs,* Proc. Coll. Rome, Theorie Combinatorie, Tomo II Acad, Nazion.'dei Lincei, Rome 1978, 19–27.

[132] P. Erdős, A. Meir, V.T. Sós and P. Turán: *On some applications of graph theory II,* Studies in Pure Math. (Dedicated to R. Rado, ed. by Mirsky) Academic Press, London, (1971), 89–100.

[133] P. Erdős, A. Meir, V.T. Sós and P. Turán: *On some applications of graph theory III,* Canad. Math. Bull. **15**(1) (1972), 27–32.

[134] P. Erdős, A. Meir, V.T. Sós and P. Turán: *On some applications of graph theory I,* Discrete Mathematics, **2** (1972), 207–228.

[135] P. Erdős, A. Rényi and V.T. Sós: *On a problem of graph theory,* Studia Sci. Math. Hung., **1** (1966), 215–235.

[136] P. Erdős and M. Simonovits: *A limit theorem in graph theory,* Studia Sci. Math. Hungar., **1** (1966), 51–57. (Reprinted in [122].)

[137] P. Erdős and M. Simonovits: *An extremal graph problem,* Acta Math. Acad. Sci. Hung., **22**(3–4) (1971), 275–282.

[138] P. Erdős, M. Simonovits: *Some extremal problems in graph theory,* Combinatorial theory and its applications, Vol. I, Proceedings Colloqium, Balatonfüred, (1969), North-Holland, Amsterdam, 1970, 377–390. (Reprinted in [122].)

[139] P. Erdős and M. Simonovits: *On a valence problem in extremal graph theory,* Discrete Math. **5** (1973), 323–334.

[140] P. Erdős and M. Simonovits: *Supersaturated graphs and hypergraphs,* Combinatorica **3**(3) (1983), 181–192.

[141] Erdős, P., M. Simonovits and V.T. Sós: *Anti-Ramsey theorems,* Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II,

633–643. Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[142] P. ERDŐS AND M. SIMONOVITS: *On the chromatic number of geometric graphs,* Ars Combin. **9** (1980), 229–246.

[143] P. ERDŐS, V.T. SÓS: *Some remarks on Ramsey's and Turán's theorem,* Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), 395–404. North-Holland, Amsterdam, 1970.

[144] P. ERDŐS AND V.T. SÓS: *On Turán-Ramsey type theorems. II,* Studia Sci. Math. Hungar. **14**(1) (1979), 27–36 (1982).

[145] P. ERDŐS AND V.T. SÓS: *On Ramsey-Turán type theorems for hypergraphs,* Combinatorica, **2**(3) (1982) 289–295.

[146] P. ERDŐS AND V.T. SÓS: *On a generalization of Turán's graph theorem,* Studies in Pure Math. (To the memory of Paul Turán) Birkhäuser Verlag, (1983), 181-185.

[147] P. ERDŐS AND J. SPENCER: *Probabilistic Methods in Combinatorics,* Acad. Press, NY, 1974

[148] P. ERDŐS AND A. H. STONE: *On the structure of linear graphs,* Bull. Amer. Math. Soc. **52** (1946), 1087-1091.

[149] P. ERDŐS AND G. SZEKERES: *A combinatorial problem in geometry,* Compositio Math. **2** (1935), 464-470.

[150] P. ERDŐS AND P. TURÁN: *On some sequences of integers* J. London Math. Soc. **11** (1936), 261–264.

[151] P. ERDŐS, AND ZS. TUZA: *Rainbow subgraphs in edge-colorings of complete graphs,* Quo vadis, graph theory?, 81–88, Ann, Discrete Math., **55**, North-Holland, Amsterdam, 1993.

[152] R.J. FAUDREE, A. SCHELTEN, AND I. SCHIERMEYER: *The Ramsey number* $r(C_7, C_7, C_7)$, Discuss. Math. Graph Theory **23**(1) (2003), 141–158.

[153] R.J. FAUDREE AND R.H. SCHELP: *Ramsey type results,* Coll. Math. Soc. J. Bolyai **10** Infinite and Finite Sets, Keszthely (Hungary, 1973) 657–665.

[154] R.J. FAUDREE AND R.H. SCHELP: *All Ramsey numbers for cycles in graphs,* Discrete Math. **8** (1974), 313–329.

[155] R.J. FAUDREE AND M. SIMONOVITS: *On a class of degenerate extremal problems,* Combinatorica, **3**(1) (1983), 83–93.

[156] R.J. FAUDREE AND M. SIMONOVITS: *Ramsey problems and their connection to Turán type extremal problems,* Journal of Graph Theory, Vol **16**(1) (1992) 25–50.

[157] A. FIGAJ AND T. ŁUCZAK: *The Ramsey number for a triple of long even cycles,* J. Combin. Theory Ser. B **97**(4) (2007), 584âĂŞ596.

[158] D.C. FISHER: *Lower bounds on the number of triangles in a graph,* J. Graph Theory **13**(4) (1989), 505–512.

[159] D.C. FISHER AND J. RYAN: *Bounds on the number of complete subgraphs,* Disc. Math. **103**(3), (1992), 313–320

[160] D. G. FON-DER-FLAASS: *A method for constructing (3,4)-graphs,* (Russian) Mat. Zametki **44**(4) (1988), 546–550, 559; translation in Math. Notes **44** (1988), 781–783 (1989).

[161] J. FOX, P.-S. LOH, AND Y. ZHAO: *The critical window for the classical Ramsey-Turán problem,* arXive 1208.3276

[162] P. FRANKL AND Z. FÜREDI: *Exact solution of some Turán-type problems,* J. Combin. Theory Ser. A **45** (1987), no. 2, 226–262.

