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Surgery formula for Seiberg–Witten invariants of negative definite plumbed 3-manifolds

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Abstract. We derive a cut-and-paste surgery formula of Seiberg–Witten invariants for negative definite plumbed rational homology 3-spheres. It is similar to (and motivated by) Okuma's recursion formula [27], 4.5, targeting analytic invariants of splice-quotient singularities. Combining the two formulas automatically provides a proof of the equivariant version [11], 5.2(b), of the Seiberg–Witten invariant conjecture [18] for these singularities.

1. Introduction

Problem 5 of the review article [30] of Ozsváth and Szabó is to develop cut-and-paste techniques for calculating the Heegaard Floer homology of 3-manifolds. In this article we obtain a possible answer at the level of the Seiberg–Witten invariant (i.e. at the level of the normalized Euler characteristic of the Heegaard Floer homology): we provide the cut-and-paste surgery formula (1.0.3) for the Seiberg–Witten invariants of plumbed rational homology 3-spheres associated with negative definite plumbing graphs. In order to state it, we fix some notations (for more details, see §3).

For any graph G, let $\mathcal{V}(G)$ denote its set of vertices. Let $|S|$ denote the size of the finite set S. Thus, $|\mathcal{V}(G)|$ is the number of vertices of G.

Let Γ be a connected plumbing graph. Each vertex $w \in \mathcal{V}(\Gamma)$ is decorated by an integer b_w. Let $\tilde{X}(\Gamma)$ be the 4-manifold with boundary obtained by plumbing from Γ, which we briefly recall. The manifold $\tilde{X}(\Gamma)$ is a tubular neighbourhood of oriented 2-spheres E_w associated with the vertices w of the graph. For every two adjacent vertices, their 2-spheres intersect transversally at one point; beside these, the 2-spheres do not intersect each other. The number b_w is the Euler number of the normal bundle of the 2-sphere of the vertex w.

The manifold $\tilde{X}(\Gamma)$ admits a canonical Spin^c structure $\widetilde{\sigma_{\mathrm{can}}}$, see (3.3.1) for its characterization.

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Set $\Sigma := \partial\tilde{X}(\Gamma)$. We assume that $H_1(\Sigma;\mathbb{Q}) = 0$, or equivalently that Γ is a tree.

Set $L := H_2\big(\tilde{X}(\Gamma);\mathbb{Z}\big)$ and $L' := H^2\big(\tilde{X}(\Gamma);\mathbb{Z}\big)$. These groups are free with bases the classes E_w of the 2-spheres and their duals E_w^*, respectively.

The graph Γ is *negative definite* if the intersection form on L is negative definite. If this is the case then the canonical map $L \to L'$ is an embedding, which is an isomorphism over $\mathbb{Q}$, thus the intersection form extends to L'. We shall write $(\cdot\,,\cdot)$ for the intersection form and $x^2 := (x,x)$ for any $x \in L'$.

For any Spin^c structure σ, let $c_1(\sigma) \in L'$ denote its first Chern class.

Finally, for any $\sigma \in \mathrm{Spin}^c(\Sigma)$ and $v \in \mathscr{V}(\Gamma)$, let $\mathscr{H}_{\sigma,v}$ be the rational function defined in 3.5, which is a Weil-type twisted zeta function. We write $\mathscr{H}^{\mathrm{pol}}_{\sigma,v}$ for its *polynomial part* which is the unique polynomial for which $\mathscr{H}_{\sigma,v} - \mathscr{H}^{\mathrm{pol}}_{\sigma,v}$ has negative degree (i.e. it is either 0 or the degree of the numerator is less than the degree of the denominator).

Theorem 1.0.1. *Let Γ be a connected negative definite plumbing graph of a rational homology 3-sphere Σ. Let v be a vertex of Γ, and let Γ_i be the components of $\Gamma\backslash v$. Let $\tilde{\sigma}$ be a* Spin^c *structure of $\tilde{X}(\Gamma)$ satisfying*

$$-1 < \left(\frac{c_1(\tilde{\sigma}) - c_1(\widetilde{\sigma_{\mathrm{can}}})}{2}, E_v^*\right) \leqq 0. \tag{1.0.2}$$

