Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
TOPOLOGY
Topology 45 (2006) 673–689
www.elsevier.com/locate/top

Milnor open books and Milnor fillable contact 3-manifolds

Clément Caubel[a], András Némethi[b,1], Patrick Popescu-Pampu[a,*]

[a] *Univ. Paris 7 Denis Diderot, Inst. de Maths.—UMR CNRS 7586, Équipe "Géométrie et Dynamique" Case 7012, 2, Place Jussieu, 75251 Paris Cedex 05, France*
[b] *Rényi Institute of Mathematics, P.O.B. 127, H-1364 Budapest, Hungary*

Received 20 September 2005

To Bernard Teissier, for his 60th birthday

Abstract

We say that an oriented contact manifold (M, ξ) is *Milnor fillable* if it is contactomorphic to the contact boundary of an isolated complex-analytic singularity $(\mathscr{X}, x)$. In this article we prove that any three-dimensional oriented manifold admits at most one Milnor fillable contact structure up to contactomorphism. The proof is based on *Milnor open books*: we associate an open book decomposition of M with any holomorphic function $f : (\mathscr{X}, x) \to (\mathbb{C}, 0)$, with isolated singularity at x and we verify that all these open books carry the contact structure ξ of (M, ξ)—generalizing results of Milnor and Giroux.

MSC: 32S55; 53D10; 32S25; 57R17

Keywords: Contact structures; Open book decompositions; Isolated singularities

1. Introduction

Let $(\mathscr{X}, x)$ be an irreducible germ of complex analytic variety, which is smooth outside x. Choose a local embedding $(\mathscr{X}, x) \subset (\mathbb{C}^s, 0)$ and intersect $\mathscr{X}$ with small euclidian spheres centered at 0. One gets like this a naturally oriented manifold whose diffeomorphism type is independent of the embedding and of

* Corresponding author. Tel.: +33 144273757.
E-mail addresses: caubel@math.jussieu.fr (C. Caubel), nemethi@renyi.hu (A. Némethi), ppopescu@math.jussieu.fr (P. Popescu-Pampu).
[1] The second author is partially supported by a Marie Curie Fellowship and an OTKA Grant.

doi:10.1016/j.top.2006.01.002

the (sufficiently small) spheres. Hence one gets (the diffeomorphism type of) a closed oriented manifold $M(\mathscr{X})$, which is called the (*abstract*) *boundary* (*or the link*) *of* $(\mathscr{X}, x)$.

Now, for any fixed embedding e and small sphere (of radius $\sqrt{\varepsilon}$), the above intersection $M_{e,\varepsilon}$—being a real one-codimensional submanifold in the complex manifold $\mathscr{X}\setminus\{x\}$—is naturally endowed with a complex hyperplane distribution $\xi_{e,\varepsilon}$, with complex multiplication $i|\xi_{e,\varepsilon}$. Then the triple $(M_{e,\varepsilon}, \xi_{e,\varepsilon}, i|\xi_{e,\varepsilon})$ is a CR-manifold. By a result of Scherk [26] it is an absolute invariant: it determines the analytic type of the germ $(\mathscr{X}, x)$. On the other hand, this CR-manifold really depends on the parameters (e, ε).

Nevertheless, one gets a well-defined intermediate invariant object by removing the complex multiplication $i|\xi_{e,\varepsilon}$ (but keeping the orientation of $\xi_{e,\varepsilon}$ induced by this multiplication). Indeed, Varchenko has showed in [29] that the couple $(M_{e,\varepsilon}, \xi_{e,\varepsilon})$ is a contact manifold, which, up to a contact isotopy, only depends on the analytic type of $(\mathscr{X}, x)$. This isotopy type is called the *contact boundary* of $(\mathscr{X}, x)$ and denoted by $(M(\mathscr{X}), \xi(\mathscr{X}))$. In this way, one associates to any isolated singular point of a complex variety a contact manifold. (For a more general definition, see Section 3.)

Definition 1.1. Let (M, ξ) be a connected closed oriented contact manifold. If there exists a germ $(\mathscr{X}, x)$ of (normal) complex analytic space with isolated singularity such that (M, ξ) is isomorphic to the contact boundary $(M(\mathscr{X}), \xi(\mathscr{X}))$ of $(\mathscr{X}, x)$, then we say that (M, ξ) *admits a Milnor filling* (or is *Milnor fillable*). The germ $(\mathscr{X}, x)$ is called a *Milnor filling* of (M, ξ).

We will use the same terminology for any oriented manifold M (forgetting the contact structure) if M is isomorphic to the abstract boundary $M(\mathscr{X})$ of some singularity $x \in \mathscr{X}$.

Above, the normality assumption is not restrictive: $(M(\mathscr{X}), \xi(\mathscr{X}))$ is isomorphic to the contact boundary $(M(\hat{\mathscr{X}}), \xi(\hat{\mathscr{X}}))$ of the normalization $(\hat{\mathscr{X}}, \hat{x})$ of $(\mathscr{X}, x)$ (see e.g. Proposition 3.3).

Some natural questions arise, the most basic one being the classification of Milnor fillable contact manifolds. In this paper we will concentrate on three-dimensional oriented manifolds. In this case the *existence* of a Milnor filling is a topological property and it is completely understood: an oriented 3-manifold M is Milnor fillable if and only if it is a graph-manifold obtained by plumbing along a weighted graph which has a negative definite intersection form (see [9]).

Our main theorem establishes the *uniqueness* property:

Theorem 1.2. *Any Milnor fillable* 3-*manifold admits a unique Milnor fillable contact structure up to contactomorphism.*

Here some comments are in order.

(i) All Milnor fillable contact 3-manifolds are Stein fillable. Indeed, if the surface singularity $(\mathscr{S}, 0)$ is smoothable, a simple application of Gray's Theorem shows that its contact boundary coincides with the contact boundary of any Milnor fiber, which is Stein. But even if $(\mathscr{S}, 0)$ is not smoothable, its contact boundary can still be filled with a complex manifold (e.g., with the resolution of the singularity) and results of Bogomolov and de Oliveira (see [1]) show that the complex structure of the filling can be made Stein without changing the contact boundary. In particular, Milnor fillable contact structures are tight, by a general theorem of Gromov and Eliashberg.

(ii) All these manifolds contain at least one incompressible torus, except for the lens spaces and some small Seifert spaces. Therefore, in general, by a theorem of Colin and Honda–Kazez–Matić (see [4]) all these manifolds admit infinitely many different tight contact structures up to isomorphism.

In particular, Theorem 1.2 indicates that Milnor fillability is a very special property of tight contact structures.

(iii) Finally notice that the above classification Theorem 1.2 is in a big contrast with the phenomenon valid for higher dimensional singularities. For instance, Ustilovsky in [27] discovered on the spheres $\mathbf{S}^{4n+1}$, $n \geqslant 1$, infinitely many different Milnor fillable contact structures. (Also, the authors know no criterions which would ensure, in general, Milnor fillability.)

In Section 2, we recall the work of Giroux on contact structures and open books. The key message is that in dimension 3 any open book *carries* a unique contact structure up to isotopy. This description of contact structures is perfectly adapted to the contact boundaries exploited in Section 3. There we start with a rather general definition of the contact boundary. Then we prove that any analytic function $f : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ (with an isolated singularity at x) defines an open book decomposition of the boundary $M(\mathscr{X})$—we call them *Milnor open books*. The point is that (see Theorem 3.9 for more details):

Theorem 1.3. *Any Milnor open book carries the natural contact structure $\xi(\mathscr{X})$ on $M(\mathscr{X})$.*

We emphasize that this result is valid in *any* dimension. We believe that it will be an essential tool in the further study of contact boundaries.

Now, the proof of Theorem 1.2 runs as follows (see the end of Section 4). For any Milnor fillable 3-manifold M, using its plumbing representation, we construct an oriented link in it which is isotopic to the binding of a Milnor open book for *any* Milnor filling of M (see 4.1). Then, using the work of Chaves [3] and Pichon [22], we show that all the possible Milnor open books with this binding are, in fact, isomorphic (see 4.8).