[163] P. FRANKL AND V. RÖDL: *Hypergraphs do not jump,* Combinatorica, **4**(2-3) (1984),

149–159.
[164] A. Frieze and R. Kannan: *Quick Approximation to Matrices and Applications,* Combinatorica, **19**(2) (1999), 175-220.
[165] A. Frieze and B. Reed: *Polychromatic Hamilton cycles,* Discrete Mathematics **118** (1993), 69-74.
[166] A. Frohmader: *More constructions for Turán's (3,4)-conjecture,* Electron. J. Combin. **15**(1) (2008), Research Paper 137, 23 pp.
[167] Z. Füredi: *Turán type problems,* Surveys in Combinatorics, (A. D. Keedwell, ed.) Cambridge Univ. Press, London Math. Soc. Lecture Note Series, **166** (1991), 253–300.
[168] Z. Füredi: *Extremal hypergraphs and combinatorial geometry,* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1343–1352, Birkhäuser, Basel, 1995.
[169] Z. Füredi: *An upper bound on Zarankiewicz' problem,* Combin. Probab. Comput. **5**(1) (1996), 29–33.
[170] Z. Füredi: *On a Turán type problem of Erdős,* Combinatorica, **11**(1) (1991), 75–79.
[171] Z. Füredi: *New asymptotics for bipartite Turán numbers,* J. Combin. Theory Ser. A **75** (1996), no. 1, 141–144.
[172] Z. Füredi and D.S. Gunderson: *Extremal numbers for odd cycles Manuscript,* submitted
[173] Z. Füredi and Komlós, J.: *The eigenvalues of random symmetric matrices,* Combinatorica **1**(3) (1981), 233–241.
[174] Z. Füredi and A. Kündgen: *Turán problems for weighted graphs, Turán problems for integer-weighted graphs,* J. Graph Theory 40 (2002), 195–225.
[175] Z. Füredi, D. Mubayi, and O. Pikhurko: *Quadruple systems with independent neighborhoods,* J. Combin. Theory Ser. A **115**(8) (2008), 1552–1560.
[176] Z. Füredi, O. Pikhurko, and M. Simonovits: *The Turán density of the hypergraph, abc,ade,bde,cde,* Electron. J. Combin. **10** (2003) R18.
[177] Z. Füredi, O. Pikhurko, and M. Simonovits: *On triple systems with independent neighbourhoods,* Combin. Probab. Comput. **14**(5-6) (2005), 795–813.
[178] Z. Füredi, O. Pikhurko, and M. Simonovits: *4-books of three pages,* J. Combin. Theory Ser. A **113**(5) (2006), 882–891.
[179] Z. Füredi and M. Simonovits: *Triple systems not containing a Fano configuration,* Combin. Probab. Comput. **14**(4) (2005), 467–484.
[180] Z. Füredi and M. Simonovits: *The history of degenerate (bipartite) extremal graph problems,* Erdős Centennial, (2013), 169–264 Springer (see also on the Arxiv).
[181] Furstenberg: *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions.* J. Analyse Math. **31** (1977), 204–256.
[182] H. Furstenberg, and Y. Katznelson: *An ergodic Szemerédi theorem for commuting transformations.* J. Analyse Math. **34** (1978), 275–291 (1979).
[183] Gerke, S. and A. Steger: *(2005) The sparse regularity lemma and its applications,* In Surveys in Combinatorics 2005, Vol. 327 of London Mathematical Society Lecture Notes, Cambridge University Press, 227–258.
[184] A. W. Goodman: *On sets of acquaintances and strangers at any party,* American Mathematical Monthly, vol. **66**(9) (1959), 778–783.
[185] W.T. Gowers: *Lower bounds of tower type for Szemerédi's Uniformity Lemma,* Geom. Funct. Anal **7**, 1997, no. 2, 322–337.
[186] W.T. Gowers: *A new proof of Szemerédi's theorem,* Geom. Funct. Anal. **11**(3), 2001,

465–588.

[187] W. T. GOWERS: *Quasirandomness, counting and regularity for 3-uniform hypergraphs,* Combin. Probab. Comput. **15** (2006), 143–184.

[188] W.T. GOWERS: *Hypergraph regularity and the multidimensional SzemerÃl'di theorem,* Ann. of Math. (2) **166** (2007), no. 3, 897âĂŞ946.

[189] W. T. GOWERS: *Erdős and arithmetic progressions,* Erdős Centennial, (2013), 265–287.

[190] R. L. GRAHAM, B. L. ROTHSCHILD AND J. H. SPENCER: *Ramsey Theory,* Second Edition, Wiley, New York, 1990.

[191] B. GREEN AND T. TAO: *The primes contain arbitrarily long arithmetic progressions,* Annals of Mathematics. Second Series **167** (2008), 481–547.

[192] J. R. GRIGGS, M. SIMONOVITS, AND G. R. THOMAS: *Extremal graphs with bounded densities of small subgraphs,* J. Graph Theory **29** (1998), 185–207.

[193] R.K. GUY: *Many facetted problem of Zarankiewicz,* The many facets of graph Theory, Proc. Conf. Western Michigan Univ., Kalamazoo, Mich., (1968), 129–148.

[194] A. GYÁRFÁS, M. RUSZINKÓ, G.N. SÁRKÖZY, AND E. SZEMERÉDI: *Three color Ramsey numbers for paths,* Combinatorica, **27**(1), 2007, 35–69.

