Important open problems in Extremal graph theory

Miklós Simonovits,
Alfréd Rényi Math Inst

Xi'an
June 26, 2021

After the lecture slightly streamlined and some extra explanations were added. The "steps" are taken out.

Which problems are important?

We can ask many questions, some of them are important, some others are less important

- Which problems are important, that is difficult to decide.
- Erdős had a talent to ask the right questions
- Erdős often looked for the first difficult problem in the area
- Turán always explained the MOTIVATION of his problems
- It is not that important if the conjecture turns out to be right or wrong:

The important thing is if it leads to UNDERSTANDING the area.

Disclaimer

In this lecture I will not go into all the details, however, soon I will post a concise pdf form of the lecture, with many references:

My homepage is www.renyi.hu/~miki
and several surveys can also be found on my homepage

The lecture will be posted in a few days:

www.renyi.hu/~miki/XianSlides2021f.pdf

Extremal Graph Theory

Extremal graph theory is one of the oldest areas of Graph Theory. In the 1960's it started evolving into a large, deep, connected theory.

In this lecture we shall start with describing some major areas in the classical extremal graph theory. Then we shall concentrate on some open conjectures, problems.

What is an extremal graph problem?

We fix some objects, they form the UNIVERSE.

The simplest case is when we consider **simple graphs:** graphs without loops and multiple edges, and a family $\mathcal{L}$ of excluded subobjects. G_n is an n-vertex graph.
The EXTREMAL PROBLEM is to determine

$$\mathbf{ex}(n, \mathcal{L}) := \max\{e(G_n) \ : \ L \not\subset G_n \quad \text{if} \quad L \in \mathcal{L}\}$$

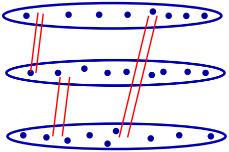
Generally there is a fixed family of objects, say

- graphs,
- multigraphs,
- digraphs,
- r-uniform hypergraphs,

= **Universe** and we maximize some parameter of these objects, say the number of edges, hyperedges, arcs, ... under the condition that the object has n vertices and does not contain some subgraphs, more generally, some sub-objects, fixed in advance.

Notation

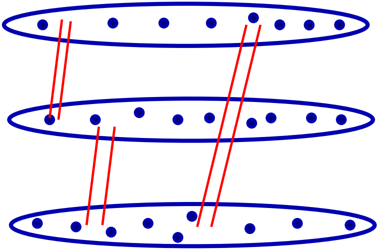
- G_n: n-vertex graph. The first subscript is always the number of vertices, e.g. in $T_{n,p}$.
- Product of two graphs: $G \bigotimes H$ Take two vertex-disjoint graphs, G and H and join each vertex of G to each vertex of H.



- $T_{n,p}$: Turán graph
- K_p, P_k, C_k: complete / path / cycle
- $\mathcal{L}$: family of forbidden subgraphs...

Turán theorem

Turán graph $T_{n,p}$: n vertices are partitioned into p classes as uniformly as possible and x, y are joined iff they belong to different classes.



Theorem (Turán 1941)

Among the graphs G_n (on n vertices) not containing a K_{p+1}, the Turán graph $T_{n,p}$ has the most edges: it is an **extremal graph** *and the* **only** *extremal graph.*

General asymptotics

Let $\mathcal{L}$ be a fixed family of excluded graphs. The SUBCHROMATIC NUMBER is

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1 \qquad (1)$$

Theorem (Erdős-Sim., Lim/ $o(n^2)$ form form)

$$\frac{\mathbf{ex}(n, \mathcal{L})}{e(T_{n,p})} \to 1, \quad i.e., \quad \mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$$

Theorem (Erdős-Sim. $\varepsilon - \delta$ form)

For every $\varepsilon > 0$ there exist a $\delta > 0$ for which if G_n does not contain any $L \in \mathcal{L}$ and has at least $\mathbf{ex}(n, \mathcal{L}) - \delta n^2$ edges, then G_n can be obtained from a $T_{n,p}$ by deleting and adding $< \varepsilon n^2$ edges.