A short preliminary version of this article appeared in [2]. In it, Theorem 1.2 was proved for rational homology spheres. In the meantime, we succeeded to replace the old proof by a more natural one and to extend the result to all Milnor fillable 3-manifolds.

2. Contact structures and open books

Let M be an oriented $(2n-1)$-dimensional manifold, where $n \geqslant 1$. A (coorientable) *contact structure* on M is a hyperplane distribution ξ in TM defined by a global 1-form α such that $\alpha \wedge (d\alpha)^{\wedge(n-1)} \neq 0$. We say that the pair (M, ξ) is a *contact manifold* and α a *contact form*. The form α is called *positive* if $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ defines the chosen orientation of M. If n is even, then the orientation defined by $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ does not depend on the choice of the defining form α, hence one can speak about *positive contact structures*.

If α is a contact form on M, the condition $\alpha \wedge (d\alpha)^{\wedge(n-1)} \neq 0$ implies that $d\alpha|_\xi$ is a symplectic form. This shows that $\ker(d\alpha)$ is a one-dimensional vector subspace of TM, transversal to ξ. Therefore, there exists a unique vector field R on M such that $\iota_R d\alpha := d\alpha(R, \cdot) = 0$ and $\alpha(R) = 1$. It is called the *Reeb vector field* associated to α.

Two contact structures ξ and ξ' on M are *isotopic* (resp. *isomorphic* or *contactomorphic*) if there is an isotopy (resp. a diffeomorphism) of M which sends ξ on ξ'.

For more about contact structures, see e.g. Eliashberg and Mishachev's book [6].

In the study of contact manifolds one of the main tools is provided by *open books carrying contact structures*.

Definition 2.1. An *open book* with *binding* N in a manifold M is a couple (N, θ), where N is a (not necessarily connected) two-codimensional closed submanifold of M with trivializable normal bundle and $\theta : M\backslash N \to \mathbf{S}^1$ is a smooth fibration which in a trivialized neighborhood $N \times \mathbf{D}^2$ of N coincides with the angular coordinate. The fibers of θ are called the *pages* of the open book.

Notice that $d\theta$ induces natural co-orientations on the binding and the pages of the open book. Thus, any fixed orientation of M induces a natural orientation on N. If N itself is oriented a priori, then we say that the open book is *compatible with the orientations of M and N* if the two orientations of N coincide.

Finally, we say that the open books (N, θ) and (N', θ') in the manifolds M, respectively M', are *isomorphic* if there exists a diffeomorphism $\phi : (M, N) \to (M', N')$ which preserves the pages and their co-orientations.

Every construction of open books we will consider rests on the following lemma. Its proof is straightforward.

Lemma 2.2. *Let M be an oriented closed manifold and let $\phi : M \to \mathbb{C}$ be a differentiable function. If there exists a number $\eta > 0$ such that*

- $d(\arg \phi) \neq 0$ *if* $|\phi| \geqslant \eta$, *and*
- $d(\phi) \neq 0$ *if* $|\phi| \leqslant \eta$

then $(\phi^{-1}(0), \arg \phi)$ *is an open book in M.*

Remark 2.3. With the exception of 3.8, where one chooses the branch $\arg \in (-\pi, \pi]$, the argument of a non-zero complex number is regarded as an element of $\mathbb{R}/2\pi\mathbb{Z} \simeq \mathbf{S}^1$.

Next we recall the relationship between contact structures and open books.

Definition 2.4 (*Giroux [7]*). We say that a contact structure ξ on an oriented manifold M is *carried by an open book* (N, θ) if it admits a defining contact form α which verifies the following:

- α restricts to a contact form on N;
- $d\alpha$ restricts to a symplectic form on each fiber F of θ;
- The orientation of N induced by α coincides with its orientation as the boundary of the symplectic manifolds $(F, d\alpha)$.

If a contact form α satisfies these conditions we say that it is *adapted* to (N, θ).

One has the following criterion for an open book to carry a contact structure:

Lemma 2.5 (*Giroux [8]*). *Let α be a positive contact form on the oriented manifold M. Suppose that there exists an open book (N, θ) in M and a neighborhood $V = N \times \mathbf{D}^2$ of $N = N \times \{0\}$ such that*:

- *θ is the normal angular coordinate in V*;
- *α restricts to a contact form on each submanifold $N \times \{*\}$ in V*;

- *$d\alpha$ restricts to a symplectic form on each fiber F' of θ in $M \setminus \mathrm{Int} V$;*
- *The orientation of $N \times \{*\} = \partial F'$ induced by α coincides with its orientation as the boundary of the symplectic manifold $(F', d\alpha)$.*

Then the open book (N, θ) carries the contact structure $\xi = \ker \alpha$.

The relevance of this notion for our study of three-dimensional contact boundaries lies in the following result:

Theorem 2.6 (*Giroux [7,8]*). *On a closed oriented* 3*-manifold, two positive contact structures carried by the same open book are isotopic.*

In particular, in order to show that two contact structures on a given 3-manifold are isomorphic, it is enough to show that they are carried by isomorphic open books.

Remark 2.7. (a) In fact, any open book in an oriented manifold carries positive contact structures. This was proven for 3-manifolds by Thurston and Winkelnkemper and in the general case by Giroux [7].

(b) Giroux [7] also proved that a version of Theorem 2.6 holds in all dimensions if one further asks that the two contact structures induce the same symplectic structure on the pages up to isotopy and completion (see again [7] and [8]). He also proved in collaboration with Mohsen that any contact structure is carried by an open book. Hence, in fact in any dimension, one can translate statements of contact geometry into properties of open books.

3. Contact boundaries and Milnor open books

Let $(\mathscr{X}, x)$ be an irreducible germ of a complex analytic space with isolated singularity. Sometimes, we will denote by $\mathscr{X}$ a sufficiently small representative of this germ. Let $m_{\mathscr{X},x} \subset \mathcal{O}_{\mathscr{X},x}$ be the ideal of germs of holomorphic functions on $(\mathscr{X}, x)$ vanishing at x.

3.1. The contact boundary associated with a holomorphic immersion

Write $\mathscr{X}^*$ for the complex manifold $\mathscr{X} \setminus \{x\}$. Let $J : T\mathscr{X}^* \to T\mathscr{X}^*$ be the operator of fiberwise multiplication by i, when $T\mathscr{X}^*$ is seen as a real vector bundle. We will also denote it by $i\,\cdot$, when no confusion is possible. Set $d^{\mathbf{c}} := J^* \circ d$, i.e. $d^{\mathbf{c}} F = dF \circ J$ for any differentiable function $F : \mathscr{X}^* \to \mathbb{R}$. Then $d^{\mathbf{c}} = i(\partial - \bar{\partial})$.

A real function F on $\mathscr{X}^*$ is called *strictly pluri-subharmonic* (*spsh*) if and only if $-dd^{\mathbf{c}}(F) > 0$, that is if $-dd^{\mathbf{c}}(F)(v, Jv) > 0$ for any non-zero tangent vector v of $\mathscr{X}^*$.

For any $\phi_1, \ldots, \phi_N \in m_{\mathscr{X},x}$ consider the holomorphic map $\Phi : (\mathscr{X}, x) \to (\mathbb{C}^N, 0)$ with components ϕ_i, and the real analytic function

$$\rho := \sum_{k=1}^{N} |\phi_k|^2 : (\mathscr{X}, x) \to (\mathbb{R}, 0).$$

For each $\varepsilon > 0$, define

$$M_{\rho,\varepsilon} := \rho^{-1}(\varepsilon).$$

Clearly, $M_{\rho,\varepsilon}$ is a smooth compact manifold for $\varepsilon > 0$ sufficiently small if and only if Φ is a *finite* analytic morphism. In the sequel we will assume that this fact holds.