[195] A. GYÁRFÁS, M. RUSZINKÓ, G.N. SÁRKÖZY, AND E. SZEMERÉDI: *Corrigendum to: "Three-color Ramsey numbers for paths",* Combinatorica **28**(4) (2008), 499–502.

[196] E. GYŐRI, V. NIKIFOROV, AND R.H. SCHELP: *Nearly bipartite graphs,* Discrete Math. **272**(2-3) (2003), 187–196.

[197] R. HÄGGKVIST: *Odd cycles of specified length in nonbipartite graphs,* Graph theory (Cambridge, 1981), 89–99, North-Holland Math. Stud., 62, North-Holland, Amsterdam-New York, 1982.

[198] G. HAHN, AND C. THOMASSEN: *Path and cycle sub-Ramsey numbers and an edge-colouring conjecture,* Discrete Math. 62 (1986) 29–33.

[199] H. HATAMI: *Graph norms and Sidorenko's conjecture,* Israel J. Math. 175 (2010), 125–150.

[200] P.E. HAXELL AND Y. KOHAYAKAWA: *On an anti-Ramsey property of Ramanujan graphs,* Random Structures Algorithms **6**(4) (1995), 417–431.

[201] P.E. HAXELL, Y. KOHAYAKAWA, AND T. ŁUCZAK: *Turán's extremal problem in random graphs: Forbidding even cycles,* J. Combin. Theory Ser. B 64 (2) (1995) 273–287.

[202] P.E. HAXELL, Y. KOHAYAKAWA, AND T. ŁUCZAK: *Turán's extremal problem in random graphs: Forbidding odd cycles,* Combinatorica **16**(1) (1996) 107–122.

[203] C. HUNDACK, H. J. PRÖMEL, AND A. STEGER: *Extremal graph problems for graphs with a color-critical vertex,* Combin. Probab. Comput. 2 (1993), **4**, 465–477.

[204] C. HYLTEN-CAVALLIUS: *On a combinatorial problems,* Colloq. Math. **6** (1958), 59-65.

[205] C. JAGGER, P. ŠTOVIČEK, AND A. THOMASON: *Multiplicities of subgraphs,* Combinatorica **16** (1996), 123–141.

[206] T. JIANG AND D.B. WEST: *On the Erdős-Simonovits-Sós conjecture about the anti-Ramsey number of a cycle,* Special issue on Ramsey theory. Combin. Probab. Comput. 12 (2003), no. 5-6, 585–598.

[207] JIN, GUO PING: *Triangle-free graphs with high minimal degrees,* Combin. Probab. Comput. **2**(4) (1993), 479–490.

[208] C.R. JOHNSON, AND M. NEWMAN: *How bad is the Hadamard determinantal bound?*

J. Res. Nat. Bur. Standards Sect. B 78B (1974), 167–169.
[209] F. Juhász: *On the spectrum of a random graph,* *in* Algebraic Methods in graph theory, (eds Lovász, at al) Coll Math Soc. J. Bolyai **25** North-Holland 1981, 313–316.
[210] J. Kahn, and J. Komlós: *Singularity probabilities for random matrices over finite fields,* Combin. Probab. Comput. **10**(2) (2001), 137–157.
[211] J. Kahn, J. Komlós, and E. Szemerédi: *On the probability that a random* ± 1*-matrix is singular,* J. Amer. Math. Soc. **8** (1995), no. 1, 223–240.
[212] Gy. Károlyi and V. Rosta: *On the Ramsey multiplicities of odd cycles,* manuscript, under publication.
[213] Gy. Katona: *Gráfok, vektorok és valószinűségszámitási egyenlőtlenségek,* Mat. Lapok, **20**(1-2) (1969), 123–127.
[214] G.O.H. Katona: *Turán's graph theorem, measures and probability theory...* This volume,
[215] Gy. Katona, T. Nemetz and M. Simonovits: *A new proof of a theorem of Turán and some remarks on a generalization of it,* (in Hungarian), Mat. Lapok, **15** (1964), 228–238.
[216] Gy. Katona and B. S. Stechkin: *Combinatorial numbers, geometrical constants and probabilistic inequalities,* Dokl. Akad. Nauk. SSSR **251** (1980), 1293–1296.
[217] P. Keevash: *A hypergraph blow-up lemma,* Random Structures Algorithms **39**(3) (2011), 275–376.
[218] P. Keevash: *Hypergraph Turán problems,* Surveys in combinatorics 2011, 83–139, London Math. Soc. Lecture Note Ser., 392, Cambridge Univ. Press, Cambridge, 2011.
[219] P. Keevash and B. Sudakov: *The Turán number of the Fano plane,* Combinatorica 25(5) (2005), 561–574.
[220] P. Keevash, and B. Sudakov: *On a hypergraph Turán problem of Frankl,* Combinatorica **25**(6) (2005), 673–706.
[221] D.J. Kleitman and B.L. Rothschild: *The number of finite topologies,* Proc. Amer. Math. Soc. **25** (1970), 276–282.
[222] D.J. Kleitman, and B.L. Rothschild: *Asymptotic enumeration of partial orders on a finite set,* Trans. Amer. Math. Soc. 205 (1975), 205–220.
[223] D.J. Kleitman, B.L. Rothschild, and J.H. Spencer: *The number of semigroups of order* n*,* Proc. Amer. Math. Soc. 55 (1976), no. 1, 227–232.
[224] D.J. Kleitman and K.J. Winston: *On the number of graphs without 4-cycles,* Discrete Mathematics **41** (1982) 167–172.
[225] Y. Kohayakawa: *Szemerédi's regularity lemma for sparse graphs,* Foundations of Computational Mathematics (Rio de Janeiro), Springer, (1997), 216–230.
[226] Y. Kohayakawa, and T. Łuczak: *Sparse anti-Ramsey graphs,* J. Combin. Theory Ser. B 63 (1995), no. 1, 146–152.
[227] Y. Kohayakawa, T. Łuczak, and V. Rödl: *On* K_4*-free subgraphs of random graphs,* Combinatorica 17 (1997), no. 2, 173–213.
[228] Y. Kohayakawa and V. Rödl: *Regular pairs in sparse random graphs. I,* Random Structures Algorithms **22**(4) (2003), 359–434.
[229] Y. Kohayakawa, and V. Rödl: *Szemerédi's regularity lemma and quasi-randomness,* Recent advances in algorithms and combinatorics, 289–351, CMS Books Math./Ouvrages Math. SMC, 11, Springer, New York, 2003.
[230] Y. Kohayakawa; V. Rödl; and M. Schacht: *The Turán theorem for random graphs,* Combin. Probab. Comput. 13 (2004), no. 1, 61–91.

