ON THE STRUCTURE OF CO-CRITICAL GRAPHS

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Abstract

Given a family $L_1, \dots, L_r$ of graphs, a graph G is called $(L_1, \dots, L_r)$–co-critical if one can colour its edges in r colours so that the subgraph defined by the ν^{th} colour contains no L_ν, however, G is saturated for this property: adding any edge to G we get a G^* with the property that arbitrarily colouring G^* in r colours for some ν we shall have a monochromatic copy of L_ν in the ν^{th} colour. (The notion comes from J. Nešetřil.) In this paper we shall investigate the structural properties of (K_3, K_3)-co-critical graphs, present various constructions of such graphs and establish some of their properties.

Notation. In this paper we shall consider only simple graphs, i.e., graphs without loops and multiple edges. Given a graph H, $v(H)$, $e(H)$ will denote the number of vertices and edges in H. Subscripts mostly will indicate the number of vertices: G_n, S_n, will always denote graphs on n vertices. However, occasionally they are just indices, e.g., in a formula, like $(L_1, \ldots, L_r)$. Given a vertex x of a graph G, $N(x)$ will denote the set of neighbours of x. If the edges of the graph are coloured, say in RED and BLUE, then $N_R(x)$ will denote the RED neighbourhood. $\chi(G)$ is the chromatic number of G.

As usually, K_m denotes the complete m-graph, $K_m(n_1, \ldots, n_m)$ is the complete m-partite graph with n_i vertices in its i^{th} class. Z_n is the graph obtained from K_n by deleting one edge. C_m and P_m are the cycle and path on m points, respectively. Given the graphs G_1, G_2, we define their **product** $G_1 \otimes G_2$ as the graph obtained from vertex-disjoint copies of G_1 and G_2 by joining every vertex x of G_1 to every vertex y of G_2.

Sometimes, according to the accepted usage, we shall say that (given the number of colours, r) the graph G **arrows** L_ν's:

$$G \to (L_1, \ldots, L_r)$$

if for arbitrary r-colouring of the edges of G for some ν the ν^{th} colour-class contains an L_ν. We shall denote by $R(L_1, \ldots, L_r)$ the (ordinary) lower Ramsey number, belonging to the graphs $(L_1, \ldots, L_r)$, where the lower Ramsey number is the maximum N for which there exists a suitable r-colouring of K_N without monochromatic L_ν in the ν^{th} colour. We shall denote by $R^*(L_1, \ldots, L_r)$ the following generalization of the Ramsey number: R^* is the largest t such that any $K_t(n, \ldots, n)$ can be coloured in r colours without getting an L_ν in the ν^{th} colour for some $\nu \in [1, r]$. Obviously, $R^*(L_1, \ldots, L_r) \le R(L_1, \ldots, L_r)$.

0. INTRODUCTION

It is well known that colouring the edges of a K_6 by two colours, a monochromatic K_3 must occur. In fact, this is the simplest graph theorem in what is called Ramsey theory (see [9]). In 1967 Erdős and Hajnal [4] asked whether there exist graphs containing a monochromatic triangle in any 2-colouring of the edges (i.e., arrowing the triangle), while K_6 does not appear as a subgraph. An affirmative answer was given by Graham [8] who observed that the graph $C_5 \otimes C_3$ arrows K_3 but contains no K_6. Graham mentioned also the existence of an unpublished example due to van Lint and another one by Pósa, the latter not containing K_5 either. In fact, Erdős and Hajnal already expected the existence of a graph without K_4 arrowing K_3 even if we do not restrict ourselves to 2-colourings. For two colours, for every positive integer m, Folkman [7] constructed graphs containing no K_{m+1} but having a monochromatic K_m in any 2-colouring of the edges. The most general result of this type was found by Nešetřil and Rödl [16] in 1976. They proved that for any graph H and positive integer r, there exists a graph G that contains a monochromatic version of H in any r-colouring of its edges, while the size of the largest clique in G is not larger than that in H.

Most of the above facts could be interpreted by saying, that "if something is not *obviously* impossible in this area, then it is possible, and can be constructed".

On the other hand, if we regard graphs which are saturated (maximal) with respect to not having those properties above, then we may face some completely new phenomena. The first such problem due to Jarik Nešetřil [15], was the following question about graphs he named *co-critical.* First we need a definition. (This definition we give in a fairly general form, although in this paper we are mainly concerned with the case where $r = 2$ and $L_1 = L_2 = K_3$.)

Definition 1. Given a family $L_1, \dots, L_r$ of graphs, a graph G which is not complete, is called $(L_1, \dots, L_r)$-co-critical if one can colour its edges in r colours so that (for $\nu = 1, \dots, r$) the subgraph defined by the ν^{th} colour contains no L_ν, however, G is saturated for this property: adding any edge to G we get a G^* with the property that arbitrarily r-colouring G^*, for some ν we shall have a monochromatic copy of L_ν in the ν^{th} colour.

Definition 2. We shall call a graph saturated for property P if it has property P but after adding any new edge it will lose this property. When given a graph F, we say that another graph G is F-saturated we mean that G is saturated for the property of not containing F.

Sometimes, when this leads to no ambiguity, we shall call $(L_1, \dots, L_r)$-co-critical graphs simply co-critical graphs. We shall use this abbreviation often in case of (K_3, K_3)-co-critical graphs.

Remark. We excluded the complete graphs. Without explicitely stating this, K_5 could have been regarded (K_3, K_3)-co-critical.

Clearly, Z_6 (a complete 6-graph minus an edge) can be coloured by RED and BLUE, without getting monochromatic triangles, but adding any (more precisely, the only missing) edge to it we get a graph G^* (now K_6) for which $G^* \to (K_3, K_3)$. How typical is this example?

Nešetřil [15] asked the following question:

> Are there infinitely many minimal co-critical graphs, i.e. co-critical graphs which lose this property, whenever a vertex is deleted? Is Z_6 the only one?