On $\mathscr{X}^*$ we consider the following natural objects associated with ρ:

$$\alpha := -d^{\mathbf{c}}\rho,$$

$$\omega := d\alpha = -dd^{\mathbf{c}}\rho,$$

$$g(u, v) := \omega(u, Jv) \quad \forall u, v \in T\mathscr{X}^*,$$

$$h := g + \mathrm{i}\omega.$$

Then, on $\mathscr{X}^*$ define

$$\xi_\rho := \ker(d\rho) \cap \ker(d^{\mathbf{c}}\rho).$$

It is a field of complex tangent hyperplanes of the real tangent bundle of $\mathscr{X}^*$ endowed with its canonical (almost) complex structure. Moreover, it is tangent to the levels $M_{\rho,\varepsilon}$ of ρ. In fact

$$\xi_{\rho,\varepsilon} := \xi_\rho|_{M_{\rho,\varepsilon}} = \ker(\alpha|_{M_{\rho,\varepsilon}}).$$

Lemma 3.1. *The following conditions are equivalent*:

(1) *The pair* $(M_{\rho,\varepsilon}, \xi_{\rho,\varepsilon})$ *is a contact manifold for* ε *sufficiently small.*
(2) *The morphism* Φ *is an immersion of* $\mathscr{X}^*$ *into* $\mathbb{C}^N$.
(3) *The function* ρ *is spsh.*

Proof. An easy computation shows that

$$-dd^{\mathbf{c}}\rho(v, Jw) = 2\sum_{k=1}^{N} \det\begin{pmatrix} \dfrac{d\phi_k(v)}{d\phi_k(v)} & \dfrac{-d\phi_k(w)}{d\phi_k(w)} \end{pmatrix}$$

for any tangent vector fields v, w of $\mathscr{X}^*$. This shows that $\ker d\Phi = \ker(-dd^{\mathbf{c}}\rho)$ and $-dd^{\mathbf{c}}\rho \geqslant 0$. The lemma is an immediate consequence of this. $\square$

From now on we fix Φ, and we assume that it induces an *immersion of* $\mathscr{X}^*$ into $\mathbb{C}^N$; we will briefly say that it is *a holomorphic immersion*. In such a case, we say that the function $\rho : \mathscr{X}^* \to \mathbb{R}$ defined above is an *euclidian rug function*. In the view of the previous lemma, the objects associated with an euclidian rug function ρ have the following properties:

- the levels $M_{\rho,\varepsilon}$ are naturally oriented as boundaries of the complex manifolds $\rho^{-1}(0, \varepsilon]$, endowed with their canonical orientations;
- α—restricted to the levels $M_{\rho,\varepsilon}$—is a positive contact form;
- ω is a symplectic form compatible with the complex structure on $\mathscr{X}^*$;

- g is a riemannian metric on $\mathscr{X}^*$;
- h is a Kähler metric.

(Notice that not all the real analytic rug functions, e.g. as in [14], have all these properties, although they are perfectly good to identify the boundary manifold $M_{\rho,\varepsilon}$.)

Now, using Gray's theorem (see [6] for instance), it is easy to prove:

Proposition/Definition 3.2. *The pair $(M_{\rho,\varepsilon}, \xi_{\rho,\varepsilon})$ is a positive contact manifold, whose contact isotopy type does not depend on the choice of the holomorphic immersion Φ and of $\varepsilon > 0$ sufficiently small. This isotopy type is called the* contact boundary of $(\mathscr{X}, x)$ *and denoted by* $(M(\mathscr{X}), \xi(\mathscr{X}))$.

This generalizes Varchenko's result [29] which corresponds to the case when Φ is an embedding. Considering arbitrary holomorphic immersions and their associated rug functions (instead of embeddings and euclidean spheres) has many advantages: it does not only increase the possibilities to realize the isotopy type of the natural contact structure, but it also gives the possibility to compare the corresponding contact structures under finite maps unramified outside the singular point:

Proposition 3.3. *Suppose that $\gamma : (\mathscr{X}, x) \to (\mathscr{Y}, y)$ is a finite map between irreducible complex analytic germs and that the restriction $\gamma : \mathscr{X}^* \to \mathscr{Y}^*$ is an unramified covering between connected smooth spaces. Then one can choose representatives of the contact boundaries $(M(\mathscr{X}), \xi(\mathscr{X}))$ and $(M(\mathscr{Y}), \xi(\mathscr{Y}))$ such that the restriction of γ to $(M(\mathscr{X}), \xi(\mathscr{X}))$ surjects onto $(M(\mathscr{Y}), \xi(\mathscr{Y}))$ and is a local contactomorphism.*

Proof. Choose an embedding $\Psi : (\mathscr{Y}, y) \to \mathbb{C}^M$ and define $v := \sum |\psi_k|^2$, $\rho := v \circ \gamma$, $\Phi := \Psi \circ \gamma$. Then compare the contact structures associated with these rug functions. □

As a particular case of the previous proposition, if G is a finite group of analytic automorphisms of $(\mathscr{X}, x)$ acting freely on $\mathscr{X}^*$ and if $(\mathscr{Y}, y)$ is the quotient of $(\mathscr{X}, x)$ by G, then the isotopy type of $(M(\mathscr{X}), \xi(\mathscr{X}))$ admits a representative on which G acts via contactomorphisms, and the quotient is a representative of $(M(\mathscr{Y}), \xi(\mathscr{Y}))$.

3.2. The Milnor open book associated with a holomorphic function

Fix a function $f \in m_{\mathscr{X},x}$ which defines an isolated singularity at x. Set

$$N_{\rho,\varepsilon}(f) := M_{\rho,\varepsilon} \cap f^{-1}(0).$$

For $\varepsilon > 0$ sufficiently small $N_{\rho,\varepsilon}(f)$ is smooth and naturally oriented. The argument of f restricted to $M_{\rho,\varepsilon} \backslash N_{\rho,\varepsilon}(f)$ gives a well-defined function

$$\theta_\varepsilon(f) := \arg\ f : M_{\rho,\varepsilon} \backslash N_{\rho,\varepsilon}(f) \to \mathbf{S}^1.$$

We then have the following generalization of Milnor's Fibration Theorem (see [15]):

Proposition/Definition 3.4. *For $\varepsilon > 0$ sufficiently small, the pair $(N_{\rho,\varepsilon}(f), \theta_\varepsilon(f))$ is an open book in the boundary $M_{\rho,\varepsilon}$ which is compatible with the orientations. Furthermore, its isotopy type does not depend on the choice of $\varepsilon > 0$ nor on the choice of holomorphic immersion Φ. It is called the* Milnor open book *of f and denoted by $(N(f), \theta(f))$. The pair $(M(\mathscr{X}), N(f))$ is called the* link *of f.*

This statement splits into two parts: a fibration and an invariance result. The first one, for Φ an embedding, appears in [10], Satz 1.6. That proof, based on Lemma 2.2, extends to our new situation once a key fact is verified. Since this fact is used by us in other places as well, we provide its complete proof in Proposition 3.8, after having stated two preliminary lemmas. For all the other details, we refer to [10]. The invariance statement can be proved similarly using the classical tools of local analytic singularities: we leave the verification to the interested reader.

Remark 3.5. (a) In fact, a fibration result has been proved in a more general context by Durfee in [5], but without specifying the fibration map: he actually proves that the complement of $f^{-1}(0)$ in the boundary of any analytic neighborhood of a (non necessarily isolated) singularity x in $\mathscr{X}$ is the total space of a fibration (with no more precision about the projection map) whose fiber is homeomorphic to the intersection of a smooth fiber of f with this neighborhood. The above Proposition 3.4 shows that, in the neighborhoods defined by euclidian rug functions, this fibration can be defined by the argument of f. A corresponding statement in the general case is not guaranteed: we use indeed the peculiar form of ρ in the computations in 3.8. We also emphasize that if one wishes to verify the compatibility of an open book with a contact structure, then one needs very precise information about θ and about (a "well chosen") contact form α. This actually explains why we need the fibration to be given by the argument of f.