[231] Y. KOHAYAKAWA, M. SIMONOVITS, AND J. SKOKAN: *The 3-colored Ramsey number of odd cycles,* Proceedings of GRACO2005, 397–402 (electronic), // Electron. Notes Discrete Math., 19, Elsevier, Amsterdam, 2005.
[232] Y. KOHAYAKAWA, M. SIMONOVITS, AND J. SKOKAN: *The 3-colored Ramsey number of odd cycles*, JCTB, accepted,
[233] PH. G. KOLAITIS, H.J. PRÖMEL, AND B.L. ROTHSCHILD: *Asymptotic enumeration and a 0-1 law for m-clique free graphs,* Bull. Amer. Math. Soc. (N.S.) **13**(2) (1985), 160–162.
[234] PH.G. KOLAITIS, H.J. PRÖMEL AND B.L. ROTHSCHILD: *$K_{\ell+1}$-free graphs: asymptotic structure and a 0-1 law,* Trans. Amer. Math. Soc., 303 (1987), 637–671.
[235] J. KOLLÁR, L. RÓNYAI, AND T. SZABÓ: *Norm graphs and bipartite Turán numbers,* Combinatorica, **16** (1996), 399–406.
[236] JÁNOS KOMLÓS: *On the determinant of (0,1) matrices,* Studia Sci. Math. Hungar. **2** (1967), 7-21.
[237] J. KOMLÓS: *On the determinant of random matrices,* Studia Sci Math. Hungar. **3** (1968), 387–399.
[238] J. KOMLÓS, J. PINTZ, AND E. SZEMERÉDI: *On Heilbronn's triangle problem,* J. London Math. Soc. (2) **24**(3) (1981), 385–396.
[239] J. KOMLÓS, J. PINTZ, AND E. SZEMERÉDI: *A lower bound for Heilbronn's problem,* J. London Math. Soc. (2) 25 (1982), no. 1, 13–24.
[240] J. KOMLÓS, G. N. SÁRKÖZY, AND E. SZEMERÉDI: *Blow-up lemma,* Combinatorica 17 (1997), 109–123.
[241] J. KOMLÓS, G.N. SÁRKÖZY, AND E. SZEMERÉDI: *Proof of the Seymour conjecture for large graphs,* Annals of Combinatorics, **2**, (1998), 43–60.
[242] J. KOMLÓS, G. N. SÁRKÖZY, AND E. SZEMERÉDI: *An algorithmic version of the blow-up lemma,* Random Struct Algorithm **12** (1998), 297–312.
[243] J. KOMLÓS, G.N. SÁRKÖZY, AND E. SZEMERÉDI: *Proof of the Alon-Yuster conjecture,* Discrete Mathematics, **235** (2001), 255–269.
[244] A. HAJNAL AND E. SZEMERÉDI: *Proof of a conjecture of Erdős,* Combinatorial Theory and its Applications vol. II (P. Erdős, A. Rényi and V.T. Sós eds.), Colloq. Math. Soc. J. Bolyai 4, North-Holland, Amsterdam, 1970, 601–623.
[245] J. KOMLÓS: *The Blow-up Lemma,* Combinatorics, Probability and Computing, **8**, (1999), 161–176.
[246] J. KOMLÓS, G.N. SÁRKÖZY, AND E. SZEMERÉDI: *On the Pósa-Seymour conjecture,* Journal of Graph Theory, **29**, (1998), 167–176.
[247] I. LEVITT, G.N. SÁRKÖZY, E. SZEMERÉDI: *How to avoid using the Regularity Lemma; Pósa's Conjecture revisited,* Discrete Math. **310**(3) (2010), 630âĂŞ641.
[248] J. KOMLÓS, A. SHOKOUFANDEH, M. SIMONOVITS, AND E. SZEMERÉDI: *The regularity lemma and its applications in graph theory,* Theoretical aspects of computer science (Tehran, 2000), 84–112, Lecture Notes in Comput. Sci., 2292, Springer, Berlin, 2002.
[249] J. KOMLÓS AND M. SIMONOVITS: *Szemerédi regularity lemma and its application in graph theory,* Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), 295–352, Bolyai Soc. Math. Stud., 2, János Bolyai Math. Soc., Budapest, 1996.
[250] G. N. KOPYLOV: *On maximal path and cycles in a graph*, Dokl. Akad. Nauk SSSR **234** (1977), no. 1, 19–21. (English translation: *Soviet Math. Dokl.* **18** (1977), no. 3, 593–596.)