Let σ, $\tilde{\sigma}_i$ and σ_i denote the restriction of $\tilde{\sigma}$ to Σ, $\tilde{X}(\Gamma_i)$ and $\Sigma_i := \partial\tilde{X}(\Gamma_i)$, respectively. Then

$$\begin{aligned} &\mathrm{sw}_\sigma(\Sigma) + \frac{c_1(\tilde{\sigma})^2 + |\mathscr{V}(\Gamma)|}{8} \\ &\qquad = -\mathscr{H}^{\mathrm{pol}}_{\sigma,v}(1) + \sum_i \left(\mathrm{sw}_{\sigma_i}(\Sigma_i) + \frac{c_1(\tilde{\sigma}_i)^2 + |\mathscr{V}(\Gamma_i)|}{8}\right). \end{aligned} \tag{1.0.3}$$

Remark 1.0.4. The Spin^c structure σ does not uniquely determine $\tilde{\sigma}$ and its restriction $\tilde{\sigma}_i$ via (1.0.2). Nevertheless, the Spin^c structure σ_i is independent of the choice of $\tilde{\sigma}$; it depends only on σ.

Remark 1.0.5. Notice that this formula differs from those obtained from surgery exact triangles (of different versions) of Floer homologies (see e.g. [29]): the surgery exact triangles involve three different 3-manifolds, while our formula only connects the plumbed 3-manifolds associated with Γ and $\Gamma\backslash v$ (and another type of invariant, namely $\mathscr{H}_{\sigma,v}$). Moreover, in general, the surgery exact triangles mix several Spin^c structures (involving all the extensions $\tilde{\sigma}$), while our formula involves only one extension $\tilde{\sigma}$ and one induced pair $(\tilde{\sigma}_i, \sigma_i)$ for any fixed σ.

The proof uses the fact (see [26], Theorem 2.4, recalled here in (3.4.1)) that the Seiberg–Witten invariant of Σ is a linear combination of the Reidemeister–Turaev torsion $\mathcal{T}$ ([35]) and the Casson–Walker invariant λ, together with explicit formulas for these invariants.

In particular, the formula above is the consequence of additivity formulas for the invariants $c_1(\tilde{\sigma})^2 + |\mathscr{V}(\Gamma)|$, $\mathcal{T}$ and λ, stated in (5.0.2), (5.0.4) and (5.0.5), which are interesting for their own sake as well.

In §8 we exemplify 1.0.1 for Seifert manifolds and surgery manifolds $S^3_{-d}(K)$. There we emphasize the arithmetical nature of $\mathscr{H}_{\sigma,v}$, too.

Any negative definite plumbed 3-manifold appears as the link of a complex surface singularity. For some singularity links, the Taylor expansion of $\mathscr{H}_{\sigma,v}$ at the origin appears as the Hilbert (Poincaré) series of a certain graded $\mathbb{C}$-algebra. In this way, $\mathscr{H}^{\mathrm{pol}}_{\sigma,v}(1)$ can be related with analytic invariants of the singularity. For applications of 1.0.1 in singularity theory, see §2 and 8.2.

2. Application in singularity theory

2.1. Seiberg–Witten invariant conjecture. Let (X, o) be an isolated complex analytic normal surface singularity whose link Σ is a rational homology sphere. Let $\pi : \tilde{X} \to X$ be a good resolution with exceptional set E (with irreducible components $\{E_w\}_w$), and Γ its dual resolution graph (for details see e.g. [14], §2.2). Then (the underlying C^∞ manifold of) $\tilde{X}$ is the plumbed 4-manifold $\tilde{X}(\Gamma)$ (for which in the sequel we will use all the above notations). The intersection form on L is automatically negative definite.

The group L can also be regarded as the group of integral cycles (divisors) of type $l = \sum_w m_w E_w$ in $\tilde{X}$ with $m_w \in \mathbb{Z}$. As customary, we denote by $\mathscr{O}_{\tilde{X}}(l)$ the line bundle associated with l. This map $l \mapsto \mathscr{O}_{\tilde{X}}(l)$ extends uniquely to a group homomorphism $L' \to \mathrm{Pic}(\tilde{X})$, denoted similarly by $l' \mapsto \mathscr{O}_{\tilde{X}}(l')$, such that the Chern class (multidegree) satisfies $c_1\big(\mathscr{O}_{\tilde{X}}(l')\big) = l'$ (see [11], 3.4–3.6).