We shall prove that the answer to Nešetřil's question is YES: there are infinitely many such graphs. Namely, using various constructions, we shall prove

Theorem 1. *There are infinitely many graphs G_n which*

(a) have 2-colourings without monochromatic triangles;

(b) adding any new edge to G_n we get a G_n^* such that in any 2-colouring of G_n^* there are monochromatic triangles.

(c) Deleting any vertex of G_n we get a graph violating (b);

(d) $K_5 \not\subseteq G_n$.

Further, we shall see, that – in some sense – being co-critical is not a monotone property: it can happen that deleting any vertex of a co-critical graph we get some non-co-critical graph, but deleting further points we get a co-critical graph again. Partly this is, why we shall relax the conditions in Theorem 1, forgetting about one of them and trying to investigate many different graphs satisfying conditions (a), (b) and (d) or (a), (b) and (c), respectively.

So, we shall construct various families of (K_3, K_3)-co-critical graphs without K_5's (and therefore not containing Z_6 either) and another family, the members of which are all minimal co-critical though they contain Z_6's. Further, we shall prove some structural theorems on co-critical graphs.

Unfortunately, though we succeded in getting quite a few constructions, our results on the structure of **all** the co-critical graphs do not seem to give as much information as we would like to have. Probably one of the most natural questions that we are still unable to answer is whether there exists a graph which does not arrow the triangle but after the addition of any new edge it does, while it contains no K_4.

Another natural question that remains open is the following. As a consequence of the above mentioned non-monotonicity, even with having infinitely many minimal co-critical graphs we cannot exclude the possibility of the existence of a finite number of graphs such that each co-critical graph contains at least one of them. (In fact, it turns out that all of our constructions contains one of two small co-critical graphs.) This leads to the problem of "finite basis": can one find infinitely many minimal co-critical graphs none of which contains any other? (This is a strengthening of Nešetřil's original question.) Or, in other words, can one find a finite set $Q_1, \ldots, Q_m$ of co-critical graphs so that every co-critical graph contains at least one of them? A list of open problems is given at the end of the paper.

We shall return to the question of many colours in another paper [18].

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1. CONSTRUCTIONS OF MINIMAL (K_3, K_3)-CO-CRITICAL GRAPHS

In this section we shall restrict our consideration to (K_3, K_3)-co-critical graphs.

Basically all our constructions originate from two basic examples, contained in Claim 1 and Claim 2, below, the proofs of which will be omitted, since they will follow from more general results.

Take two copies of a C_5 and join them completely. Denote the obtained graph by Q_{10}. (Using the "product" notation: $Q_{10} = C_5 \otimes C_5$.)

Claim 1. *$Q_{10} = C_5 \otimes C_5$ contains no Z_6 and is (K_3, K_3)-co-critical. However, deleting any vertex of $C_5 \otimes C_5$ we get a non-(K_3, K_3)-co-critical graph. Moreover, $K_5 \not\subseteq C_5 \otimes C_5$.*

Claim 2. *$\overline{C}_{11}$, the complement of the cycle C_{11}, is a co-critical graph, while for any vertex v, the graph $\overline{C}_{11} - v$ is not (K_3, K_3)-co-critical. (However, $Z_6 \subseteq \overline{C}_{11}$.)*

Theorem 1 will easily follow from a corollary of our next theorem.

Theorem 2. *If G_1 and G_2 are two non-bipartite K_3-sturated graphs, then $G_1 \otimes G_2$ is (K_3, K_3)-co-critical.*

If, in addition, the G_i's $(i = 1, 2)$ have the property that for any vertex z of G_i there exists a pair of independent vertices x, y in G_i for which z is their only common neighbour in G_i, then $G_1 \otimes G_2$ is minimal co-critical.

Remarks. (a) If G is saturated for not containing K_3, then any pair x, y of independent vertices have distance 2, i.e., have a common neighbour, otherwise we could add the edge xy to G without getting a K_3. However, mostly, non-adjacent vertices of our graphs have a lot of common neighbours.

(b) If G is saturated for not containing K_3, then either it is a complete bipartite graph or it contains a C_5.

(c) Mostly we shall consider graphs G which, besides being saturated for not containing K_3, satisfy the following condition:

$(*)$ $\alpha(G) < \frac{1}{2}v(G)$.

Proof of Theorem 2. (a) Obviously, if we colour $G_1 \otimes G_2$ so that the edges belonging to the same G_i's are RED and the edges connecting G_1 to G_2 are BLUE, then we get no monochromatic K_3.

(b) To prove that $G_1 \otimes G_2$ is co-critical, we have to show that if we add any edge to a G_i, then in any 2-colouring we get a monochromatic triangle. For some reasons, which will become clear in Section 4, we shall not choose the most direct approach. Instead, first we shall assume that the G_i's satisfy condition $(*)$.

Let us add an arbitrary edge to the product. This edge is added either to G_1 or G_2. Assume, we added it to G_1. Thus we got a G_1^* containing a triangle abc. We wish to show that there must be a monochromatic triangle in $G_1^* \otimes G_2$.

Assuming the contrary we get that, say, two of the edges of abc are RED, one is BLUE. Assume that ab, ac are RED. We show that all the edges between G_2 and a are BLUE. If for some $x \in G_2$ xa were RED, then xb and xc were BLUE, yielding a BLUE triangle xbc. This contradiction shows that all the edges xa are BLUE. Now, for each $x \in G_2$ at least one of xb and xc must be RED, since bc is BLUE. Therefore either b or c is joined to at least $\lceil \frac{1}{2}v(G_2) \rceil$ vertices of G_2 in RED. Assume it is b. Then $N_R(b) \cap V(G_2)$ contains an edge uv which cannot be BLUE, since both its endvertices are joined to a in BLUE, and cannot be RED since both its endvertices are joined to b in RED. This contradiction completes the proof of the statement that $G_1 \otimes G_2$ is co-critical.