(b) Fix the boundary M of an analytic germ $(\mathscr{X}, x)$ as above. In general it is extremely difficult to verify that an open book (N, θ) in M (or a link N in M) is isotopic with a Milnor open book (or link) of a function germ f on $(\mathscr{X}, x)$. Even in the surface case it can happen that some open book is determined by a function germ for some analytic structure of $(\mathscr{X}, x)$, but the same fact is not true if one modifies the analytic structure of $(\mathscr{X}, x)$ (see [17], (2.15)).

For any real function F defined on $\mathscr{X}^*$, its gradient ∇F will be taken with respect to the riemannian metric g. If $\phi \in m_{\mathscr{X},x}$, we also denote by $\nabla\phi$ its gradient with respect to the hermitian metric h, that is $d\phi = h(\nabla\phi, \cdot)$ (ϕ being holomorphic, this field is well-defined).

The proof of the following lemma is straightforward:

Lemma 3.6. *If $\phi \in m_{\mathscr{X},x}$, one has*

(1) $\nabla|\phi|^2 = 2\phi\nabla\phi$;
(2) *In* $\mathscr{X}\backslash\phi^{-1}(0)$, $d \arg\phi = \operatorname{Im}(d\phi/\phi)$ *and* $\nabla \arg\phi = \mathrm{i}(\nabla\phi/\bar{\phi})$.

Lemma 3.7. *Let $p : (\mathbb{R}_+, 0) \to (\mathscr{X}, x)$ be a non-constant real analytic arc, and set $\dot{p} := \mathrm{d}p/\mathrm{d}t$. Then*:

$$\lim_{t\to 0} h\left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t)), \frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right) = 1.$$

Proof. Let h_0 be the canonical hermitian form on $\mathbb{C}^N$. If $\rho_0 = \sum_{k=1}^N |z_k|^2$, then its imaginary part is $\omega_0 = -dd^{\mathbf{c}}\rho_0$. As $\rho = (\rho_0|_{\Phi(\mathscr{X})}) \circ \Phi$, we get:

$$h = \Phi^*(h_0|_{\Phi(\mathscr{X})}),$$

$$\Phi_*(\nabla\rho) = \nabla_0(\rho_0|_{\Phi(\mathscr{X})}),$$

$$\Phi_*(\dot{p}) = \dot{q},$$

where $q := \Phi \circ p$ and ∇_0 is the gradient with respect to the riemannian metric $g_0 := \operatorname{Re} h_0|_{\Phi(\mathscr{X})}$. Hence, we get the following equalities of functions of t (where we write briefly $\nabla\rho$ instead of $(\nabla\rho)(p(t))$):

$$h\left(\frac{\nabla\rho}{\|\nabla\rho\|}, \frac{\dot p}{\|\dot p\|}\right) = \Phi^* h_0\left(\frac{\nabla\rho}{\|\nabla\rho\|}, \frac{\dot p}{\|\dot p\|}\right) = h_0\left(\Phi_*\left(\frac{\nabla\rho}{\|\nabla\rho\|}\right), \Phi_*\left(\frac{\dot p}{\|\dot p\|}\right)\right)$$
$$= h_0\left(\frac{\nabla_0(\rho_0|_{\Phi(\mathscr{X})})}{\|\nabla_0(\rho_0|_{\Phi(\mathscr{X})})\|}, \frac{\dot q}{\|\dot q\|}\right).$$

Take now a semi-analytic neighborhood $U(p)$ of the image of p in $\mathscr{X}^*$ which is embedded in $\mathbb{C}^N$ by Φ. Then the pair $(\Phi(U(p)), \{0\})$ verifies Whitney's condition (b) at 0 (see e.g. [14]), which shows that the angle between the vector $q(t) \in \mathbb{C}^N$ and the tangent space to $\Phi(U(p))$ at the point $q(t)$ converges to 0 when $t \to 0$. This implies that

$$\frac{\nabla_0(\rho_0|_{\Phi(\mathscr{X})})}{\|\nabla_0(\rho_0|_{\Phi(\mathscr{X})})\|}(q(t)) = \frac{\nabla_0\rho_0}{\|\nabla_0\rho_0\|}(q(t)) + o(1) = \frac{q}{\|q\|}(t) + o(1),$$

where $\nabla_0\rho_0$ denotes the gradient on the ambient space $\mathbb{C}^N$. From the previous computation we get

$$h\left(\frac{\nabla\rho}{\|\nabla\rho\|}, \frac{\dot p}{\|\dot p\|}\right) = h_0\left(\frac{q}{\|q\|}, \frac{\dot q}{\|\dot q\|}\right) + o(1). \tag{1}$$

Let $q(t) = q_0 t^Q + o(t^Q)$ be the beginning of the series expansion of $q(t)$, where $q_0 \in \mathbb{C}^N - \{0\}$ and $Q \in \mathbb{N}^*$. Then $\dot q(t) = Q q_0 t^{Q-1} + o(t^Q)$, which implies:

$$\frac{q}{\|q\|} = \frac{q_0}{\|q_0\|} + o(1), \qquad \frac{\dot q}{\|\dot q\|} = \frac{q_0}{\|q_0\|} + o(1).$$

Using Eq. (1), the lemma is proved. □

The next proposition—which is the key step in the proof of 3.4—generalizes Lemma 4.3 of [15] to the case of singular ambient spaces and *immersions* Φ (which are not necessarily embeddings). Our proof runs rather similarly, with the difference that our computations are intrinsic, they do not depend on any choice of local coordinates.

Proposition 3.8. *Let $\phi \in m_{\mathscr{X},x}$. For any $\theta_0 \in (0, \pi/2)$, there exists a neighborhood U_{θ_0} of* 0 *in $\mathscr{X}$ such that inside $U_{\theta_0} \backslash \phi^{-1}(0)$ the following implication holds*:

$$\nabla \arg\phi = \mathrm{i}\lambda\nabla\rho \quad \textit{with } \lambda \in \mathbb{C}^* \Longrightarrow |\arg\lambda| < \theta_0. \tag{$*$}$$

Proof. Suppose, by contradiction, that $(*)$ does not hold. Then, by the Curve Selection Lemma (see [14]), there exists an analytic arc $p : (\mathbb{R}_+, 0) \to (\mathscr{X}, x)$ such that along it we have the equality:

$$\nabla(\arg\phi)(p(t)) = \mathrm{i}\lambda(t)\nabla\rho(p(t)), \tag{2}$$

with $|\arg\lambda(t)| \geqslant \theta_0$ for any sufficiently small t.