As usual, $h^1(\mathscr{L})$ denotes $\dim_{\mathbb{C}} H^1(\tilde{X}, \mathscr{L})$. In this way, the *geometric genus* is $p_g := h^1(\mathscr{O}_{\tilde{X}})$. More generally, for the special set of representatives

$$R := \left\{ \sum_w r_w E_w \in L' : -1 < r_w \leqq 0 \right\} \subset L'$$

of the classes L'/L, we get the *equivariant geometric genera* $\big\{h^1\big(\mathscr{O}_{\tilde{X}}(l')\big)\big\}_{l' \in R}$ of (X, o) (the $L'/L = H_1(\Sigma; \mathbb{Z})$ eigen-decomposition of the geometric genus of the universal abelian cover of (X, o), see [11], 3.7, and [27], 2.2(3)). They are subtle analytic invariants of (X, o), which guide crucial analytic aspects (e.g. equisingular deformations). In general, they are not topological; nevertheless, in [11], 5.2(b), the second author formulated essentially the following conjecture, which predicts that in special cases, these invariants can be recovered from the *link* Σ:

Conjecture 2.1.1 (Seiberg–Witten invariant conjecture [11]). *Set L_e for the effective integral cycles, i.e. $L_e := \left\{ \sum_w m_w E_w : m_w \geqq 0 \text{ for all } w \right\}$. Set $R + L_e := \bigcup_{l' \in R} (l' + L_e) \subset L'$.*

If the analytic structure of (X,o) *is* '*nice*', *then for all* $l'\in R+L_e$ *one has*

$$-h^1\big(\mathscr{O}_{\tilde X}(l')\big)=\mathrm{sw}_{[l']*\sigma_{\mathrm{can}}}(\Sigma)+\frac{\big(c_1(\widetilde{\sigma_{\mathrm{can}}})+2l'\big)^2+|\mathscr{V}(\Gamma)|}{8}. \tag{2.1.2}$$

(*For the definition of the* Spin$^{\mathrm{c}}$ *structure* $[l']*\sigma_{\mathrm{can}}$ *of* Σ, *see* §3.3.)

Remark 2.1.3. (1) It is part of the conjecture to clarify the meaning of 'nice'. In the original version [11], [18] the conjecture was formulated for all $\mathbb{Q}$-Gorenstein singularities, but counterexamples are given in [10], §4. On the other hand, the conjecture holds for all rational singularities ([11], [17], see also [16]), and, in fact, here we shall prove it for all splice-quotient singularities, see 2.2.4. Restricted to the case of the canonical Spin$^{\mathrm{c}}$ structure, it was verified for elliptic Gorenstein singularities (by combining [13] and [16]), singularities with good $\mathbb{C}^*$ action ([19]), and suspension hypersurface singularities defined by $f(x,y)+z^n=0$ with f irreducible ([20]). For a review of related problems, see [14], [17]. For related results, see [1], [2], [3], [7], [6], [23], [25].

(2) As a byproduct of the main theorem 1.0.1, in Theorem 2.2.1 we provide a criterion which characterizes the singularities satisfying (2.1.2).

(3) The special case of the canonical Spin$^{\mathrm{c}}$ structure was conjectured in [18]. It generalizes the Casson invariant conjecture of Neumann and Wahl formulated for any isolated complete intersection with *integral* homology sphere link [23].

(4) In fact, (2.1.2) essentially consists of (only) $|H_1(\Sigma;\mathbb{Z})|$ different identities. The reason is that the expression

$$h^1\big(\mathscr{O}_{\tilde X}(l')\big)+\frac{\big(c_1(\widetilde{\sigma_{\mathrm{can}}})+2l'\big)^2+|\mathscr{V}(\Gamma)|}{8}$$

depends only on $[l']\in H_1(\Sigma;\mathbb{Z})$ for $l'\in R+L_e$ by [11], 5.3(c). Therefore, it is enough to verify the identity (2.1.2), say, for all $l'\in R$.

2.2. Application. Using the main theorem 1.0.1, the above Seiberg–Witten invariant conjecture 2.1.1 may be transformed into an additivity property of analytic invariants $h^1(\mathscr{L})$. In order to state it, we need the following notation.