[251] A. V.KOSTOCHKA: *A class of constructions for Turán's (3,4)-problem,* Combinatorica **2**(2) (1982), 187–192.

[252] T. KŐVÁRI, V.T. SÓS AND P. TURÁN: *On a problem of Zarankiewicz,* Coll. Math. **3** (1954), 50–57.

[253] I.N. KOVALENKO AND A.A. LEVITSKAYA: *Limiting behavior of the number of solutions of a system of random linear equations over a finite field and a finite ring,* Dokl. Akad. Nauk SSSR **221** (1975), 778–781. In Russian.

[254] I.N. KOVALENKO, A.A. LEVITSKAYA, AND M.N. SAVCHUK,: *Selected Problems in Probabilistic Combinatorics,* Naukova Dumka, Kiev. (1986), In Russian.

[255] D. KÜHN AND D. OSTHUS: *Embedding large subgraphs into dense graphs,* Surveys in Combinatorics, Cambridge University Press, 2009, 137–167.

[256] F. LAZEBNIK AND V.A. USTIMENKO: *Some algebraic constructions of dense graphs of large girth and of large size,* Expanding graphs (Princeton, NJ, 1992), 75–93, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 10, Amer. Math. Soc., Providence, RI, 1993

[257] F. LAZEBNIK, V. A. USTIMENKO, AND A.J. WOLDAR: *A new series of dense graphs of high girth,* Bull Amer Math Soc 32 (1995), 73–79.

[258] F. LAZEBNIK, V.A. USTIMENKO, A.J. WOLDAR: *Polarities and 2k-cycle-free graphs,* 16th British Combinatorial Conference (London, 1997). Discrete Math. 197/198 (1999), 503–513.

[259] H. LEFMANN, Y. PERSON, V. RÖDL, AND M. SCHACHT: *On colourings of hypergraphs without monochromatic Fano planes,* Combin. Probab. Comput. 18 (2009), no. 5, 803–818.

[260] D. LONDON: *Inequalities in quadratic forms,* Duke Math. J. **83** (1966), 511–522.

[261] L. LOVÁSZ: *Independent sets in critical chromatic graphs,* Acta Math. Acad. Sci. Hungar. **8** (1973), 165–168.

[262] L. LOVÁSZ: *Kneser's conjecture, chromatic number, and homotopy,* J. Combin. Theory Ser. A **25**(3) (1978), 319–324.

[263] L. LOVÁSZ: *Large graphs, graph homomorphisms and graph limits,* 2012 (AMS BOOK)

[264] L. LOVÁSZ AND M. SIMONOVITS: *On the number of complete subgraphs of a graph,* Proc. Fifth British Conf. on Combinatorics, Aberdeen (1975), 431-442.

[265] L. LOVÁSZ AND M. SIMONOVITS: *On the number of complete subgraphs of a graph II,* Studies in Pure Math. (dedicated to P. Turán) (1983), 458–495 Akadémiai Kiadó+Birkhäuser Verlag.

[266] L. LOVÁSZ AND B. SZEGEDY: *Szemerédi's lemma for the analyst,* J. Geom Func Anal 17 (2007), 252–270.

[267] A. LUBOTZKY, R. PHILLIPS, AND P. SARNAK: *Ramanujan graphs,* Combinatorica 8 (1988), 261–277.

[268] T. ŁUCZAK: *On the structure of triangle-free graphs of large minimum degree,* Combinatorica 26 (2006), **4**, 489–493.

[269] T. ŁUCZAK: $R(C_n; C_n; C_n) \leq (4 + o(1))n$, Journal of Combinatorial Theory, Ser. B, 75, 1999, 174–187.

[270] T. ŁUCZAK: *The Ramsey number for a triple of long even cycles,* J. Combin. Theory Ser. B **97**(4) (2007) 584–596.

[271] T. ŁUCZAK, M. SIMONOVITS: *On the minimum degree forcing F-free graphs to be (nearly) bipartite,* Discrete Math. 308 (2008), no. 17, 3998–4002.

[272] T. Łuczak, M. Simonovits, and J. Skokan: *On the multi-colored Ramsey numbers of cycles,* J. Graph Theory 69 (2012), no. 2, 169–175. `arXiv:1005.3926v1 [math.CO]`.

[273] W. Mantel: *Problem 28,* Wiskundige Opgaven **10** (1907), 60–61.

[274] J.J. Montellano-Ballesteros, V. Neumann-Lara: *An anti-Ramsey theorem on cycles,* Graphs Combin. **21**(3) (2005), 343–354.

[275] J. W. Moon: *On independent complete subgraphs in a graph,* Canad. J. Math., **20** (1968), 95–102. also in: International Congress of Math. Moscow, (1966), vol 13.

[276] J. W. Moon and Leo Moser: *On a problem of Turán,* Magyar. Tud. Akad. Mat. Kutató Int. Közl. (Publ. Mathematical Institute of the Hungarian Academy of Sciences), **7** (1962), 283–286.

[277] T. S. Motzkin and E. G. Straus: *Maxima for graphs and a new proof of a theorem of Turán,* Canad, J. Math. **17** (1965), 533–540.

[278] M. Mörs: *A new result on Zarankiewicz problem,* J. Combin. Theory A, **31** (1981), 126–130.

[279] D. Mubayi: *Counting substructures I: color critical graphs,* Adv. Math. **225**(5) (2010), 2731–2740.

[280] D. Mubayi: *The co-degree density of the Fano plane,* J. Combin. Theory Ser B 95 (2005), 333–337.

