EXTREMAL GRAPH PROBLEMS

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<u>Notations</u>. $v(G)$, $e(G)$, $\chi(G)$ denote the number of vertices, edges and the chromatic number of the graph G. Here the graphs have no directed, multiple or loop edges. $\overset{d}{\underset{i=1}{\times}} G_i$ denotes the product of graphs G_i, i.e. the graph, obtained by joining vertices of G_i to the vertices of the other G_i-s.

Generalizing a well-known theorem of Turán [1] Erdös and I have proved independently [3], [4] that for any given graph $M_1,\dots,M_k$ and fixed n if K^n has maximum number of edges among graphs of n vertices, not containing any M_i as a subgraph, then

<u>Theorem A</u>. *There exist graphs* $N_1,\dots,N_d$, $(d+1 = \min \chi(M_i))$ *such that* K^n *can be obtained from* $\overset{d}{\underset{i=1}{\times}} N_i$ *omitting* $0(n^{2-\frac{1}{r}})$ *edges from it. Here is an integer depending only on* $M_1,\dots,M_\mu$ *and*

(1) $v(N_i) = \frac{n}{d} + 0(n^{1-\frac{1}{r}})$, $e(N_i) = 0(n^{2-\frac{1}{r}})$

(2) *any vertex of* N_i *has valence* $\geq \frac{n}{d}(d-1) + 0(n^{1-\frac{1}{r}})$

(3) *the number of vertices of* N_i *joined to at least* εn *vertices of the same* N_i *is* $0_\varepsilon(1)$.

The graph K^n is called the *extremal graph* for $M_1,\dots,M_\mu$. Theorem A shows that the extremal graphs for $M_1,\dots,M_\mu$ are fairly well determined by $\min \chi(M_i)$, they depend loosely on the structure of M_i-s.

How the structure of M_i-s influence the structure of the extremal graphs? Erdös and I have proved [5] that the extremal graphs for $K(3,r_1,\dots,r_d)$ are products: $K^a = \overset{d}{\underset{i=1}{\times}} N_i$ where $3 \leq r_1 \leq r_d$ and

(1) $v(N_i) = \frac{n}{d} + O(n^{2/3})$

(2) N_i is an extremal graph for $K(3,r_1)$.

(3) $N_2,\ldots,N_d$ are extremal graphs for $(1,r_2)$.

Here 3 can be replaced by 2 or 1 as well.

I have found the following generalization of this latest theorem:

Notation.

(1) $f(n,M_1,\ldots,M_k)$ denotes the number of edges of the extremal graphs for $M_1,\ldots,M_\mu$.

(2) Let $\chi(M) = 2$ and colour both M and $K(n, n)$ by two colours: red and blue. We consider subgraphs G^{2n} of $K(n,n)$ such that if M is the subgraph of G^{2n}, then the class of blue vertices of M is not contained by the class of blue vertices of $K(n,n)$. The maximum of $e(G^{2n})$ will be denoted by $h(n,G^{2n})$.

Definition. $x \in M_1$ is a *weak point* for $M_1,\ldots,M_\mu$ if $\chi(M_1) = 2$ and $h(n;M_1 - x) = o(f(n;M_1,\ldots,M_\mu))$.

Remark. If there exists an automorphism of $M_1 - x$ changing the colours, then our condition with $f(n;M_1 - x) = o(f(n;M_1,\ldots,M_\mu))$.

Examples.

(1) $K(r_0,\ldots,r_d)$ has weak points if either $r_0 \leq 3$, or if $r_0^2 - 3r_0 + 3 > r_1$. [5] Probably it always has.

(2) If M is not a tree, but $M - x$ is, $\chi(M) = 2$ then $x \in M$ is a weak point of it.

(3) Let $C(2l)$ be a circuit of $2l$ vertices, $x \notin C(2l)$ and let x be joined to 5 or more vertices of $C(2l)$ so that the obtained graph M be two-chromatic. Then $x \in M$ is a weak point of it.

(4) Let M be a graph, obtained from two $C(2l)$ or from two $K(r,r)$ by joining them by a path of length 2. Then M has *no* weak point.

Theorem 1. *Let M be a d+1 chromatic graph and let us colour it by* 1,2,...,d+1. *L_{ij} denotes the subgraph of M spanned by the vertices of the ith and jth colours. If $x \in L_{ij}$ is a weak point of $\{L_{ij}\}$ and K^n is an extremal graph for M, then K^n can be obtained from a suitable product $N^n = \times_{i=1}^{d} N_i$ omitting $o(n)$ edges from it. Here*

(1) $v(N_i) = \frac{n}{d} + o(n)$

(2) *N_i is almost an extremal graph for $\{L_{ij}\}$ it has $f(n;\ldots,L_{ij},\ldots) + O(n)$ edges, but it does not contain any L_{ij}.*

(3) *The vertices of N_i $(i=2,\ldots,d)$ are joined to less than s other vertices of N_i, if x is joined to s vertices of the 3rd colour.*

Theorem 2. *If in Theorem 1. $r \leq 3$, then $o(n)$ can be replaced by $o(1)$. If $r \leq 2$, then there exists an extremal graph K^n such that $K^n = \times_{i=1}^{d} N_i$ whenever n is large enough.*

Remarks.

(1) Similar theorems hold if M is replaced by $M_1,\ldots,M_\mu$. The only change is that L_{ij}-s must be replaced by those subgraphs of $M_1,\ldots,M_\mu$, for which $\chi(M_j - L_t) = \min \chi(M_j) - 2$ if $L_t \subseteq M_j$.

(2) Theorem 1 has "assymptotic" character, but it has many corollaries of "exact" character. One of them is the theorem of Erdös and mine about the extremal graphs for $K(3,r_1,\ldots,r_d)$. Another one is

Theorem 3. *Let $\Gamma(3k)$ be the graph, having the vertices $x_1,\ldots,x_k$; $y_1,\ldots,y_k$; $z_1,\ldots,z_k$ and defined by*

(i) *$x_i \to y_i \to z_i \to x_i$ is an automorphism of $\Gamma(3k)$.*

(ii) *$x_1,\ldots,x_k$, $y_1,\ldots,y_k$ determine a $C(2l)$.*

Then for $n > n_0$ any extremal graph K^n for $\Gamma(3k)$ is a product: $K^n = k_1 \times k_2$ where $v(K_i) = \frac{n}{2}$, $e(K_2) = 0$ and K_1 is an extremal graph for $\{\ldots,C(2l),\ldots\}$ $\frac{k}{2} \leq l \leq k$.

References

2. Turán, P., *Matematikai Lapik*, 48 (1941), 436-452. (in Hungarian).

3. Erdös, P., On some new inequalities concerning extremal properties of graphs. Theory of Graphs, Proc. Coll. held at Tihany, Hungary, 1966.

4. Simonovits, M., A method for solving extremal problems. Stability problems. Theory of Graphs, Proc. Coll. held at Tihany, Hungary, 1966.

5. Erdös, P. and Simonovits, M., An extremal graph problem. *Acta Math. Acad., Sci. Hungar.* (forthcoming).