Extremal problems in general

PROBLEM

Given a UNIVERSE $\mathcal{U}$, a property $\mathcal{P}$ and some parameters f, g, e on the Universe, and we try to maximize $e = e(G)$ on $\mathcal{U} \cap \mathcal{P}$ under the assumption that $f(G) = x_f$, $g(G) = x_g$. $e(G)$ is mostly the number of edges.

A simple example, Erdős 1962,... On "Rademacher-Turán"

How many edges can G_n have if it does not contain a K_3 and the chromatic number $\chi(G_n) \geq 3$.

Turán type extremal problems (again)

UNIVERSE = Ordinary, simple graphs.

$$\mathbf{ex}(n, \mathcal{L}) = \max_{L \in \mathcal{L}} e(G_n).$$

$\mathbf{EX}(\mathbf{n}, \mathcal{L})$ is the family of n-vertex graphs attaining the maximum: the family of EXTREMAL graphs.

For us the **structure** of extremal graphs is often more important than the **maximum** number of edges.

Origins in Geometry, Logic and Number theory

- MANTEL'S theorem (1907)
- ERDŐS, Multiplicative Sidon problem, Tomsk

Multiplicative Sidon condition for $a_1, \dots, a_m \in [n]$:

If $a_i a_j = a_k a_\ell$ then $\{i, j\} = \{k, \ell\}$. How large can m be?

- ERDŐS-SZEKERES theorem → Ramsey → Turán

Esther Klein, Erdős and Szekeres:

How many points of the plane $\mathbb{E}^2$ guarantee a convex k-gone?

CONJECTURE (ERDŐS-SZEKERES)

$n = 2^{k-2} + 1$ *points guarantee a convex k-gone.*

For $k = 4$ this is easy, for $k = 5$ this is difficult (older E. Makai), generally open, one of the most important question in COMBINATORIAL GEOMETRY

Unit distances:

Consider a metric space $\mathcal{M}$ and n points in it, $x_1, \ldots, x_n$ and join two of them iff their distance is 1.

Problem (Erdős, for $\mathbb{E}^d$ (Q1))

How many edges can have such a graph G_n?

Erdős Lemma: $O(n^{3/2})$

Problem (Hadwiger-Nelson (Q2))

How large can the chromatic number $\chi(G_n)$ be (as a function of d and n)?

Erdős-Stone-Sim.

Theorem (Erdős-Sim.)

Let

$$p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1.$$

Then

$$\mathbf{ex}(n, \mathcal{L}) = \mathbf{ex}(n, K_{p+1}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

So the maximum number of edges primarily depends on the **minimum chromatic number.**

Stability

Theorem (Extremal Structure, Erdős-Sim.)

Let

$$p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1.$$

If S_n is extremal for $\mathcal{L}$ then one can change $o(n^2)$ edges of S_n to get $T_{n,p}$.

Theorem (Almost extremal structure, Stability, Erdős-Sim.)

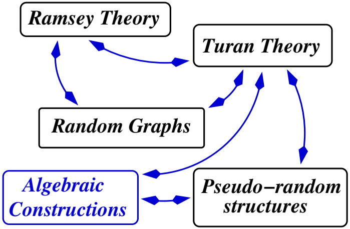
Let $p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1$. If (G_n) is an almost extremal graph sequence for $\mathcal{L}$, i.e.

no $L \in G_n$ and

$$e(G_n) > \mathbf{ex}(n, \mathcal{L}) - o(n^2)$$

then one can change $o(n^2)$ edges of G_n to get $T_{n,p}$.

Turán and Ramsey problems: very closely related



- Turán was motivated by Ramsey theorem
- Turán misjudged the symmetric Ramsey
- Random graph method emerged this way
- Stability method came from Extremal problems
- Szemerédi Regularity Lemma (new) came from extremal graph problems
- Applications of Ramsey Thm: (Erdős)
- Applications of Turán Thm (Turán,Katona, Sidorenko, Erdős-Meir-Sós-Turán)
- → Ramsey-Turán (T. Sós, Erdős-Hajnal-T. Sós-Szemerédi, (...+Sim) + Bollobás-Erdős) . . .