[281] D. Mubayi and Y. Zhao: *Co-degree density of hypergraphs,* J. Combin Theory Ser A 114 (2007), 1118–1132.

[282] H. P. Mulholland and C. A. B. Smith: *An inequality arising in genetical theory,* Amer. Math. Monthly 66 (1969), 673–683.

[283] B. Nagle and V. Rödl: *The asymptotic number of 3-graphs not containing a fixed one,* Discrete Math. 235 (2001), 271–290.

[284] B. Nagle; V. Rödl; M. Schacht: *Extremal hypergraph problems and the regularity method,* Topics in discrete mathematics, 247–278, Algorithms Combin., 26, Springer, Berlin, 2006.

[285] B. Nagle; Poerschke, Annika; V. Rödl; M. Schacht: *Hypergraph regularity and quasi-randomness,* Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, 227–235, SIAM, Philadelphia, PA, 2009.

[286] V. Nikiforov: *The number of cliques in graphs of given order and size,* Trans. Amer. Math. Soc. 363 (2011), no. 3, 1599–1618. (also http://arxiv.org/abs/0710.2305v2.)

[287] Y. Person and M. Schacht: *Almost all hypergraphs without Fano planes are bipartite,* In: Claire Mathieu (editor): Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 09), 217–226. ACM Press.

[288] H. J. Prömel and A. Steger: *Excluding induced subgraphs: quadrilaterals,* Random Structures Algorithms **2**(1) (1991), 55–71.

[289] H.J. Prömel and A. Steger: *Excluding induced subgraphs. III, A general asymptotic,* Random Structures Algorithms **3**(1) (1992), 19–31.

[290] H.J. Prömel and A. Steger: *Excluding induced subgraphs. II, Extremal graphs,* Discrete Appl. Math. **44**(1-3) (1993), 283–294.

[291] H.J. Prömel and A. Steger: *The asymptotic number of graphs not containing a fixed color-critical subgraph,* Combinatorica, **12** (1992) 463–473.

[292] R. Rado: *Anti-Ramsey theorems,* Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. III, 1159–1168. Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[293] A. Razborov: *Flag algebras,* J. Symbolic Logic **72** (2007), 1239–1282.

[294] A.A. Razborov: *On 3-hypergraphs with forbidden 4-vertex configurations,* SIAM J. Discrete Math. **24**(3) (2010), 946–963.

[295] A. A. Razborov: *On the minimal density of triangles in graphs,* Combin. Probab. Comput. 17 (2008), **4**, 603–618.

[296] A.A. Razborov: *Flag Algebras: an Interim report,* in "The Mathematics of Paul Erdős II" Graham, Ronald L.; Nešetřil, Jaroslav; Butler, Steve (Eds.) 2nd ed. 2013, XIX, 607 p. 66 illus., 2 illus. in color.

[297] C. Reiher: *Minimizing the number of cliques in graphs of given order and edge density.* arXiv:1212.2454, 2012 - arxiv.org

[298] V. Rödl, B. Nagle, J. Skokan, M. Schacht, and Y. Kohayakawa: *The hypergraph regularity method and its applications.* Proc. Natl. Acad. Sci. USA 102 (2005), no. 23, 8109–8113.

[299] V. Rödl: *Note on a Ramsey-Turán type problem,* Graphs Combin. **1**(3) (1985), 291–293.

[300] V. Rödl and A. Ruciński: *Perfect matchings in ε-regular graphs and the Blow-up Lemma,* Combinatorica, **19** (1999), 437–452.

[301] V. Rödl and A. Ruciński: *Dirac-type questions for hypergraphs - a survey (or more problems for Endre to solve),* An Irregular Mind (Szemerédi is 70), Bolyai Soc. Math. Stud. **21** (2010).

[302] V. Rödl and M. Schacht: *Regularity lemmas for graphs,* Fete of combinatorics and computer science, 287–325, Bolyai Soc. Math. Stud., 20, János Bolyai Math. Soc., Budapest, 2010.

[303] V. Rödl and M. Schacht: *Extremal results in random graphs,* Erdős Centennial, (2013), (eds L. Lovász, I. Ruzsa and V.T. Sós) Springer Verlag 535–583.

[304] V. Rödl and A. Sidorenko: *On the jumping constant conjecture for multigraphs,* J. Combin. Theory Ser. A 69 (1995), no. 2, 347–357.

[305] V. Rödl and Zs. Tuza: *Rainbow subgraphs in properly edge-colored graphs,* Random Structures Algorithms 3 (1992), no. 2, 175–182.

[306] Rosta, V.: *On a Ramsey type problem of Bondy and Erdős, I and II,* J. Comb. Theory B **15** (1973), 94–120.

[307] V. Rosta and L. Surányi: *A note on the Ramsey multiplicity of the circuit,* Period. Math. Hungar. **7** (1976), 223–227

[308] K.F. Roth: *Sur quelques ensembles d'entiers,* C. R. Acad. Sci. Paris **234** (1952), 388–390.

[309] K.F. Roth: *On certain sets of integers,* J. London Math. Soc. **28** (1953), 104–109.

[310] H. Ruben: *The volume of a random simplex in an n-Ball is asymptotically normal,* J. Appl. Probability, **14** (1977), 647-653.

[311] I. Z. Ruzsa and E. Szemerédi: *Triple systems with no six points carrying three triangles,* Combinatorics, (Keszthely, 1976), (1978), **18**, Vol. II., 939–945. North-Holland, Amsterdam-New York.