(c) We should still prove that if both G_i's have the property that for any vertex z of G_i there exists a pair of independent vertices x, y in G_i which have no other common neighbour in G_i but z, then $G_1 \otimes G_2$ is minimal co-critical. Indeed, delete any vertex z of (say) G_1. Then we can find two vertices x, y in G_1 with the above property. Adding the edge xy to $(G_1 \otimes G_2) - \{z\}$ we get an $H \otimes G_2$, where H contains no triangles, therefore $H \otimes G_2$ can be RED-BLUE-coloured without monochromatic triangles.

(d) Now we would like to get rid of condition $(*)$. But this is trivial: the argument of (b) shows that $C_3 \otimes C_5 \to K_3$. So, add now an edge to G_1 and observe that thus we get a $G_1^* \supseteq K_3$, and we do have a $C_5 \subseteq G_2$. Hence

$$G_1^* \otimes G_2 \supseteq C_3 \otimes C_5 \to K_3.$$

This proves Theorem 2 in full generality. ∎

Remark. Actually, the fact that $C_3 \otimes C_5 \to K_3$ (as we have already mentioned in the introduction) was found by R. L. Graham. We could have shortened part (b) of the proof by referring to this fact. Still we wanted a self-contained proof. Further, we feel that this more general setting helps to understand the phenomenon better.

Theorem 2 enables us to get many co-critical graphs. Notice that Claim 1 is now already proved by the foregoing. The first question that arises is this: which graphs can replace C_5 in Claim 1. One of them is the famous Kneser graph [10].

Definition 3. The Kneser graph $KN(m,k)$ is a graph the vertices of which are the $\binom{m}{k}$ k-tuples of an m–element set and two of these k-tuples are joined if they are disjoint.

(The Petersen graph is also a Kneser graph, namely, $KN(5,2)$.)

Corollary 1. *Fix two integers q, r $(q, r \geq 2)$ and let*

$$G_1 = KN(3q-1, q) \qquad \text{and} \qquad G_2 = KN(3r-1, r).$$

Then $G = G_1 \otimes G_2$ is a (K_3, K_3)-co-critical graph, and deleting any vertex of it we get a non-co-critical graph.

Proof. We have chosen the above parameters $3q-1, q$ and $3r-1, r$ to ensure that the corresponding Kneser graphs will be triangle-saturated: it is triangle-free but adding any edge we get triangles in them. The only thing to be checked is that they are not bipartite. However, one can easily show that every $KN(3m-1, m) \supseteq C_5$ (whenever $m \geq 2$).

We still have to prove that deleting any vertex of such a G_n we get a non-co-critical graph. As we have seen, it is enough to show that for any vertex z of the Kneser graph there are 2 other vertices which are joined by a path of length 2 only through this z. This is trivial: z is a vertex of (say) $KN(3q-1, q)$, hence it corresponds to a q-tuple Z of a corresponding $3q-1$-element set U. Now, choose any pair (X, Y) of q-tuples in $U \setminus Z$ intersecting each other in exactly one element. Clearly, $|X \cup Y| = 2q - 1$, therefore only Z will be disjoint from both of these two sets, whence z will be the only common neighbour of the vertices x, y corresponding to the q-tuples X, Y. ∎

Clearly, Corollary 1 implies Theorem 1.

Remark. The Kneser graph $KN(3m-1, m)$ (as it easily follows from the Erdős-Ko-Rado Theorem [5]) satisfies property $(*)$.

2. OTHER PRODUCT CONSTRUCTIONS OF (K_3, K_3)-CO-CRITICAL GRAPHS

Often we shall have graphs satisfying – instead of $(*)$ above – the following much stronger condition.

Condition $(**)$. For an infinite family of graphs G assume that

$$\alpha(G) = o(v(G)) \qquad \text{as} \qquad v(G) \to \infty. \tag{$**$}$$

At this point one should ask if there are such K_3-free graphs at all. It is well known that the Ramsey number $R(k, 3) > ck^2/\log^2 k$ (see Spencer [19], and also [2], [3].). This implies that there exist graphs G_m on m vertices, containing no K_3, with $\alpha(G_m) = O(\sqrt{m}\log m)$. And, obviously, this implies

Claim 3. *There exist graphs S_m on m vertices, $(m \to \infty)$ containing no K_3, saturated for not containing K_3, and with $\alpha(S_m) = O(\sqrt{m}\log m)$.*

The following is not a real construction, only a "random graph construction". (This is because the above bound on $R(k, 3)$ is obtained by random graph arguments.)

Construction 1. Take any graph G_n described by Claim 3. Then $G_n \otimes G_n$ is (K_3, K_3)-co-critical. ∎

Construction 2. (Margulis–Lubotzky–Phillips–Sarnak [14], [11]) The non-bipartite Lubotzky-Philips-Sarnak graphs have the property that their girth is large. (Besides, they satisfy $(**)$.) They are not K_3-saturated, but we can add edges to them until they become K_3-saturated. Take two such graphs, their product is (K_3, K_3)-co-critical by Theorem 2. ∎

Remark. As to the independence number of the above graph, one can be more specific: taking the parameters in the LPS graph in [11] to be $q \approx p^3$ we get a graph X_n with $\alpha(X_n) \approx O(n^{5/6})$.

At this point it is worthwhile to emphasize what we have implicitly already observed, that each of the above constructions contains a $C_5 \otimes C_5$. Indeed, we have already mentioned, that any K_3-saturated graph but the $K(n_1, n_2)$ contains a C_5. We can already see, however, that co-criticality is not a monotone property: e.g., taking the Petersen graph $KN(5,2)$, $G_{20} = KN(5,2) \otimes KN(5,2)$ is co-critical, deleting any vertex we get a non-co-critical graph, but deleting half of its vertices we may arrive at $C_5 \otimes C_5$ which is again co-critical. The same phenomenon can be observed in case of $\overline{C}_{11}$ (see Claim 2), or more generally, in the "cyclic construction" of the next section, providing a second proof of Theorem 1, apart from that it will contain K_5's.