Ramsey, simplified

Given n,

for how large ℓ do we have a K_ℓ either in every G_n or in its complementary graph $\overline{G}_n$?

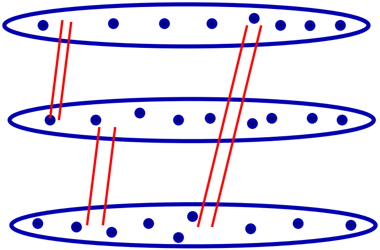
Given ℓ,

for how large n do we have a K_ℓ either in G_n or $\overline{G}_n$?

Problem (Ramsey-extremal graphs? (Q3))

V. T. Sós: Are Ramsey-extremal *graphs randomlike in some sense?*

The Random method enters



n vertices $p = [\sqrt{n}]$ classes

Turán conjectured: this is RAMSEY-EXTREMAL: ...

Let $m = \sqrt{n}$. Each G_n contains a K_m or m independent vertices

Erdős: the Random graph is much "better":

Let $m = (2 + \varepsilon) \log n$. Most G_n contains no K_m neither m independent vertices.

This way Extremal Graph Theory lead to Random Graphs.

Stability and Exact results

The stability methods help to prove exact results.

- We have an extremal problem with a conjectured simple extremal structure.

If we have stability, then

- First we show that in the important subcases the extremal structure is near to the conjectured one.

- Using this structural information we prove the conjectured extremal structure

Degenerate problems

We call the problem of $\mathbf{ex}(n, \mathcal{L})$ degenerate if

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2).$$

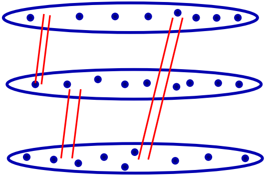
By Erdős-Sim. Theorem, or by Kővári-Sós-Turán thm

$\mathbf{ex}(n, \mathcal{L}) = o(n^2)$ iff $\mathcal{L}$ contains a bipartite L.

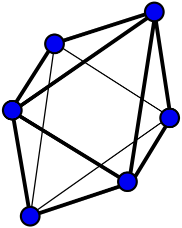
The Product conjecture tries to reduce general extremal problems to Degenerate extremal graph problems, showing that an $S_n \in \mathbf{EX}(\mathbf{n}, \mathcal{L})$ is the product of p graphs G_i which are extremal for some Degenerate problems $\mathbf{ex}(n, \mathcal{M}_i)$.

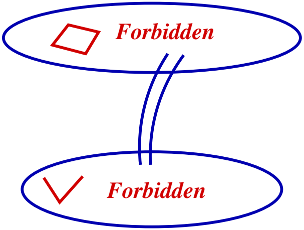
Octahedron Theorem

Erdős-Sim.:



Erdős-Sim.: Octahedron Theorem, see next page.





See Erdős-Rényi-T. Sós, and Füredi for C_4-free graphs, ...

The Product conjecture

We start with an illustration. Let $O_6 = K(2,2,2)$ be the octahedron graph.

Theorem (Octahedron Theorem, Erdős and Sim. (1971))

If S_n is an extremal graph for the octahedron O_6 for n sufficiently large, then there exist extremal graphs G_1 and G_2 for the circuit C_4 and the path P_3 such that $S_n = G_1 \bigotimes G_2$ and $|V(G_i)| = \frac{1}{2}n + o(n)$, $i = 1, 2$.

If G_1 does not contain C_4 and G_2 does not contain P_3, then $G_1 \bigotimes G_2$ does not contain O_6. Thus, if we replace G_1 by any $H_1 \in \mathbf{EX}(\mathbf{v}(\mathbf{G_1}), C_4)$ and G_2 by any $H_2 \in \mathbf{EX}(\mathbf{v}(\mathbf{G_2}), P_3)$, then $H_1 \bigotimes H_2$ is also extremal for O_6.