[312] T. Sanders: *On Roth's theorem on progressions,* Ann. of Math. (2) **174**(1) (2011), 619–636.

[313] I. Schiermeyer: *Rainbow 5- and 6-cycles: a proof of the conjecture of Erdős Simonovits and Sós,* Preprint, TU Bergakademie Freiberg, 2001.

[314] J.-C. Schlage-Puchta: *An inequality for means with applications.* Arch. Math. (Basel) **90**(2) (2008), 140–143.

[315] A. SCHRIJVER: *Vertex-critical subgraphs of Kneser graphs,* Nieuw Arch. Wisk. (3) **26**(3) (1978), 454–461.
[316] ALEXANDER SCOTT: *Szemerédi's Regularity Lemma for Matrices and Sparse Graphs,* Combin. Probab. Comput. 20 (2011), no. 3, 455–466.
[317] A. F. SIDORENKO: *Boundedness of optimal matrices in extremal multigraph and digraph problems,* Combinatorica, **13**(1) (1993), 109–120.
[318] A. F. SIDORENKO: *What do we know and what we do not know about Turán Numbers,* Graphs Combin. **11**(2) (1995), 179–199.
[319] A.F. SIDORENKO: *Inequalities for functionals generated by bipartite graphs* (Russian) Diskret. Mat. **3** (1991), 50–65; translation in Discrete Math. Appl. 2 (1992), 489–504.
[320] A.F. SIDORENKO: *A correlation inequality for bipartite graphs,* Graphs and Combin. **9** (1993), 201–204.
[321] M. SIMONOVITS: *A method for solving extremal problems in graph theory,* Theory of graphs, Proc. Coll. Tihany, (1966), (Ed. P. Erdős and G. Katona) Acad. Press, N.Y., (1968), 279–319.
[322] M. SIMONOVITS: *On colour critical graphs,* Studia Sci.Math. Hungar. **7** (1972), 67–81.
[323] M. SIMONOVITS: *On the structure of extremal graphs,* thesis for "candidate degree", (≈ PhD. Thesis) 1969.
[324] M. SIMONOVITS: *The extremal graph problem of the icosahedron,* J. Combinatorial Theory, (B) **17** (1) (1974), 69-79.
[325] M. SIMONOVITS: *Extremal graph problems with symmetrical extremal graphs, additional chromatic conditions,* Discrete Math., **7** (1974), 349–376.
[326] M. SIMONOVITS: *On Paul Turán's influence on graph theory,* J. Graph Theory **1**(2) (1977), 102–116.
[327] M. SIMONOVITS: *Extremal Graph Theory,* Selected Topics in Graph Theory, (ed. by Beineke and Wilson) Academic Press, London, New York, San Francisco, 161–200. (1983)
[328] M. SIMONOVITS: *Extremal graph problems, Degenerate extremal problems and Supersaturated graphs,* Progress in Graph Theory (Acad Press, ed. Bondy and Murty) (1984), 419–437.
[329] M. SIMONOVITS: *Extremal graph problems and graph products,* Studies in Pure Mathematics, (to the memory of P. Turán), 1983 Akad. Kiadó+Birkhäuser V., 669-680.
[330] M. SIMONOVITS: *How to solve a Turán type extremal graph problem? (linear decomposition).* Contemporary trends in discrete mathematics (Štiřin Castle, 1997), 283–305, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., **49**, Amer. Math. Soc., Providence, RI, 1999.
[331] M. SIMONOVITS: *Paul Erdős' influence on extremal graph theory,* The mathematics of Paul Erdős, II, 148–192, Algorithms Combin., **14**, Springer, Berlin, 1997.
[332] M. SIMONOVITS: *Paul Erdős' Influence on Extremal Graph Theory* (Updated/Extended version of [331]), Springer, 2013.
[333] M. SIMONOVITS AND V.T. SÓS: *Szemerédi's partition and quasirandomness,* Random Structures Algorithms **2**(1) (1991), 1–10.
[334] M. SIMONOVITS AND V.T. SÓS: *On restricted colourings of K_n,* Combinatorica **4**(1) (1984), 101–110.
[335] M. SIMONOVITS AND V.T. SÓS: *Ramsey-Turán Theory,* Discrete Math. **229** (2001),