So in general, we should distinguish between two notions of minimality:

Definition 4. We will call an $(L_1, \dots, L_r)$-co-critical graph G **minimal** co-critical if deleting any vertex u of G we get a non-$(L_1, \dots, L_r)$-co-critical graph $G - u$. We will call an $(L_1, \dots, L_r)$-co-critical graph G **strongly minimal** co-critical if it contains no smaller $(L_1, \dots, L_r)$-co-critical graphs.

It should be emphasized, that, apart from Z_6, the only strongly minimal co-critical graph we know about is $C_5 \otimes C_5$ and it remains one of the most intriguing open problems whether there are infinitely many *strongly* minimal co-critical graphs.

Before leaving the product constructions we should still examine one question: which are the possible RED-BLUE colourings of the above graphs without monochromatic triangles. One could ask if the co-critical graphs described in Theorem 2 have only one RED-BLUE-colouring without monochromatic triangles, namely, colouring the edges of G_1 and of G_2 by the same colour, say, by RED, and colouring the edges joining G_1 to G_2 by the other colour (now, by BLUE). This is almost true, but not quite.

Lemma 1. *If G_1 and G_2 are connected non-bipartite graphs, then in every RED-BLUE colouring of $G_1 \otimes G_2$ with no monochromatic K_3 both G_1 and G_2 must be monochromatic, moreover, they must have the same colour. Further, if the edges of G_1 are RED, say, then for every $x \in V(G_2)$ all but $\alpha(G_1)$ edges $xy : y \in V(G_1)$ must be BLUE (and the same holds if we change the role of G_1 and G_2).*

Proof. (a) First we show that in any good colouring of the above product, (i.e., one without monochromatic triangles) all the edges of G_1 must have the same colour. Assume this is not so. Since G_1 is connected, there exists a node a connected to two nodes in different colours. Let these be b and c. Assume ab is RED and ac is BLUE. Those vertices in $V(G_2)$ that are connected to a by RED should be connected to b by BLUE, so neither RED nor BLUE edges can occur among them. Similarly, all other vertices in $V(G_2)$ are connected to a by BLUE and to c by RED, i.e., no edge can appear among these vertices either. This implies that G_2 is bipartite, a contradiction. Obviously, all edges in G_2 must have the same colour, too.

(b) Next we prove the last statement. Assume that all the edges in $V(G_1)$ are RED. Now we show that for any $x \in V(G_2)$ the number of RED edges joining x to G_1 is at most

$\alpha(G_1)$. Indeed, $|N_R(x) \cap V(G_1)|$ contains no RED edges, so it contains no edges at all:

$$|N_R(x) \cap V(G_1)| \leq \alpha(G_1).$$

By the way, this inequality holds not only for the whole graphs G_1 but every connected induced subgraph $H_1 \subseteq G_1$:

$$|N_R(x) \cap V(H_1)| \leq \alpha(H_1).$$

(c) Since the G_i's are non-bipartite, they both contain odd cycles. Let $H_i \subseteq G_i$ be odd cycles. (We do not assume them to be induced cycles, though taking the shortest odd cycles we could achieve that.) By (b), if G_1 is RED, then the number of RED edges between H_1 and H_2 is strictly greater than the number of BLUE edges. Now, if G_2 were BLUE, then the number of BLUE edges between H_1 and H_2 were again strictly greater than the number of RED edges. These two facts exclude each other: G_1 and G_2 have the same colour. ∎

Remark. We cannot assert that all the edges joining G_1 to G_2 are of the same colour: Fix a matching between nodes of G_1 and G_2, colour it RED, colour all the edges of G_1 and G_2 also RED but colour all the other edges BLUE: we still will have no monochromatic triangles.

The next short argument is a detour from our main line. Here we investigate whether one can reverse Theorem 2.

Problem. *Is it true that if $G_1 \otimes G_2$ is (K_3, K_3)-co-critical, then G_1, G_2 are K_3-saturated?*

The answer is NO:

Claim 4. *There exist (K_3, K_3)-co-critical graphs of form $G_1 \otimes G_2$ where G_1 is bipartite, G_2 contains triangles.*

Proof. Indeed, let $G_1 = K(m, m)$, G_2 be any non-bipartite K_3-saturated graph.

(a) Then $G_1 \otimes G_2$ is *not* (K_3, K_3)-co-critical, since we can add an edge uv to G_2 and still 2-colour the obtained $G_1 \otimes H_2$ without monochromatic triangles. Indeed, colour the original $G_1 \otimes G_2$ as usual, G_1, H_2 in RED, the new edge uv in BLUE, and change the colour of the edges xu to RED if x is in the first class of $K(m, m)$ and the edges yv also to RED if y is in the second class of $K(m, m)$. There will be no monochromatic triangles.

(b) At the same time, adding any edge to G_1, (by the argument of the Proof of Theorem 2), we immediately get a graph "arrowing" (K_3, K_3).

(c) Now add edges to $G_1 \otimes G_2$ until the whole graph becomes co-critical. This example will be a product $G_1 \otimes H_2^*$ where $H_2^* \supseteq K_3$, G_1 is bipartite. ∎

3. THE CYCLIC CONSTRUCTION

Generalizing the construction of Claim 2, now we give a second proof that there are infinitely many (K_3, K_3)-co-critical graphs G_n which are minimal: deleting any vertex of G_n we get a G_{n-1} which is not co-critical any more. This construction is weaker than that of Theorem 2, since it contains (many) K_5's. Still, it has a nice and simple structure, completely different from the previous ones.