More generally, on $K_{p+1}(n_1, \dots, n_p)$

Theorem (Erdős–Sim.)

Let L be a complete $(p+1)$-partite graph, $L := K(a, b, r_3, r_4, ..., r_{p+1})$, where $r_{p+1} \geq r_p \geq \cdots \geq r_3 \geq b \geq a$ and $a = 2, 3$. There exists an $n_0 = n_0(a, b, \dots, r_{p+1})$ such that if $n > n_0$ and $S_n \in \mathbf{EX(n, L)}$, then $S_n = U_1 \bigotimes U_2 \bigotimes \dots \bigotimes U_p$, where

1. $v(U_i) = n/p + o(n)$, *for* $i = 1, \dots, p$.
2. U_1 *is extremal for* $K_{a,b}$
3. $U_2, U_3, \dots, U_p \in \mathbf{EX(n, K(1, r_3))}$.

This theorem is indeed a reduction theorem.

Conjecture (The Product Conjecture, Sim. (Q4))

Assume that $p(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(L) - 1 > 1$. If for some constants $c > 0$ and $\varepsilon \in (0,1)$

$$\mathbf{ex}(n,\mathcal{L}) > e(T_{n,p}) + cn^{1+\varepsilon}, \qquad (1)$$

then there exist p forbidden families $\mathcal{M}_i$, with

$$p(\mathcal{M}_i) = 1 \qquad \text{and} \qquad \max_{M\in\mathcal{M}_i} v(M) \le \max_{L\in\mathcal{L}} v(L),$$

such that any $S_n \in \mathbf{EX}(\mathbf{n},\mathcal{L})$ is a product: $S_n = G_1 \bigotimes \dots \bigotimes G_p$, where G_i are extremal for $\mathcal{M}_i$.

This means that the extremal graphs S_n are "products" of extremal graphs for some degenerate extremal problems (for $\mathcal{M}_i$), and therefore we may reduce the general case to degenerate extremal problems.

Explaining Condition (1)

This is equivalent with

There exist no $p+1$-colouring of any $L \in \mathcal{L}$ for which the first two colour classed span a tree.

Without this condition the "product" conjecture is not necessarily true:

in some cases, e.g. in Turán's theorem it does hold, in a complicated case Simonovits found a counterexample to it.

Remarks

(a) If we allow infinite families $\mathcal{L}$, then one can easily find counterexamples to this conjecture.
(b) If we allow linear error-terms, i.e. do not assume (1), then one can also find counterexamples, using a general theorem of Simonovits Sim74Symm; however, this is not trivial at all, see Sim83ProdBirk.
(c) A weakening of the above conjecture would be the following: for arbitrary large n, in Conjecture **??** there are several extremal graphs, and for each $n > n_{\mathcal{L}}$, some of them are of product form, (but maybe not all of them) and the families $\mathcal{M}_i$ also may depend on n a little.

Assume that for some $\gamma > 0$,

$$\mathbf{ex}(n, \mathcal{L}) > \left(1 - \frac{1}{p}\right)\binom{n}{2} + n^{1+\gamma}$$

for $n > n_0$. Then for every $n > n_1$ each extremal graph S_n is the product:

$$S_n := \prod_{i=1}^{p} H_i$$

where H_i is extremal for some degenerate extremal graph problem.

Open problems connected to degenerate extremal graphs

Theorem (Kővári–T. Sós–Turán, (1954))

Let $K_{a,b}$ denote the complete bipartite graph with a and b vertices in its color-classes. Then

$$\mathbf{ex}(n, K_{a,b}) \le \frac{1}{2}\sqrt[a]{b-1} \cdot n^{2-(1/a)} + O(n).$$

Kővári, T. Sós, and Turán conjectured that this is sharp:

Conjecture (Kővári–T. Sós–Turán (Q5))

For any integers $a \ge 2$ and $b \ge a$, there exists a constant $c_{a,b} > 0$ for which

$$\mathbf{ex}(n, K_{a,b}) \ge c_{a,b} \cdot n^{2-(1/a)}.$$

Sharpness

This conjecture was proved

- for $a = 2$ by Erdős Erd38Tomsk,
- for $a = 2, 3$ by W.G. Brown Brown66Thomsen,
- and Kollár, Rónyai, and T. Szabó KollRonyaiSzab96,
- improved by Alon, Rónyai, and Szabó AlonRonyaiSzab99:
  - it holds for $b > (a-1)!$.