For a fixed vertex v of the graph Γ, let Γ_i be the components of $\Gamma\backslash v$, and let $\tilde X_i$ be a small tubular neighbourhood of $E_i:=\bigcup_{w\in\mathscr{V}(\Gamma_i)}E_w$ in $\tilde X$. Let (X_i,o) be the normal surface singularity (with dual resolution graph Γ_i) obtained by collapsing the curve $E_i\subset\tilde X_i$ to a point.

Theorem 2.2.1. *Consider a family of singularities which satisfy the next property*: *for any non-rational* (X,o) *in the family*, *there exists at least one vertex* v (*called* splitting *vertex*) *in its* (*minimal*) *resolution graph* Γ *such that all the singularities* (X_i,o) *are in the family.*

Then, *for such a family*, *the validity of* 2.1.1 *for all the members of the family is equivalent to the next additivity property*: *every non-rational singularity* (X,o) *in the family has a*

splitting vertex v satisfying

$$h^1\big(\mathscr{O}_{\tilde{X}}(l')\big) = \mathscr{H}^{\mathrm{pol}}_{\sigma,v}(1) + \sum_i h^1\big(\mathscr{O}_{\tilde{X}_i}\big(R_i(l')\big)\big) \quad \textit{for } l' \in R, \tag{2.2.2}$$

where R_i is the natural cohomological restriction defined in 3.6.1(2).

Note that the above additivity property (2.2.2) does not involve any part of Seiberg–Witten theory.

Remark 2.2.3. For fixed (X, o) and v, the validity of (2.2.2) for all $l' \in R$ implies its validity for all $l' \in R + \sum_{w \neq v} \mathbb{Z}_{\geqq 0} E_w$.

The reason is that $[l'] = \left[l' + \sum_{w \neq v} m_w E_w\right]$ and $[R_i(l')] = \left[R_i\left(l' + \sum_{w \neq v} m_w E_w\right)\right]$ for any integers m_w, hence Remark 2.1.3(4) and equation (5.0.2) applies to show the desired implication.

For splice-quotient singularities, the additivity formula (2.2.2) was proved by T. Okuma in [27]. In fact, Okuma's formula gave the idea of the existence of the set of purely topological identities (1.0.3), and was the starting point of our investigation.

As an application, we verify Conjecture 2.1.1 for splice-quotients. These singularities were introduced recently by Neumann and Wahl [24], [25]. Since their definition is rather involved, we omit it. The interested reader may consult [24], [25], [27].

Splice-quotients include rational and minimal elliptic singularities (see [28]), and also the singularities which admit a good $\mathbb{C}^*$ action. For splice-quotient singularities and for the canonical Spin^c structure, the conjecture was verified in [21], [22] (for some sporadic cases, see also [34]). Here, as a byproduct, we get the general case:

Corollary 2.2.4. *Conjecture* 2.1.1 *is true for any splice-quotient singularity.*

Theorem 2.2.1 and Corollary 2.2.4 are proved in §7.

3. Preliminaries and notations

3.1. Notations regarding the plumbing representation. In the sequel we fix a negative definite tree Γ as in §1. Notice that L' can be identified with the dual lattice of L. It is generated by the elements E_w^*, where $(E_w^*, E_u) = \delta_{wu}$ is the Kronecker delta function. The matrix I of the inclusion $L \hookrightarrow L'$ in the basis $\{E_w\}_w$ of L and the basis $\{E_w^*\}_w$ of L' is exactly the matrix of the intersection form in the basis $\{E_w\}_w$, namely, $I_{ww} = b_w$ for all w, and for $u \neq w$, we have $I_{uw} = 1$ if u and w are adjacent, and $I_{uw} = 0$ otherwise.

By duality, $L' \cong H_2\big(\tilde{X}(\Gamma), \Sigma; \mathbb{Z}\big)$, and $L'/L \cong H_1(\Sigma; \mathbb{Z})$. We denote the latter group by H. Let $|H|$ and $\hat{H}$ denote its order and Pontrjagin dual $\mathrm{Hom}(H, \mathbb{C}^*)$, respectively. Sometimes we write $d = \det(\Gamma)$ for $\det(-I) = |H|$. We define

$$a_{uw} := -|H| \cdot (E_u^*, E_w^*) = -|H| \cdot (I^{-1})_{uw}. \tag{3.1.1}$$

Notice that every a_{uw} is a *positive* integer.