Construction 3. Let for $n = 6k - 1$, $(k \geq 3)$ the vertices of the ("cyclic") graph G_n be $0, 1, 2, \ldots, 6k - 2$, and put these vertices onto the circumference of a circle of perimeter $(6k - 1)$. We join two vertices i and j by an edge iff their distance is at least k on the circumference of the circle.

Theorem 3. *The graph defined in Construction 3 is minimal (K_3, K_3)-co-critical, i.e., it is co-critical while deleting any vertex of it the resulting graph is not co-critical any more.*

Remark. Since G_n is a different graph for every $k \geq 3$, Construction 3 provides infinitely many minimal co-critical graphs, by Theorem 3. (The construction in Claim 2 is the special case when $k = 3$.)

Proof. (a) First we show that G_n has a two-colouring without monochromatic triangles. We colour those edges RED which join vertices of distance $k, k + 1, \ldots, 2k - 1$ and by BLUE those joining vertices of distance $2k, 2k + 1, \ldots, 3k - 1$. It is easy to verify that no monochromatic triangle occurs in this colouring.

(b) If we add any missing edge to G_n then a K_6 will occur already ensuring that every 2-colouring of the resulting graph has monochromatic triangles. This proves that G_n is co-critical.

(c) To see that G_n is minimal co-critical, let us delete a vertex. Because of the symmetry of G_n we may assume that vertex $3k$ is deleted. We claim that now we can add new edges to the graph and still two-colour it without monochromatic triangles. Let us add the edge $(5k, 0)$ and colour it RED, while (for the moment) colour all the other edges with the colour it had in the colouring described in (a). We will show that by changing the colour of three appropriately chosen edges we get a good colouring. First observe that after the addition of our new edge in RED only two monochromatic triangles occured: $(5k, 0, k)$ and $(4k, 5k, 0)$ (obviously both RED). Now change the colour of the edges $(5k, k)$ and $(4k, 0)$ to BLUE. Now we do not have RED triangles any more but BLUE triangles may occur. Notice that no BLUE triangle occurs because of the recolouring of the edge $(5k, k)$. (In fact, the only such triangle would be $(5k, k, 3k)$ but the vertex $3k$ is now deleted.) There occurs only one BLUE triangle as a result of recolouring $(4k, 0)$: this is $(4k, 0, 2k)$. Change the colour of $(2k, 4k)$ to RED, then we have no BLUE triangles any more, but RED triangles may occur again. But actually they do not (again, the only possible RED triangle would have the deleted $3k$ as its third vertex), so we have found a good colouring, proving our statement. ∎

Remarks. (a) Having deleted $3k$, besides adding $(5k,0)$, three more edges can still be added to our graph without ruining its colourability: adding the edges $(1,k)$, $(k+1,2k)$, $(4k,5k-1)$ and changing the colour of the edge $(1,2k)$ from RED to BLUE in the colouring given above one can easily check that no monochromatic triangle occurs.

(b) The above proof of Theorem 3 is easy to check, but does not explain the reason why are we so "lucky" that changing the colour of only very few edges gives us a good colouring of the truncated graph G_{n-1}. Here we sketch another proof of the statement that the truncated graph G_{n-1} is well colourable, because, we believe, this argument "gives a reason for it". Consider the graph we left with after the deletion of vertex $3k$ and the addition of the four edges $(1,k),(k+1,2k),(4k,5k-1),(5k,0)$. Let's call this graph H_{n-1}. One can easily check that the pairs of vertices $(0,1),(k,k+1),(2k,2k+1),(4k-1,4k)$ and $(5k-1,5k)$ became twins in this graph. (Two vertices are twins if they are adjacent to exactly the same vertices in the graph, meaning also that they are not adjacent to each other.) Identifying the vertices that form twins we get a graph on $n-6$ vertices that we call F_{n-6}. Now notice that this graph is nothing but the G_{n-6} from Construction 3, i.e., the graph Construction 3 gives with k being 1 smaller than before. So we know (from the beginning of the proof of Theorem 3) that F_{n-6} has a valid colouring. But duplicating a vertex of a graph we could colour without monochromatic triangles cannot ruin this property.

Remark. It is worth noticing that the graphs of Construction 3 contain no $C_5 \otimes C_5$. On the other hand, it is not difficult to see that the G_{6k-1} of Construction 3 contains $G_{6(k-1)-1}$ and thereby all G_{6j-1}'s with $j<k$ of the same construction. Indeed, if we leave the vertices $1, k+1, 2k+1, 3k, 4k, 5k$ from G_{6k-1} we obtain exactly a $G_{6(k-1)-1}$.

One more observation should be made. In all the cases we know, having a co-critical graph G and replacing a vertex x of it by a set of independent vertices $\{x_1,\dots,x_r\}$ joined to the neighbours of x we get a co-critical graph again. Unfortunately, we cannot prove this in general (cf. Problem 5).

4. STRUCTURAL RESULTS

Co-critical graphs with high edge-density

One can ask for the extremal values of the basic parameters of co-critical graphs: what are the extremal values of the sizes (= edge-number) of a co-critical G_n, or the extremal values of the degrees, and so on. Some of these questions are more interesting, some others prove to be routine.

We start with an easy question.

We introduce a new notation. For given n and p, $T_{n,p}$ denotes the p-chromatic graph with the most edges (also called the Turán graph on n vertices, with p classes): n vertices are partitioned into p classes as equally as possible and all the edges joining points of different classes are present.