These constructions used basically (?!?!) the Unit Distance Graphs.

The simplest unknown case is

Conjecture ((Q6))

Prove that there exist a constant $c > 0$ for which, for $n > n_0$,

$$\mathbf{ex}(n, K(4,4)) > cn^{2-(1/4)}.$$

Space $\mathbb{E}^3$:

Claim (Erdős)

The unit distance graph G_n does not contain $K(3,3)$ therefore, by Kővári-T. Sós-Turán,

$$e(G_n) = O(n^{2-(1/3)}).$$

Connection to the Unit distance graph problem

Problem (Erdős (Q7))

How many unit distances can occur in the unit distance graph U_n in $\mathbb{R}^d$?

Connection to extremal graph problems.

Remark (Erdős)

The plane unit distance graph does not contain $K(2,3)$, therefore in the plane we may have at most $O(n^{3/2})$ unit distances.

Conjecture (Erdős (Q8))

For any $\varepsilon > 0$, if $n > n_0(\varepsilon)$, then a plane unit distance graph U_n has at most $O(n^{1+\varepsilon})$ unit distances.

Even cycles

When we consider excluded bipartite graphs, we should mention

Theorem (Bondy and Sim. 1974)

$\mathbf{ex}(n, C_{2k}) = O(n^{1+(1/k)})$.

Erdős, and Bondy and Sim. conjectured that this (i.e. the exponent) is sharp. This sharpness is known for C_4, C_6 and C_{10}, however it is not known for any other C_{2k}.

Conjecture ((Q9))

There exists a constant $c_8 > 0$ such that $\mathbf{ex}(n, C_8) > c_8 n^{5/4}$.

Conjecture ((Q10))

There exists a constant $c_{2k} > 0$ such that

$$\mathbf{ex}(n, C_{2k}) > c_{2k} n^{1+(1/k)}.$$

Even cycles II

Claim (Erdős)

In each G_n we can find a bipartite H_n with at least $\frac{1}{2}e(G_n)$ edges.

Claim (Győri lemma, approximately)

In each C_6-free G_n we can find a subgraph H_n without C_4 and with at least $\frac{1}{2}e(G_n)$ edges.

Conjecture (Győri compactness (Q11))

In each C_{2k}-free G_n we can find a subgraph H_n without $C_{2k-2}, C_{2k-4}, \dots, C_4$ and with at least $c_k e(G_n)$ edges.

Compactness

Theorem (Erdős, Klein)

$$\mathbf{ex}(n, C_4) = \left(\frac{1}{2} + o(1)\right) n\sqrt{n}.$$

Theorem (Erdős, compactness (Q12))

$$\mathbf{ex}(n, \{C_4, C_3, C_5, C_7, \dots\}) = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n\sqrt{n}.$$

Actually, the story is longer and more complicated.

Conjecture (Erdős, compactness (Q13))

$$\mathbf{ex}(n, \{C_3, C_4\}) = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n\sqrt{n}.$$

An easier result is

Theorem (Erdős-Sim. compactness)

$$\mathbf{ex}(n, \{C_5, C_4\}) = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n\sqrt{n}.$$

(Exluding a C_5 is a much stronger assumption than excluding C_3.)

Many related results. e.g.:

Allen, Peter; Keevash, Peter; Sudakov, Benny; Verstraëte, Jacques:

Turán numbers of bipartite graphs plus an odd cycle. J. Combin. Theory Ser. B 106 (2014), 134–162.