As $\nabla\arg\phi = \mathrm{i}\nabla\phi/\bar\phi$ by Lemma 3.6, part 2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(p(t)) = h\left(\nabla\phi, \frac{\mathrm{d}p}{\mathrm{d}t}\right) = h(-\mathrm{i}\bar\phi\nabla\arg\phi, \dot p) = h(\bar\phi\lambda\nabla\rho, \dot p) = \phi\bar\lambda h(\nabla\rho, \dot p).$$

This implies

$$\lambda(t) = (\overline{\phi(p(t))})^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\overline{\phi(p(t))} \cdot \|\nabla\rho\|^{-1} \cdot \|\dot{p}\|^{-1} \cdot h\left(\frac{\nabla\rho}{\|\nabla\rho\|}, \frac{\dot{p}}{\|\dot{p}\|}\right)^{-1}. \tag{3}$$

Eq. (3) shows that the function $\lambda(t)$ has a Laurent expansion of type

$$\lambda(t) = lt^L + o(t^L), \quad l \neq 0, \quad L \in \mathbb{Z}. \tag{4}$$

Consider now the other two expansions as well

$$\phi(p(t)) = at^A + o(t^A), \quad a \neq 0, \quad A \in \mathbb{N}^*,$$

$$\|\nabla\rho\|^{-1} \cdot \|\dot{p}\|^{-1} = bt^B + o(t^B), \quad b > 0, \quad B \in \mathbb{Z}.$$

From the first one we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\phi(p(t))} = A\overline{a}t^{A-1} + o(t^{A-1}).$$

Combining them with Lemma 3.7 and with Eq. (3), we get:

$$l = A \cdot b > 0. \tag{5}$$

Since (4) and (5) contradict the hypothesis $|\arg\lambda(t)| \geqslant \theta_0$, the proposition is proved. □

3.3. The Milnor open books carry the natural contact structure on the boundary

Let us summarize: we have associated to any isolated singularity $(\mathscr{X}, x)$ a well-defined contact structure on its boundary $M_{\rho,\varepsilon}$, and to any function $f \in m_{\mathscr{X},x}$ with an isolated singularity an open book $(N_{\rho,\varepsilon}(f), \theta_\varepsilon(f))$ on $M_{\rho,\varepsilon}$. These two objects are naturally related, as is shown in the following theorem, which is a more precise version of Theorem 1.3 stated in the introduction:

Theorem 3.9. *Let $(\mathscr{X}, x)$ be an irreducible complex analytic germ having an isolated singularity at x, and let $f : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a holomorphic function having also an isolated singularity. Then the Milnor open book $(N_{\rho,\varepsilon}(f), \theta_\varepsilon(f))$ of f carries the natural contact structure on the boundary $M_{\rho,\varepsilon}$ of $\mathscr{X}$.*

Remark 3.10. (a) Theorem 3.9 stengthens a result of Giroux (see [8] for more details) and generalizes it to a singular ambient space: Giroux's original proof is valid only up to isotopy, the contact boundary $M_{\rho,\varepsilon}$ being replaced there by one of its deformations—a level of the function $\rho_c := \rho + c|f|^2$, for $c \gg 1$.

(b) The particular case of the function $z_0 \in m_{\mathscr{X}_k,0}$ with $\mathscr{X}_k = \{z_0^k + z_1^2 + \cdots + z_n^2 = 0\} \subset \mathbb{C}^{n+1}$ has been studied by Van Koert and Niederkrüger in [28], in relation with Ustilovsky's spheres.

(c) Theorem 3.9 has the following consequence. Let us fix the analytic germ $(\mathscr{X}, x)$. Then all the open books associated with all the possible holomorphic function germs f (with isolated singularity at x) determine (up to isotopy) the same contact structure. Notice also that function germs f with isolated singularity always exist. Indeed, once an embedding of $(\mathscr{X}, x)$ into $(\mathbb{C}^N, 0)$ is chosen, it is enough to take the restriction to $\mathscr{X}$ of a linear form whose kernel is not a limit of tangent hyperplanes to $\mathscr{X}\setminus\{x\}$ (see Lê and Teissier [13] for details).

We start the proof of 3.9 with some lemmas.

Fix an euclidian rug function $\rho : \mathscr{X} \to \mathbb{R}$. For ε sufficiently small, the 1-form $\alpha = -d^{\mathbf{c}}\rho$ defines the natural contact structure on the smooth level $M_{\rho,\varepsilon} = \rho^{-1}(\varepsilon)$. Denote $R \in \Gamma(M_{\rho,\varepsilon}, TM_{\rho,\varepsilon})$ its Reeb vector field.

Lemma 3.11. *The Reeb vector field R of α is given by $R = \mathrm{i}\nabla\rho/\|\nabla\rho\|^2$. Moreover, the contact distribution $\xi_{\rho,\varepsilon}$ on $M_{\rho,\varepsilon}$ is exactly the orthogonal complement of $\mathbb{C}.R = \mathbb{C}.\nabla\rho$ in $T\mathscr{X}^*|_{M_{\rho,\varepsilon}}$ with respect to the hermitian form h associated with ρ.*

Proof. Since R is a generator of the kernel of ω restricted to $TM_{\rho,\varepsilon}$, on $T\mathscr{X}^*|_{M_{\rho,\varepsilon}}$ we have $\iota_R\omega = kd\rho$ for some $k \in \mathscr{C}^\infty(M_{\rho,\varepsilon}, \mathbb{R})$. This shows that on $T\mathscr{X}^*|_{M_{\rho,\varepsilon}}$ one has

$$\iota_R\omega = g(k\nabla\rho, \cdot) = \omega(-k\mathrm{i}\nabla\rho, \cdot).$$

Since ω is non-degenerate on $T\mathscr{X}^*|_{M_{\rho,\varepsilon}}$, this shows that $R = -k\mathrm{i}\nabla\rho$. Hence

$$1 = \alpha(R) = -d\rho(\mathrm{i}R) = -d\rho(k\nabla\rho) = -k\|\nabla\rho\|^2,$$

which proves the first statement. For the second statement, it suffices to notice that

$$h(R, v) = \omega(R, \mathrm{i}.v) + \mathrm{i}\omega(R, v) = 0$$

for any section v of $\xi_{\rho,\varepsilon}$ (here we use the fact that $\mathrm{i}.v$ is also a section of $\xi_{\rho,\varepsilon}$). □

Lemma 3.12. *Fix $c > 0$, and put $\alpha_c := e^{-c|f|^2}\alpha$. If R_c denotes the Reeb vector field of α_c, on $M_{\rho,\varepsilon}\backslash N_{\rho,\varepsilon}(f)$ one has*:

$$d\theta(R_c) = e^{c|f|^2}(d\theta(R) + 2c|f|^2\|\mathrm{pr}_\xi\nabla\theta\|^2),$$

where $\mathrm{pr}_\xi : T\mathscr{X}^*|_{M_{\rho,\varepsilon}} \to \xi$ denotes the projection parallel to $\mathbb{C}.R$.

Proof. Put $H := e^{-c|f|^2}$. Put also $R_c := k(R + S_c)$, where $S_c \in \Gamma(M_{\rho,\varepsilon}, \xi_{\rho,\varepsilon})$ and $k \in \mathscr{C}^\infty(M_{\rho,\varepsilon}, \mathbb{R})$. In fact,

$$1 = \alpha_c(R_c) = H\alpha(k(R + S_c)) = kH\alpha(R) = kH,$$

hence one has $k = 1/H$. Now,

$$d\alpha_c = dH \wedge \alpha + Hd\alpha$$

which, when applied to $R_c = (1/H)(R + S_c)$ and restricted to $\xi := \xi_{\rho,\varepsilon} = \ker\alpha$ gives

$$(\iota_{S_c}d\alpha)|_\xi = \left.\frac{dH}{H}\right|_\xi = -c\,d|f|^2|_\xi.$$

But on $T\mathscr{X}^*$, Lemma 3.6 implies that

$$d|f|^2 = g(\nabla|f|^2, \cdot) = \omega(-\mathrm{i}\nabla|f|^2, \cdot) = \omega(-2|f|^2\nabla\theta, \cdot).$$

In particular, $(\imath_{S_c} d\alpha)|_\xi = \omega(2c|f|^2\nabla\theta, \cdot)|_\xi$. But $d\alpha|_\xi = \omega|_\xi$ is non-degenerate and $\imath_v\omega|_\xi = \imath_{\mathrm{pr}_\xi(v)}\omega|_\xi$ for any $v \in T\mathscr{X}^*|_{M_{\rho,\varepsilon}}$. Hence, we get

$$S_c = \mathrm{pr}_\xi(2c|f|^2\nabla\theta),$$

which shows that

$$d\theta(S_c) = 2c|f|^2 d\theta(\mathrm{pr}_\xi\nabla\theta) = 2c|f|^2 g(\nabla\theta, \mathrm{pr}_\xi\nabla\theta) = 2c|f|^2\|\mathrm{pr}_\xi\nabla\theta\|^2,$$

the last equality being a consequence of the second statement of the preceding Lemma 3.11. Since $d\theta(R_c) = d\theta(e^{c|f|^2}(R + S_c))$, we are done. □