Theorem 4. *Let $(L_1, \dots, L_r)$ be complete graphs. If G_n is $(L_1, \dots, L_r)$-co-critical, and $R = R(L_1, \dots, L_r)$ is the corresponding Ramsey number, then*

$$e(G_n) \le e(T_{n,R}).$$

Proof. If $e(G_n) > e(T_{n,R})$, then there is a $K_{R+1} \subseteq G_n$. If we r-colour the edges of G_n, then this K_{R+1} is also r-coloured and gives (by definition) a monochromatic L_ν (for some $\nu \le r$). This proves Theorem 4. ∎

Remark. Clearly, Theorem 4 is sharp: $T_{n,R}$ is really $(L_1, \dots, L_r)$-co-critical. The above estimate is valid also for arbitrary sample graphs $(L_1, \dots, L_r)$, since we have not used the fact that the L_i's are complete when proving the inequality. However, in the general case Theorem 4 is not sharp.

One can generalize the above theorem to arbitrary graphs L_i, to get a sharp form. First we recall the definition of the generalized Ramsey numbers. Let $(L_1, \dots, L_r)$ be arbitrary graphs. Let $R = R^*(L_1, \dots, L_r)$ be the largest integer satisfying the following condition (see Burr, Erdős, Lovász, [1]): for arbitrary large h, $K_R(h, \dots, h)$ can be coloured in r colours without getting an L_ν in the ν^{th} colour.

Theorem 4*. *Let $L_1, \dots, L_r$ be arbitrary graphs. If G_n is $(L_1, \dots, L_r)$-co-critical, with maximum number of edges (for fixed $L_1, \dots, L_r$ and n), and $R = R^*(L_1, \dots, L_r)$ is the corresponding generalized Ramsey number, then*

$$e(T_{n,R^*}) \le e(G_n) \le e(T_{n,R^*}) + o(n^2), \qquad \text{as} \qquad n \to \infty.$$

Proof of Theorem 4*.

(a) The lower bound of Theorem 4* is trivial again: by definition, T_{n,R^*} can be r-coloured without monochromatic L_ν in its ν^{th} colour.

(b) To prove the upper bound of Theorem 4*, we shall use the Erdős–Stone theorem [6], according to which, for every ν and $\varepsilon > 0$ there exists an n_0 such that for $n > n_0$ a graph G_n not containing $K_{t+1}(\nu, \dots, \nu)$ has at most

$$\left(1 - \frac{1}{t}\right) \binom{n}{2} + \varepsilon n^2$$

edges. By definition, we can fix a ν so large that

$$K_{R^*+1,(R^*+1)\nu} = K_{R^*+1}(\nu, \dots, \nu) \to (L_1, \dots, L_r).$$

Hence a co-critical G_n cannot contain a $K_{R^*+1}(\nu, \dots, \nu)$. So, by the Erdős–Stone theorem,

$$e(G_n) \le e(T_{n,R^*}) + o(n^2), \qquad \text{as} \qquad n \to \infty. \qquad \blacksquare$$

Co-critical graphs with low maximum degree

It is much more interesting to ask if there are co-critical graphs with low edge-density. Still, here we have to be slightly careful:

It is easy to construct (K_3, K_3)-co-critical graphs with a linear number of edges: $K(1, 1, 1, 1, n-4)$ will do. It will have $4n - 10$ edges. Or, if we want a co-critical graph not containing K_5's, and having $O(n)$ edges, take $G_1 \otimes G_2$ as in Theorem 2, however, keep $G_1 = C_5$ fixed, and let $G_2 = C_5(1, 1, 1, 1, n-9)$, where $C_5(1, 1, 1, 1, k)$ denotes the graph we obtain when substituting a vertex of C_5 by an independent set of size k.. This G_n is co-critical, and $e(G_n) = 7n - 35$. Thus one should ask for constructing "almost regular" co-critical graphs with low edge-density, or for low maximum degree.

Theorem 5. *There exists an infinite sequence of (K_3, K_3)-co-critical graphs with*

$$d_{max}(G_n) = O(n^{3/4} \log n).$$

Proof. Let $\Delta(G)$ denote the maximum degree of a graph G. Let $k|n$. Take a graph S_k, described in Claim 3 (or any other K_3-saturated graph with $\alpha(S_k) = O(\sqrt{k} \log k)$). To get a graph on n vertices, replace each vertex of S_k by n/k independent vertices: D_i is the class of $m = n/k$ vertices corresponding to the vertex x_i $(i = 1, \dots, k)$. Join every $x \in D_i$ to every $y \in D_j$ iff the original vertices corresponding to D_i and D_j have been joined. We shall denote the obtained graph with $G_n^0 = G_k \odot I^{n/k}$. Let H be any K_3-saturated graph on $m = n/k$ vertices, satisfying $\alpha(H_m) = O(\sqrt{m} \log m)$. (E.g. let H also be a graph described in Claim 3.) Put into each class of G_n^0 such an H. Denote the obtained graph by H_n^*.

Now we add a few new edges to this graph: repeatedly add edges to the graph until it becomes saturated for the property of not arrowing (K_3, K_3). Call the resulting graph U_n. We show that U_n is (K_3, K_3)-co-critical and its maximum degree is

$$\Delta(U_n) \le \Delta(S_k) \cdot m + k \cdot \alpha(H) + \Delta(H). \tag{2}$$

Here, on the right hand side, the first term estimates the degrees in the full connections between different copies of H, the last one the degrees within the graphs H and – as it is explained below, – the middle term estimates the degrees coming from the saturating edges.

The first statement, namely, that U_n is co-critical, is trivial, by its construction: by the fact that we obtained it by a saturation procedure. (In fact, we also have to show that the graph we start to saturate is not yet arrowing the triangle. But this is obvious: colouring all edges within the same D_i by RED and all others by BLUE, we get no monochromatic triangles in that graph.)

To prove (2) on the maximum degree we will show that

(a) we could not possibly add edges joining vertices of the same D_j and

(b) if D_i and D_j correspond to independent vertices of S_m, then each $x \in D_i$ is joined to at most $\alpha(H)$ vertices of D_j.