A weaker question is

Conjecture (Erdős, compactness (Q14))

Does there exist a constant $c > 0$ such that

$$\mathbf{ex}(n, \{C_3, C_4\}) < \left(\frac{1}{2} - c\right) n\sqrt{n} \ ?$$

A general conjecture:

Conjecture (Simonovits, compactness (Q15))

For any finite family $\mathcal{L} = \{L_1, \ldots, L_h\}$ of bipartite graphs there exists an $L \in \mathcal{L}$ for which

$$\mathbf{ex}(n, \mathcal{L}) = O(\mathbf{ex}(n, L)).$$

I.e. Excluding one of them does the same as excluding all of them.

Remark

Some research of FAUDREE and SIMONOVITS suggest that after all, this conjecture is not always true. So: decide if this is true or not, perhaps by finding a counterexample.

CONJECTURE (RATIONAL EXPONENTS (Q16))

For any finite family $\mathcal{L}$ of bipartite excluded graphs $L \in \mathcal{L}$, there exist a rational $\gamma \in [0,1]$ such that

$$\frac{\mathbf{ex}(n, \mathcal{L})}{n^{1+\gamma}}$$

converges to a $c = c_{\mathcal{L}} > 0$.

This does not hold for 3-uniform hypergraphs:

Remark (The famous Ruzsa-Szemerédi theorem provides a counterexample for hypergraphs.)

Consider 3-uniform hypergraphs and $\mathcal{L}$ be the family of 3-uniform 6-vertex hypergraphs with three hyperedges dots

The positioned exclusion problem

Let $\mathbf{ex}^*(n, L)$ be the maximum number of edges in a Red-Blue $K(n, n)$ not containing a Red-Blue L whose Red vertices are in the Red class of $K(n, n)$.

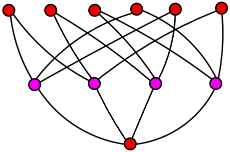
PROBLEM (POSITIONED EXCLUSION (Q17))

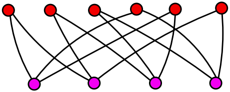
Let L be a bipartite (connected) Red-Blue excluded graph. Is it true that $\mathbf{ex}^*(n, L) = O(\mathbf{ex}(n, L))$*?*

Perhaps the simplest unknown case is that of $K(4, 5)$.

More complicated bipartite excluded subgraphs

When the (degenerate) extremal problems of the path P_k, of the complete bipartite graphs $K(a, b)$ were solved (at least good upper bounds were found) and then the BONDY-SIMONOVITS upper bound was also proved, for even cycles, the FAUDREE-SIMONOVITS upper bound was found on Θ-graphs, the researchers looked for more complicated degenerate extremal graph problems. Here we mention only three of them: M_{10}, M_{11}, and the cube Q_8.





Füredi: $\mathbf{ex}(n, M_{11}) = \Theta(N\sqrt{n})$

Cube excluded

Q_8: the cube graph on 8 vertices with 12 edges.

Theorem (Erdős-Simonovits (1970))

$$\mathbf{ex}(n, Q_8) = O(n^{8/5}).$$

Later some alternative proofs were given on this theorem. Erdős and Simonovits conjectured that

Conjecture (Cube, lower bound (Q18))

There exist a constant $c > 0$ such that

$$\mathbf{ex}(n, Q_8) > cn^{8/5}.$$

We know that $\mathbf{ex}(n, C_4) \approx \frac{1}{2}n\sqrt{n}$, however, we cannot even prove

Conjecture (Cube, much weaker lower bound (Q19))

$$\frac{\mathbf{ex}(n, Q_8)}{n^{3/2}} \to \infty.$$

Another annoying problem is that we do not have reasonable upper bound for higher dimensional cube graphs.

Erdős-Simonovits-Sidorenko type problems

Here we formulate only the ERDŐS-SIMONOVITS type problems, (see SIMONOVITS (1984)) The SIDORENKO (1991) problems are their formulation with integrals.
We restrict ourselves to the simplest versions.