Proof of Theorem 3.9. Fix a sufficiently small representative of $\mathscr{X}$ so that $\mathrm{Re}\,\lambda > 0$ on $\mathscr{X}\backslash f^{-1}(0)$ whenever a relation of the form $\nabla\theta = \mathrm{i}\lambda\nabla\rho$ holds (see 3.8). Consider also $\varepsilon > 0$ sufficiently small. Now let $\eta > 0$ be sufficiently small to ensure that all the fibers $f^{-1}(t) \subset \mathscr{X}$ cut $M_{\rho,\varepsilon}$ transversally for $|t|^2 \leqslant \eta$. Denote

$$V_\eta := M_{\rho,\varepsilon} \cap \{|f|^2 \leqslant \eta\}$$

the corresponding tubular neighborhood of the binding $N_{\rho,\varepsilon}(f)$ in $M_{\rho,\varepsilon}$. Then clearly θ is a normal angular coordinate on this neighborhood $V_\eta \simeq N_{\rho,\varepsilon}(f) \times \mathbf{D}^2_{\sqrt{\eta}}$. Moreover, each submanifold $(f^{-1}(t) \cap M_{\rho,\varepsilon}) \subset V_\eta$, being a level of the strictly plurisubharmonic function ρ on the complex manifold $f^{-1}(t)\backslash\{0\}$, is a contact submanifold of $M_{\rho,\varepsilon}$. Thus, Lemma 2.5 will imply Theorem 3.9 if we can find a convenient $c > 0$ such that $d\alpha_c$ restricts to a symplectic form on each fiber of θ in $M_{\rho,\varepsilon}\backslash\mathrm{Int}\,V_\eta$ which induces the same orientation on its boundary as the contact form α_c. But this is exactly equivalent to the inequality $d\theta(R_c) > 0$.

Now, there is a $m > 0$ such that $d\theta(R) \geqslant -m$ on the compact set $M_{\rho,\varepsilon}\backslash\mathrm{Int}\,V_\eta$. Put

$$Z_\varepsilon := (M_{\rho,\varepsilon}\backslash\mathrm{Int}\,V_\eta) \cap \{d\theta(R) \leqslant 0\} \quad \text{and} \quad k := \min_{Z_\varepsilon}(|f|^2\|\mathrm{pr}_\xi\nabla\theta\|^2).$$

Since Z_ε is compact, k is well defined. If $k > 0$, then for $c := m/k$ we will always have $d\theta(R_c) > 0$ on $M_{\rho,\varepsilon}\backslash\mathrm{Int}\,V_\eta$ by Lemma 3.12, and the theorem is proved.

Next, assume $k = 0$. This means that there is a $p \in Z_\varepsilon$ so that $\mathrm{pr}_{\xi_p}\nabla\theta(p) = 0$. But this implies that $\nabla\theta(p) = \mathrm{i}\lambda\nabla\rho(p)$ for some $\lambda \in \mathbb{C}$. Our initial choice of the representative $\mathscr{X}$ (see 3.8) implies that $\mathrm{Re}\,\lambda > 0$. But then

$$\begin{aligned} d\theta(p)(R(p)) &= g(\nabla\theta(p), \mathrm{i}\nabla\rho(p)/\|\nabla\rho(p)\|^2) \\ &= \frac{1}{\|\nabla\rho(p)\|^2}\, g(i\lambda\nabla\rho(p), i\nabla\rho(p)) \\ &= \mathrm{Re}\,\lambda > 0, \end{aligned}$$

which is impossible since $p \in Z_\varepsilon$. □

4. Construction of an ubiquitous Milnor open book

In this section we will restrict ourselves to normal surface singularities (see [17] for further references) and we construct for any given 3-manifold M—which is the boundary of some Milnor filling—an open book decomposition, which is isomorphic to a Milnor open book for *any* Milnor filling of M.

4.1. A sufficient condition to be the exceptional part of the divisor of a function

We start with some notations. Let $(\mathscr{S}, 0)$ be the germ of a normal complex analytic surface singularity. Fix a *good* resolution $p : (\Sigma, E) \to (\mathscr{S}, 0)$. Namely, Σ is smooth, p is proper and realizes an isomorphism over $\mathscr{S} - \{0\}$ and finally the set-theoretical fiber $E := p^{-1}(0)$ is a normal crossing divisor in Σ having smooth irreducible components $E_1, \ldots, E_r$. For each i, we denote by g_i the genus, respectively by $v_i := E_i \cdot (E - E_i)$ the valency of E_i seen as a vertex of the dual graph of E. In general, we prefer to fix a Stein representative $\mathscr{S}$ of $(\mathscr{S}, 0)$ and to set $\Sigma = p^{-1}(\mathscr{S})$.

As usual, $|D|$ denotes the support of the divisor D of Σ. Then, for any D, there exists a unique decomposition $D = D_e + D_s$ such that $|D_e| \subset E$ and $\dim(|D_s| \cap E) < 1$. Notice that $f \in m_{\mathscr{S},0}$ defines an isolated singularity at 0 if and only if $\operatorname{div}(f \circ p)_s$ in Σ is reduced. The next theorem guarantees the existence of a function germ f with prescribed exceptional part $\operatorname{div}(f \circ p)_e$ which is resolved by p, that is, such that $\operatorname{div}(f \circ p)$ is a normal crossing divisor. Notice that in such a case, the number n_i of components of $\operatorname{div}(f \circ p)_s$ (all of them smooth) intersecting E_i is determined by $D := \operatorname{div}(f \circ p)_e$: indeed, $n_i = -D \cdot E_i$ for any i, as $\operatorname{div}(f \circ p) \cdot E_i = 0$.

Theorem 4.1. *Let $p : (\Sigma, E) \to (\mathscr{S}, 0)$ be a good resolution of a normal surface singularity $(\mathscr{S}, 0)$ as above. Assume that the effective divisor $D = \sum m_i E_i \neq 0$ satisfies*

$$(D + E + K_\Sigma) \cdot E_i + 2 \leqslant 0 \quad \text{for any } i \in \{1, \ldots, r\}.$$

Then there exists a function $f \in m_{\mathscr{S},0}$, with an isolated singularity at 0, *such that* $\operatorname{div}(f \circ p)$ *is a normal crossing divisor on Σ with* $\operatorname{div}(f \circ p)_e = D$. *Moreover, for each i, the number of intersection points $n_i = \operatorname{div}(f \circ p)_s \cdot E_i$ is strictly positive.*

Proof. We refer to [25] for an introduction to the methods used in this proof.

First notice that

$$n_i = -D \cdot E_i \geqslant v_i + E_i^2 + K_\Sigma \cdot E_i + 2 = v_i + 2g_i. \qquad (*)$$

The right member is strictly positive, except when $E = E_1$ is just a rational curve. But in this case, $D = m_1 E_1$ and $E_1^2 < 0$, proving that $n_1 > 0$. This answers the last statement.

For the existence of f, consider the exact sequence

$$0 \to \mathscr{O}_\Sigma(-D - E) \to \mathscr{O}_\Sigma(-D) \to \mathscr{O}_E(-D) \to 0.$$

Clearly, $H^1(\mathscr{O}_\Sigma(-D - E)) = H^1(\mathscr{O}_\Sigma(K_\Sigma + A))$, where $A := -D - E - K_\Sigma$, with $A \cdot E_i \geqslant 2$ for any i. Therefore, this last group is vanishing by the Laufer–Ramanujam theorem (see [12] (3.2) or [24]; it is also called the 'generalized' Grauert–Riemenschneider vanishing theorem). This shows that $\pi : H^0(\mathscr{O}_\Sigma(-D)) \to H^0(\mathscr{O}_E(-D))$ is onto.