The proof of (a) is easy from Theorem 2: For each D_j we consider a D_ℓ completely joined to D_j. They span a (K_3, K_3)-co-critical subgraph of U_n. So no edges can be added to D_j without arrowing a monochromatic K_3.

To prove (b) observe first that since S_k is K_3-saturated, there must exist a D_k joined to both D_i and D_j. By Lemma 1, we know that if U_n is coloured in RED and BLUE without monochromatic triangles, then all the edges joining vertices of the same group D_t must have the same colour, and this is the same for every D_t. We shall assume that this colour is RED. In particular, all the edges xy, $x, y \in D_i$ are RED, and the same holds for D_j and D_ℓ. We also know that for fixed $u \in D_i$ all but at most $\alpha(H)$ edges uw $w \in D_j$ are BLUE. Let now $x \in D_i$ be joined to the vertices of $X \subseteq D_j$. If there is an edge uv, $u, v \in X$, then we may assume – by the exclusion of RED triangles – that xu is BLUE. Both x and u are joined to D_ℓ in RED by at most $\alpha(H)$ edges, by Lemma 1. Therefore there exists a $w \in D_\ell$ joined to both x and u in BLUE, providing a BLUE K_3, a contradiction. So X is a set of independent vertices in D_j, proving that x is joined to at most $\alpha(H)$ vertices of D_j.

Observe, that in a triangle-free graph U the degree $d(x)$ of a vertex is at most $\alpha(U)$: the neighbours of x form an independent set. So the estimates above on the independence number apply also to the maximum degree. Put

$$m = k, \qquad \Delta(H) \approx \Delta(S_k) < c_1\sqrt{k}\log k.$$

Now we can estimate the maximum degree as follows:

$$\Delta(U_n) \le \Delta(S_k)k + k\alpha(H) + k = O(n^{3/4}\log n).$$

So $\Delta(U_n) \le c_2 n^{3/4} \log n$. ∎

Restrictions on the minimum degree

In this section we will show that the minimum degree of a (K_3, K_3)-co-critical graph is at least 4. To prove this we will make use of the following easy statement.

Claim 5. *G is (K_3, K_3)-co-critical, if and only if in any good RED-BLUE edge-colouring of G (i.e., one without monochromatic triangles) for any non-edge (v, w) there must be a 2-path between v and w in both colours.*

Proof. To see this, assume that G is co-critical and there is given a good RED-BLUE colouring of G. If there is no RED 2-path between v and w, then joining v and w by a RED edge would not create a monochromatic triangle contradicting that G is co-critical. The proof of the opposite direction is similar. ∎

Another easy observation we will use is the following.

Claim 6. *If $\chi(G) \leq 5$ then the edges of G can be 2-coloured without having monochromatic triangles.* ∎

For the sake of completeness we formulate the latter statement (in fact, a consequence of it) in a more general form.

Lemma 2. *If G is $(L_1, \ldots, L_r)$-co-critical, then $\chi(G) \geq R(L_1, \ldots, L_r)$. Further, if $\chi(G) = R(L_1, \ldots, L_r)$, then G is a complete R-partite graph: $G = K_R(n_1, \ldots, n_R)$.* ∎

Now we are ready to prove our statement about the minimum degree.

Theorem 6. *For every (K_3, K_3)-co-critical graph G the minimum degree $d(G) \geq 4$.*

Proof. Assume that there exists a co-critical G with a vertex v of degree 3. (The same argument works for smaller minimum degree.) Let the neighbours of v be x, y, and z and the set of remaining vertices be denoted by U.

Consider a "good colouring" of G. It follows immediately from Claim 5 that the edges vx, vy, vz cannot all have the same colour. Then we can assume that two of them are RED and one of them is BLUE. Let vz be the BLUE one.

Again, by Claim 5, z should be connected to each $u \in U$ by a BLUE edge, otherwise there would be no BLUE 2-path between v and some $u \in U$. Hence U contains no BLUE edges at all.

Now U can be partitioned into $A = U \cap (N_R(x) \cap N_R(y))$, $B = U \cap (N_R(x) \setminus N_R(y))$ and $C = U \cap (N_R(y) \setminus N_R(x))$, since, again by Claim 5 every $u \in U$ is connected to either x or y by RED.

Here $A \cup B$ and $A \cup C$ are independent sets of vertices, since for each of them there is a vertex outside, completely joined to them in RED, and another vertex, (namely z) completely joined to them in BLUE. Hence they can contain neither RED, nor BLUE edges.

Thus we succeded in covering the whole set $V(G)$ by 5 independent sets: $\{v\} \cup A \cup B$, C and $\{x\}$, $\{y\}$ and $\{z\}$. Assume first that B and C are nonempty. Join (say) v to a vertex $u \in C$. The resulting G^* is still covered by the same 5 groups of independent vertices. Thus $\chi(G^*) \leq 5$. Since any 5-chromatic graph can be coloured in RED and BLUE avoiding monochromatic triangles, and adding an edge to G we still could obtain an at most 5-chromatic graph, we have shown that G is not co-critical. The case when e.g. B is empty is even easier, for now G is covered by 4 classes of independent vertices: $\{v\} \cup U$, $\{x\}$ $\{y\}$ $\{z\}$, and joining v to a $u \in U$ we obtain a 5-chromatic graph, contradicting again to the assumption that G is co-critical. ∎

Remark. Of course, this theorem is sharp, as shown by Z_6, or more generally, by $K_5(1,1,1,1,n-4)$.

Claim 7. *If there is a vertex of degree 4, then there exists a* $K_4 \subseteq G_n$.

Proof. Given a RED-BLUE-colouring of a graph G, we shall say that a vertex x has degree $(p.q.)$ if it is adjacent to p RED and q BLUE edges.

There are two cases: either we can colour G_n having (3.1.) or (1.3.) RED-BLUE degree at the degree four vertex x or it must be (2.2.). Consider the second case first.