PROBLEM (ERDŐS-SIM. (Q20))

Let $\chi(L) = 2$. There exists a large constant c and a small constant $\gamma = \gamma_L > 0$ such that If $e(G_n) > c\,\mathbf{ex}(n, L)$ then G_n contains at least $\gamma n^{v(L)}$ copies of L.

Some open problems in Ramsey-Turán theory

Large area, see the survey of Simonovits and Sós SimSosV01RT.
Start in the middle!

Problem (General Question)

Given an L, estimate $\mathbf{RT}(n, L, o(n))$.

In other words, consider an L-free graph sequence (G_n) with $\alpha(G_n) = o(n)$. Estimate $e(G_n)$ from above.
Find good constructions = lower bounds.

Theorem (Triviality)

$RT(n, K_3, o(n)) = o(n^2)$

Research started by Erdős, Simonovits, and Sós (1973)

Problem (Erdős (19xx) (Q21))

If (G_n) is a sequence of $K(2,2,2)$-free graphs and $\alpha(G_n) = o(n)$, does this imply that $e(G_n) = o(n^2)$?

Theorem (Szemerédi)

If (G_n) is K_4-free and $\alpha(G_n) = o(n^2)$, then

$$e(G_n) \leq \frac{n^2}{8} + o(n^2).$$

Theorem (Bollobás-Erdős)

There exist a K_4-free (G_n) with $\alpha(G_n) = o(n^2)$, for which

$$e(G_n) \geq \frac{n^2}{8} + o(n^2).$$

General asymptotics?

Problem ((Q22))

Find some analogue of Erdős-Sim. asymptotics.

Problem ((Q23))

Find some analogue of Erdős-Sim. *asymptotics: is it true that there exists a generalized matrix graph providing almost extremal graphs, where the densities are* $0, \frac{1}{2}, 1$

Explanation: Matrix graphs

The n vertices arepartitioned into r classes $C_1, \ldots, C_r$ and the edges between C_i and C_j are either random, or (at least) ε-regular (in the sense of Szemerédi regularity lemma) with edge-density $a_{i,j} \in [0, 1]$.

Hypergraph Extremal problems

Erdős generalized the Kővári-T. Sós-Turán Theorem to hypergraphs.

Theorem (Erdős)

$$\mathbf{ex}(n, K_p^{(p)}(t, \dots, t)) < cn^{p-(1/t^{p-1})}.$$

Problem ((Q24))

Provide a reasonable lower bound.

Remark

Erdős stated that for two suitable constants, $c_1, c > 0$,

$$\mathbf{ex}(n, K_p^{(p)}(t, \dots, t)) > c_1 n^{p-(c/t^{p-1})}.$$

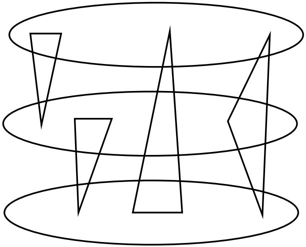
However, the proof was not reconstructed. (???)

Complete 4-graph is excluded

Conjecture (Turán (Q25))

Consider 3-uniform simple hypargraphs and let $\mathcal{H}_4^{(3)}$ be the excluded hypergraph. Then

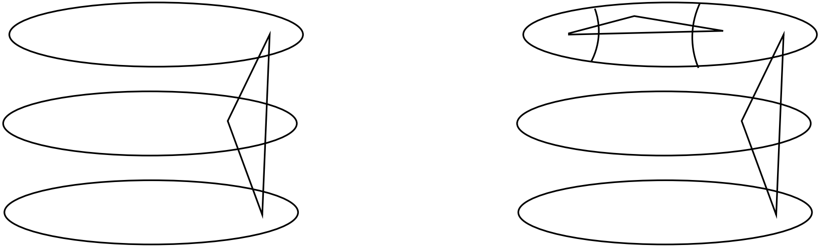
$$\mathbf{ex}(n, \mathcal{H}_4^{(3)}) = \frac{5}{9}\binom{n}{3} + o(n^3).$$



4-3-graph is excluded

Problem (4/3 problem (Q26))

Consider 3-uniform hypergraphs. Let $\mathcal{H}_4^{(3)}$ be a 3-uniform hypergraph on 4 vertices with 3 hyperedges. Extimate $\mathbf{ex}(n, \mathcal{H}_4^{(3)})$.