Next, we show that $\mathscr{O}_E(-D) = \omega_E(A|_E)$ is generated by global sections. Firstly, if q is a smooth point of E, then one has to show that there exists a global section s of $\omega_E(A|_E)$ with $s(q)$ nonzero. For this it is enough to verify that $H^0(\omega_E(A|_E)) \to \omega_E(A|_E) \otimes \mathscr{O}_q$ is onto, or $H^1(\omega_E(A|_E - q)) = H^0(\mathscr{O}_E(-A|_E + q))$ is zero. But this last group is vanishing indeed, since $-A + q$ is still negative on each component E_i. Similarly, if $q \in E_i \cap E_j$, with local coordinates (x, y) such that $(E, q) = \{xy = 0\}$, then it is convenient to consider the ideal sheaf I generated by $x + y$. Then $\omega_E(A|_E) \otimes I$ is still locally free and its degree drops by one on both E_i and E_j, hence again $H^1(\omega_E(A|_E) \otimes I) = 0$.

Finally notice that the projective morphism $\varphi : E \to \mathbb{P}^N$ induced by the globally generated $\mathcal{O}_E(-D)$ is finite. Indeed, assume that $\varphi(E_i)$ is a point. Then the general hyperplane section of $\mathbb{P}^N$ misses this point, which implies $D \cdot E_i = 0$ contradicting $(*)$.

Hence, if one takes $s \in H^0(\mathcal{O}_\Sigma(-D))$ with $\pi(s)$ a generic element of $H^0(\mathcal{O}_E(-D))$, then $s = f \circ p$ for some f with the wanted properties. □

Remark 4.2. A divisor D which verifies the hypothesis in this theorem always exists (since the intersection form is negative definite). Moreover, one can also assume that D is fixed by the automorphism group of the weighted dual graph of p.

4.2. Vertical links and horizontal open books in plumbed manifolds

Let Γ be a finite connected weighted (plumbing) graph with r vertices $A_1, \ldots, A_r$. Each vertex A_i is weighted with two integers (g_i, e_i), with $g_i \geqslant 0$. Let $M(\Gamma)$ be the oriented closed 3-manifold obtained from Γ by plumbing (see [16,19]). Briefly, this construction runs as follows. We associate with each vertex A_i an oriented circle bundle $p_i : M_i \to S_i$ with Euler number e_i, where S_i is an oriented compact connected real surface of genus g_i, then we glue these 3-manifolds according to the edges of the graph.

Definition 4.3. Let $M(\Gamma)$ be a plumbed 3-manifold. To any r-tuple $\underline{n} = (n_1, \ldots, n_r)$ of non-negative integers one associates an oriented link $N(\underline{n})$ in $M(\Gamma)$ as follows. For each i, consider n_i generic fibers of the circle bundle $M_i \to S_i$, then $N(\underline{n})$ is their union, where $i \in \{1, \ldots, r\}$. Any such link is called *vertical*.

Any vertical link is naturally oriented by the orientations of the fibers.

Definition 4.4. Let $M(\Gamma)$ be a plumbed manifold. A *horizontal open book* in $M(\Gamma)$ is an open book $(N(\underline{n}), \theta)$ whose binding is a vertical link, whose (open) pages are transversal to the fibers of the bundles $M_i \to S_i$, and which is compatible with the orientations.

Example 4.5. The main example is provided by normal surface singularities and germs of analytic functions (see [19]). Indeed, for any fixed good resolution p, the *dual resolution graph* $\Gamma(p)$ serves as a plumbing graph, and the boundary $M(\mathcal{S})$ is diffeomorphic to the plumbed manifold $M(\Gamma(p))$. Moreover, if $f \in m_{\mathcal{S},0}$ defines an isolated singularity at 0, and $\mathrm{div}(f \circ p)$ is a normal crossing divisor, then the Milnor open book $(N(f), \theta(f))$ defined in $M(\mathcal{S})$ is isomorphic to a horizontal open book $(N(\underline{n}), \theta)$ in $M(\Gamma(p))$, where each n_i is exactly the number of components of the strict transform $\mathrm{div}(f \circ p)_s$ which cut E_i.

Proposition 4.6. *Let Γ be a connected weighted plumbing graph with r vertices whose associated intersection form $I(\Gamma)$ is non-degenerate. Let $\underline{n} = (n_1, \ldots, n_r)$ be a r-tuple of* strictly positive *integers. Then any two horizontal open books in $M(\Gamma)$ with binding $N(\underline{n})$ are isomorphic.*

Proof. In order to prove the proposition, we only have to collect some existing results from the literature (all of them stated in [22]). The sequence of arguments is the following.

Any isomorphism class of open book can be characterized completely by the conjugacy class of the monodromy h acting on the page F of the open book. In the case of plumbed (or graph) manifolds, one can take for the monodromy a quasi-periodical homeomorphism. In [20] and [21] Nielsen associated with such a homeomorphism the (so called) Nielsen graph. From this graph, in general, one cannot recover the conjugacy class of h; but Chaves in [3] completed this graph by some additional decorations—in this way constructing the 'completed Nielsen graph'—and proved that this completed graph characterizes completely the conjugacy class of the quasi-periodical homeomorphism h.

On the other hand, Pichon in [22] (see especially Section 4) describes the combinatorial relationship connecting the completed Nielsen graph and the plumbing graph Γ' of the vertical link $N(\underline{n}) \subset M$. (Here Γ' is obtained from the plumbing graph Γ of M by adding arrowheads corresponding to the link components.) In general, the completed Nielsen graph contains more information, and cannot be recovered from Γ' (see 4.7,b). Nevertheless, if each n_i is strictly positive, then Γ' determines completely the completed Nielsen graph (see Algorithm 4.8 of [22]). This ends our proof as well.

For the convenience of the reader we provide a few more details. Recall that Γ codifies the intersection matrix $I(\Gamma) = \{E_i \cdot E_j\}_{i,j}$ represented in a fixed basis $\{E_i\}_i$. In the proof of 4.1, $(*)$ provides n_i as $-E_i \cdot \left(\sum_j m_j E_j\right)$. These 'multiplicities' $\{m_j\}_j$ constitute a part of the decorations of the Nielsen graphs. Since $I(\Gamma)$ is non-degenerate, they can be computed from the integers $\{n_i\}_i$.

The plumbing construction provides a decomposition $M(\Gamma)=\bigcup_i V_i$ of $M(\Gamma)$, where V_i is an S^1-bundle over $S_i \backslash (v_i$ discs). Let F be the page of an open book with binding $N(\underline{n})$, and set $F_i := F \cap V_i$ for each i. Let r_i be the number of connected components of F_i. It turns out that r_i divides m_i. The point is that, basically, the additional information contained in the Nielsen graph (associated with the monodromy of the open book), compared with Γ', is exactly the collection of the integers $\{r_i\}_i$. It may help to think about this in the following intuitive picture: one may construct a 'covering graph' of Γ' (which is equivalent with the Nielsen graph) by putting r_i vertices above the vertex A_i of Γ' (and there is a similar covering procedure for edges as well). If the integers r_i are larger than one, it can happen that for the same collection of $\{r_i\}_i$'s more covering types can appear. This global twisting data is the additional information in the completed Nielsen graph, but which is superfluous as soon as $r_i = 1$ for all i.

In our case, since $n_i > 0$ for all i, analyzing tubular neighbourhoods of the link components, we easily realize that each F_i is connected, i.e. $r_i = 1$. Therefore, the completed Nielsen graph and Γ' codify the same amount of information. □

Remark 4.7. (a) If some components of $\underline{n}$ are allowed to vanish, then 4.6 fails (see e.g. [18] and [22]).

(b) In [2], the main result 1.2 was established only for Milnor fillable 3-manifolds which are rational homology spheres because the authors were not conscious of 4.6. Instead, they used the fact, easily deducible from results of Stallings and Waldhausen, that in such a 3-manifold an open book is determined up to isotopy by its binding alone.