(a) Assume that we regard the vertex x joined to a, b, c and d. Assume that ax and dx are BLUE, and bx, cx are RED. Since bx (alone) cannot be recoloured by BLUE, therefore either ab or db must be BLUE. Assume that ab is BLUE. Since xa cannot be recoloured by RED, therefore either ba or ca must be RED. Thus ca is RED. Applying this argument to the recolouring of xc we get that dc is BLUE. Applying it to xd we get that bd is red. Let U denote the set of remaining vertices. Every $u \in U$ is connected either to a or to d by BLUE (to be joined to x by a BLUE 2-path). Therefore joining ad in RED cannot create a monochromatic K_3. The same argument shows that joining bc in BLUE cannot create BLUE triangles. Hence we have got even a K_5 in G_n.

(b) The other case is when there is a (3.1) colouring, say xa is RED, xb, xc, xd are BLUE. Hence a is joined in RED completely to U. Hence ab, ac, ad can be added in BLUE. If any of bc, bd, or cd is in the graph then we are home: we have a K_4. In the remaining case we really can assume that the edges ab, ac, ad are BLUE. Therefore each of them can be joined to U in RED: this contradicts to the BLUE 2-path condition. ∎

We have seen that all of our constructions contain either Z_6 or $C_5 \otimes C_5$ as subgraphs. Both of these graphs contain Z_5, so it is natural to ask whether a co-critical graph must always contain a Z_5, or even stronger, is it true that adding any new edge to a co-critical graph a K_5 appears. Knowing the results by Folkman and by Nešetřil and Rödl, one is inclined to believe in the existence of Z_5-free co-critical graphs. But we have no such examples. The next step is Z_4, a complete graph on four vertices minus one edge. (Z_4 is often called as *diamond.*) Next we prove that a diamond always appears in co-critical graphs.

Theorem 7. *Every* (K_3, K_3)*-co-critical graph contains a diamond, i.e., a* $Z_4 = (K_4 - e)$.

Proof. Assume there is no diamond in a (K_3, K_3)-co-critical graph G, and fix a good RED-BLUE edge-colouring, i.e., one without monochromatic triangles. Observe, that every edge belongs to at most one triangle in this graph (otherwise there would be a diamond). Let a and b be two non-adjacent vertices of our graph G. If $N(a) \cap N(b)$ contains an edge, we have an (induced) diamond, as a subgraph. If $N(a) \cap N(b)$ is a set of independent vertices, then we change all the BLUE edges between $\{a, b\}$ and $N(a) \cap N(b)$ RED. Some monochromatic K_3's of form

$$\{axy \;:\; x \in N(a) \cap N(b), \quad y \in N(a) \setminus N(b)\}$$

or

$$\{bxy \;:\; x \in N(a) \cap N(b), \quad y \in N(b) \setminus N(a)\}$$

can arise. However, since each edge is in at most one triangle, the edges ay (or by, respectively) belonging to RED triangles could be recoloured by BLUE, without getting further monochromatic triangles. Now, adding ab as a BLUE edge would not create a monochromatic triangle, so G was not co-critical. ∎

Remark. The above proof does not show whether there must be an induced Z_4 in a co-critical graph, because the "independent triangle colourability" is ruined also if we have a K_4. It could be interesting to know the answer to the following

Problem. *Is it true that every non-edge of a (K_3, K_3)-co-critical graph is the non-edge of a diamond? (cf. Open Problem 3.)*

5. CONNECTION TO DIRAC'S EXAMPLE IN THE THEORY OF COLOUR-CRITICAL GRAPHS

Some of the ideas of this paper are coming or motivated from the theory of 4-chromatic-edge-critical graphs. Hence we include a " short survey" of these graphs.

Gallai, Erdős and Dirac started investigating the structure of colour-edge-critical graphs, i.e. graphs, which are k-chromatic for some given, fixed k, but deleting any edge of the graph the chromatic number drops (by 1). Erdős asked (among others) if the number of such graphs can be large or not.

For $k = 3$ the odd cycles are the colour-critical graphs, therefore we shall assume above that $k \geq 4$. Dirac observed that $C_{2k+1} \otimes C_{2\ell+1}$ is 6-chromatic, colour-critical. Further, if $k = \ell$ and $n = 4k + 2$, then the minimum degree of this graph is also $\approx \dfrac{1}{2}n$.

For $k = 2, \ell = 2$ we get $C_5 \otimes C_5$, the graph of Claim 1. One feels that the two problems have some deeper connection. (B. Toft later constructed a graph G_n which was 4-chromatic colour-critical and had $\approx n^2/16$ edges. This settled the problem of the existence of colour-critical graphs with many edges. The maximum number of edges such a graph can have is still open (apart from some estimates) and one of the most intriguing problems in the field asks if the minimum degree of a 4-critical graph can be $\geq cn$ for some $c > 0$ as $n \to \infty$. The best result is the construction of 4-chromatic colour-critical graphs G_n with mimimum degree $\geq cn^{1/3}$ ([20], [21], [17].))

6. OPEN PROBLEMS

1. Are there infinitely many *strongly minimal* co-critical graphs.

2. Can one get a construction of a (K_3, K_3)-co-critical graph G_n without K_4?

3. Is it true that for every (K_3, K_3)-co-critical G_n adding any new edge we get a K_5? Or at least a K_4? (Instead of the latter question we can ask whether every non-edge of a co-critical graph is in a diamond.)

4. Assume that a (K_3, K_3)-co-critical G_n contains a K_5. Does this imply that G_n contains also a Z_6?

5. Is it always true that duplicating a vertex of a co-critical graph (cf. the Remark after Theorem 3) we get a co-critical graph?

6. Can one fix the parameters of a LPS-graph to get a (K_3, K_3)-co-critical graph?

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