Frankl-Füredi: NO, they improved this construction.

References

AlonRonyaiSzab99 N. Alon, Lajos Rónyai, and Tibor Szabó: *Norm-graphs: Variations and application, J. Combin. Theory Ser. B* **76** **(1999)**, *280–290.*

BondySim74C2k J.A. Bondy and M. Simonovits: *Cycles of even length in graphs, J. Combin. Theory Ser. B* **16** **(1974)** *97–105.*

Brown66Thomsen William G. Brown: *On graphs that do not contain a Thomsen graph, Can. Math. Bull.* **9** **(1966)**, *281–285.*

ChungGrahamErdLegacy F.R.K. Chung, and R.L. Graham: *Erdős on Graphs: His Legacy of Unsolved Problems, A. K. Peters Ltd., Wellesley, MA, 1998.*

ErdhomepagePaul Erdős: *Erdős homepage (his scanned in papers up to 1989):*

www.renyi.hu/~p_erdos.

Erd38Tomsk P. Erdős: *On sequences of integers no one of which divides the product of two others, and some related problems, Mitt. Forschungsinst. Math. u. Mech. Tomsk* **2** *(1938), 74–82.*

Erd62Radem P. Erdős: *On a theorem or Rademacher-Turán, Illinois J. Math.* **6** **(1962)**, *122–127. 1962-09*

ErdSim71Octa P. Erdős and M. Simonovits: *An extremal graph problem, Acta Math. Acad. Sci. Hungar. 22 (1971/72), 275–282.*

ErdSim70Cube P. Erdős and M. Simonovits: *Some extremal problems in graph theory, Combinatorial Theory and Its Applications, I. (Proc. Colloq. Balatonfüred, 1969), North Holland, Amsterdam, 1970, pp. 377–390. (1969) MR0300924 (Reprinted in* **[Erd73ArtOfCount]***).*

ErdSimSos73 P. Erdős, M. Simonovits, and V.T. Sós: *Anti-Ramsey theorems, In: Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), vol. II, pp. 633–643. Colloq. Math. Soc. János Bolyai, vol. 10. North-Holland, Amsterdam (1975) MR0379258*

KollRonyaiSzab96 János Kollár, L. Rónyai, and Tibor Szabó: *Norm-graphs and bipartite Turán numbers. Combinatorica* **16 (1996)**, *399–406.*

KovSosTur54 Tamás Kővári, V.T. Sós, and P. Turán: *On a problem of Zarankiewicz, Colloq Math. 3 (1954), 50–57.*

Sim66Tihany M. Simonovits: *A method for solving extremal problems in graph theory, Theory of Graphs, Proc. Colloq. Tihany, (1966), (Ed. P. Erdős and G. Katona) Acad. Press, N.Y., (1968) pp. 279–319.*

Sim74Symm M. Simonovits: *Extermal graph problems with symmetrical extremal graphs, additional chromatic conditions, Discrete Mathematics* **7 (1974)**, *349–376.*

[Sim83ProdBirk] M. Simonovits: *Extremal graph problems and graph products, in Studies in Pure Mathematics, Birkhäuser, Basel, 1983, pp. 669–680.*

[Sim84Wat] M. Simonovits: *Extremal graph problems, Degenerate extremal problems and Supersaturated graphs, Progress in Graph Theory (Acad. Press, ed. Bondy and Murty) (1984) 419–437.*

[TuranML] Paul Turán: *On an extremal problem in graph theory (in Hungarian). Mat. Fiz. Lapok* **48** **(1941)** *436–452. (For its English version see* [*Tur89Coll.)*]

[Tur89Coll] P. Turán: *Collected papers. Akadémiai Kiadó, Budapest, 1989. Vol.* **1–3,** *(with comments of Simonovits on these topics, and by others on other topics).*

- my hompage: www.renyi.hu/˜miki