Corollary 4.8. *Let M be a closed connected oriented 3-manifold which is Milnor fillable. Then there exists an open book (N, θ) in M, which can be completely characterized by the topology of M, such that, for any germ $(\mathscr{S}, 0)$ of normal complex surface with $M \simeq M(\mathscr{S})$, there exists a function $f \in m_{\mathscr{S},0}$ having an isolated singularity at 0 whose Milnor open book $(N(f), \theta(f))$ is isomorphic to (N, θ).*

Proof. First notice that by the work of Neumann [19] the homeomorphism type of M determines the dual weighted graph $\Gamma(p_0)$ of the minimal good resolution $p_0 : (\Sigma_0, E_0) \to (\mathscr{S}, 0)$ of *any* Milnor filling $(\mathscr{S}, 0)$ of M.

Fix any of these fillings. Then choose an exceptional divisor D in Σ_0 as in 4.1. This choice only depends on $\Gamma(p_0)$, that is, on the topology of M. Then apply 4.1 and 4.6. □

Remark 4.9. The notion of *good resolution* used in [19] does not coincide with the one we use here. The difference is that we ask the components E_i to be smooth, while in [19] self-intersections are allowed. Nevertheless, minimal good resolutions exist and are unique for both definitions, and the way to pass from one to the other can be completely described using the associated weighted dual resolution graphs. Hence, Neumann's result quoted before and proved in [19] for his definition implies the analogous result for our definition.

4.3. Proof of Theorem 1.2

Let M be a closed connected oriented 3-manifold. By Corollary 4.8, there exists an open book (N, θ) in M which is isomorphic to a Milnor open book $(N(f), \theta(f))$ in any Milnor filling $M(\mathscr{S})$ of M. Now each of these Milnor open books carries the corresponding contact structure $(M(\mathscr{S}), \xi(\mathscr{S}))$ by Theorem 3.9. Since they are all isomorphic to (N, θ), Theorem 2.6 shows that these contact structures are also all isomorphic. □

Remark 4.10. Notice that the homotopy type (as an *unoriented* 2-plane field) of a Milnor fillable contact structure is well defined (it is invariant up to isotopy by the group $Diff^+(M)$ of self-diffeomorphisms of M which preserve the orientation). Firstly, two oriented plane fields which are positively transversal to the oriented fibers of the plumbing structure are homotopic (as one can see by taking an auxiliary Riemannian metric and rotating at each point one plane into the other by the *unique* shortest path). Secondly, the plumbing decomposition (with *unoriented fibers*) which corresponds to the minimal good resolution is unique up to isotopy, which shows that it is invariant by $Diff^+(M)$, up to isotopy (see [23]).

Using this last invariance and the results of [2], we see that, on a Milnor fillable three-dimensional rational homology sphere, all Milnor fillable contact structures are *isotopic*, not only contactomorphic. *Is this true in general*?

Acknowledgements

We are grateful to János Kollár for the simplification of the proof of Proposition 4.1 and to Françoise Michel and Anne Pichon for conversations that led to the proof of Proposition 4.6. We would also like to thank Michel Boileau, Etienne Ghys, Emmanuel Giroux and Bernard Teissier for having kindly answered our questions.

References

[1] F.A. Bogomolov, B. de Oliveira, Stein Small Deformations of Strictly Pseudoconvex Surfaces, Contemp. Math. 207 (1997) 25–41.
[2] C. Caubel, P. Popescu-Pampu, On the contact boundaries of normal surface singularities, C.R. Acad. Sci. Paris, Ser. I 339 (2004) 43–48.
[3] N. Chaves, Variétés graphées fibrées sur le cercle et difféomorphismes quasi-finis de surfaces, Thèse, Univ. de Genève, 1996.

[4] V. Colin, E. Giroux, K. Honda, On the coarse classification of tight contact structures, Proceedings of the 2002 Georgia International Topology Conference, Proc. Symp. Pure Maths. 71 (2003) 109–120.
[5] A.H. Durfee, Neighborhoods of algebraic sets, Trans. Am. Math. Soc. 276 (1983) 517–530.
[6] Y. Eliashberg, N. Mishachev, Introduction to the *h*-principle, Graduate Studies in Mathematics, vol. 48, American Mathematical Society, Providence, RI, 2002.
[7] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, Proceedings of the ICM 2002, vol. II, Higher Ed. Press, Beijing, pp. 405–414.
[8] E. Giroux, Contact structures and symplectic fibrations over the circle, Notes of the summer school Holomorphic curves and contact topology, Berder, June 2003, available at: ⟨http://www-fourier.ujf-grenoble.fr/~eferrand/berder.html⟩.
[9] H. Grauert, Über Modifikationen und exceptionelle analytische Mengen, Math. Ann. 146 (1962) 331–368.
[10] H. Hamm, Lokale topologische Eigenschaften komplexer Räume, Math. Ann. 191 (1971) 235–252.
[12] H. Laufer, On rational singularities, Am. J. Math. 94 (1972) 597–608.
[13] D.T. Lê, B. Teissier, Limites d'espaces tangents en géométrie analytique, Comment. Math. Helv. 63 (1988) 540–578.
[14] E.N. Looijenga, Isolated singular points on complete intersections, London Mathematical Society Lecture Notes Series, vol. 77, Cambridge University Press, Cambridge, MA, 1984.
[15] J. Milnor, Singular Points of Complex Hypersurfaces, Princeton University Press, Princeton, NJ, 1968.
[16] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Inst. Hautes Études Sci. Publ. Math. 9 (1961) 5–22.
[17] A. Némethi, Five lectures on normal surface singularities, Lectures delivered at the Summer School in Low Dimensional Topology, Budapest, Hungary, 1998; Bolyai Society Mathematical Studies, Low Dimensional Topology, vol. 8, 1999, pp. 269–351.
[18] A. Némethi, Resolution graphs of some surface singularities. I. Cyclic coverings, in: Singularities in Algebraic and Analytic Geometry, San Antonio, TX, 1999, pp. 89–128, Contemp. Math. 266 (2000).
[19] W. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Am. Math. Soc. 268 (2) (1981) 299–344.
[20] J. Nielsen, Die Struktur periodischer transformationen von Flächen, Mat.-Fys. Medd. Danske Vid. Selsk. 15 (1937), Collected Papers 2, Birkhäuser, Basel, 1986.
[21] J. Nielsen, Surface transformation classes of algebraically finite type, Mat.-Fys. Medd. Danske Vid. Selsk. 21 (1944), Collected Papers 2, Birkhäuser, Basel, 1986.
[22] A. Pichon, Fibrations sur le cercle et surfaces complexes, Ann. Inst. Fourier Grenoble 51 (2) (2001) 337–374.
[23] P. Popescu-Pampu, The geometry of continued fractions and the topology of surface singularities, Singularities-Sapporo 2004, Advanced Studies in Pure Mathematics, Kinokuniya, Tokyo, 2006, to appear.
[24] C.P. Ramanujam, Remarks on the Kodaira vanishing theorem, J. Indian Math. Soc. 36 (1972) 41–51.
[25] M. Reid, Chapters on algebraic surfaces, in: J. Kollár (Ed.), Complex Algebraic Geometry, American Mathematical Society, Providence, RI, 1997, pp. 3–159.
[26] J. Scherk, CR structures on the link of an isolated singular point, Can. Math. Soc. Conf. Proc. 6 (1986) 397–403.
[27] I. Ustilovsky, Infinitely many contact structures on S^{4m+1}, Internat. Math. Res. Notices 14 (1999) 781–792.
[28] O. Van Koert, K. Niederkrüger, Open book decompositions for contact structures on Brieskorn manifolds, Proc. Am. Math. Soc. 133 (12) (2005) 3679–3686.
[29] A.N. Varchenko, Contact structures and isolated singularities, Moscow Univ. Math. Bull. 35 (2) (1980) 18–22.