293–340.
[336] R. Singleton: *On minimal graphs of maximum even girth,* Journal of Combinatorial Theory **1** (1966), 306–332.
[337] J. Solymosi: Regularity, uniformity, and quasirandomness, Proc. Natl. Acad. Sci. USA 102 (2005), no. 23,
[338] V.T. Sós: *On extremal problems in graph theory,* Combinatorial Structures and Their Applications, (Gordon and Breach, N. Y., 1970), 407–410.
[339] V.T. Sós: *Some remarks on the connection between graph-theory, finite geometry and block designs,* Theorie Combinatorie, Acc. Naz.dei Lincei (1976), 223–233.
[340] V.T. Sós: *Turbulent years: Erdős in his correspondence with Turán from 1934 to 1940,* Paul Erdős and his mathematics, I (Budapest, 1999), 85–146, Bolyai Soc. Math. Stud., **11**, János Bolyai Math. Soc., Budapest, 2002.
[341] J. H. Spencer: *From Erdős to algorithms,* Trends in Discrete Mathematics, Discrete Math. **136**(1-3) (1994), 295–307.
[342] B. Sudakov: *A few remarks on Ramsey-Turán-type problems,* J. Combin. Theory Ser. B **88**(1) (2003), 99–106.
[343] B. Sudakov: *Recent developments in extremal combinatorics: Ramsey and Turán type problems.* Proceedings of the International Congress of Mathematicians. Volume IV, 2579–2606, Hindustan Book Agency, New Delhi, 2010.
[344] G. Szekeres: *Tournaments and Hadamard matrices,* Enseignement Math **15** (1969), 269–278.
[345] G. Szekeres: *The average value of skew Hadamard matrices,* Proceedings of the First Australian Conference on Combinatorial Mathematics (Univ. Newcastle, Newcastle, 1972), 55–59.
[346] G. Szekeres: *The graph algebra of skew Hadamard determinants,* Sets, graphs and numbers (Budapest, 1991), 691–698, Colloq. Math. Soc. János Bolyai, **60**, (North-Holland, Amsterdam) 1992.
[347] G. Szekeres and P. Turán: *On an extremal problem in the theory of determinants (in Hungarian),* Matem. és Természettud. Értesítő, (1937), 796-806. (Reprinted in English in [368]).
[348] E. Szemerédi: *On sets of integers containing no k elements in arithmetic progression,* Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Jurii Vladimirovič Linnik.
[349] E. Szemerédi: *Regular partitions of graphs,* Problemes Combinatoires et Théorie des Graphes (ed. J. Bermond et al.), CNRS Paris, 1978, 399–401.
[350] E. Szemerédi: *On graphs containing no complete subgraphs with 4 vertices* (in Hungarian) Mat. Lapok, **23** (1972), 111–116.
[351] T. Tao: *A variant of the hypergraph removal lemma,* Journal of Combinatorial Theory, Ser. A 113, 2006, 1257-1280.
[352] Terry Tao, Vu, Van: *On random* ± 1 *matrices: singularity and determinant,* STOC'05: Proceedings of the 37th Annual ACM Symposium on Theory of Computing, 431–440, ACM, New York, 2005.
[353] A.G. Thomason: *Pseudorandom graphs,* In North-Holland Mathematical Studies, Vol. 144 (Random graphs'85, Poznań), North-Holland, Amsterdam, 1987, 307–331.
[354] A.G. Thomason: *A disproof of a conjecture of Erdős in Ramsey theory,* J. London Math. Soc. 39 (1989), 246–255.
[355] C. Thomassen: *On the chromatic number of triangle-free graphs of large minimum*

degree, Combinatorica 22 (2002), 591–596.
[356] P. TURÁN: *On a theorem of Hardy and Ramanujan,* J. London Math. Soc. **9** (1934), 274-276.
[357] P. TURÁN: *Extremal problems for determinants,* (in Hungarian) Math. és Term. Tud. Értesítő, **59** (1940), 95–105. (= Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, (1940), 95–105.)
[358] P. TURÁN: *On an extremal problem in graph theory,* Matematikai Lapok, **48** (1941), 436–452 (in Hungarian), (see also [359], [368]).
[359] P. TURÁN: *On the theory of graphs,* Colloq. Math., **3** (1954), 19–30, (see also [368]).
[360] P. TURÁN: *On a problem in the theory of determinants,* Acta Sinica (1955), 411–423.
[361] P. TURÁN: *Research problem,* MTA Mat. Kutató Int. Közl. **6** (1961), 417–423.
[362] P. TURÁN: *On some questions concerning determinants,* Annales Polon. Matem. 12 (1962), 49–53.
[363] P. TURÁN: *Applications of graph theory to geometry and potential theory,* Proc. Calgary International Conf. on Combinatorial Structures and their Application, (1969), 423–434 (see also [368]).
[364] P. TURÁN: *Remarks on the packing constants of the unit sphere,* (in Hungarian) Mat. Lapok, **21** (1970), 39–44. (Reprinted in English in [368].)
[365] P. TURÁN: *On some applications of graph theory to analysis,* Proc. Internal. Confer, on Constructive Th. of Functions, Varna (1970).
[366] P. TURÁN: *A general inequality in Potential theory,* Proc. NRD Conference on Classical Function Theory (1970), 137–141.
[367] P. TURÁN: *A Note of Welcome,* Journal of Graph Theory, **1** (1977), 7–9.
[368] COLLECTED PAPERS OF PAUL TURÁN: Akadémiai Kiadó, Budapest, 1989. Vol 1–3, (with comments of Simonovits on Turán’s graph theorem pp241-256, its applications pp1981–1985 and the Szekeres-Turán matrix results p88–89).
[369] P. TURÁN: *On a new method of analysis and its applications.* With the assistance of G. Halász and J. Pintz. With a foreword by V.T. Sós. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984. xvi+584 pp. ISBN: 0-471-89255-6
[370] VAN VU: *Random Discrete Matrices,* Horizons of combinatorics, 257–280, Bolyai Soc. Math. Stud., 17, Springer, Berlin, 2008. (arXiv:math.CO/0611321v1)
[371] R. WENGER: *Extremal graphs with no C^4, C^6 and C^{10},* J. Combin Theory Ser B **52** (1991), 113–116.
[372] D. WILSON AND D. KLEITMAN: *On the number of graphs which lack small cycles* Manuscript, 1996.
[373] D. R. WOODALL: *Sufficient conditions for circuits in graphs,* Proc. London Math. Soc. (3) **24** (1972), 739-755.
[374] R. YUSTER: *The number of edge colorings with no monochromatic triangle,* J. Graph Theory 21 (1996), **4**, 441–452.
[375] A. A. ZYKOV: *On some properties of linear complexes* Mat.Sbornik n.s. 24 **66** (1949), 163-188, (in Russian), translated into English, Amer. Math. Soc. Transl. no. 79, 33 (1952).

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