For any $u \in \mathscr{V}(\Gamma)$ we write δ_u for the degree of u in Γ and we set:

$$\alpha_u := \sum_{w \in \mathscr{V}(\Gamma)} (\delta_w - 2) a_{uw}, \tag{3.1.2}$$

$$\beta_u := \sum_{w \in \mathscr{V}(\Gamma)} (\delta_w - 2) a_{uw}^2. \tag{3.1.3}$$

Next we consider some topological/combinatorial invariants of Σ and Γ.

3.2. The Casson–Walker invariant. Let $\lambda(\Sigma)$ denote the Casson–Walker invariant of Σ, normalized as in [9], (4.7). Then from [33] one has:

$$-24\frac{\lambda(\Sigma)}{|H|} = \sum_{w \in \mathscr{V}(\Gamma)} b_w + 3|\mathscr{V}(\Gamma)| + \frac{1}{|H|} \cdot \sum_{w \in \mathscr{V}(\Gamma)} (\delta_w - 2) a_{ww}. \tag{3.2.1}$$

3.3. Spin^c structures. As it is well-known, see e.g. [8], (2.4.16), the set of Spin^c structures is an H^2 torsor for any manifold admitting a Spin^c structure. Let $*$ denote the action of H^2 on the set of Spin^c structures. Recall that for any $h \in H^2$ and Spin^c structure σ, the action and the Chern class interact as $c_1(h * \sigma) = c_1(\sigma) + 2h$.

For our plumbed manifold $\tilde{X}(\Gamma)$, there is a canonical Spin^c structure $\widetilde{\sigma_{\mathrm{can}}}$, whose Chern class is characterized by (see [18], 2.7–2.9)

$$\big(c_1(\widetilde{\sigma_{\mathrm{can}}}), E_w\big) = b_w + 2 \quad \text{for all } w \in \mathscr{V}(\Gamma). \tag{3.3.1}$$

Hence, there is a bijection between L' and the set of Spin^c structures of $\tilde{X}(\Gamma)$ which assigns $l' \in L'$ to $l' * \widetilde{\sigma_{\mathrm{can}}}$.

Similarly, the set of Spin^c structures of the boundary Σ is an H torsor. The restriction of Spin^c structures commute with the action via the canonical map $L' \to H$. Since this homomorphism is surjective, every Spin^c structure of Σ extends to $\tilde{X}(\Gamma)$.

By definition, the *canonical* Spin^c structure σ_{can} on Σ is the restriction of the canonical Spin^c structure $\widetilde{\sigma_{\mathrm{can}}}$ of $\tilde{X}(\Gamma)$.

3.4. The Reidemeister–Turaev torsion and the Seiberg–Witten invariant. For any $\sigma \in \mathrm{Spin}^c(\Sigma)$, we consider the *Reidemeister–Turaev torsion* $\mathcal{T}_\sigma = \sum_{h \in H} \mathcal{T}_\sigma(h) h \in \mathbb{Q}[H]$ from [35]. We will write $\mathcal{T}_\sigma(\Sigma)$ for $\mathcal{T}_\sigma(0)$. Then, by [26], Theorem 2.4, the *Seiberg–Witten invariant* $\mathrm{sw}_\sigma(\Sigma)$ of Σ associated with $\sigma \in \mathrm{Spin}^c(\Sigma)$ equals (note our sign convention):

$$\mathrm{sw}_\sigma(\Sigma) = \frac{\lambda(\Sigma)}{|H|} - \mathcal{T}_\sigma(\Sigma). \tag{3.4.1}$$

By [18], 3.8, 5.7, $\mathcal{T}_\sigma(\Sigma)$ can be determined from the graph Γ via Fourier transform as follows.

First, for any $\rho \in \hat{H}$ and fixed vertex $u \in \mathscr{V}(\Gamma)$, we define a rational function in t:

$$P_{\rho,u}(t) := \prod_{w \in \mathscr{V}(\Gamma)} \big(1 - \rho([E_w^*]) t^{a_{wu}}\big)^{\delta_w - 2}, \tag{3.4.2}$$

where $[E_w^*]$ is the class of E_w^* in $H = L'/L$. Take also $h_\sigma \in H$ such that $h_\sigma * \sigma_{\text{can}} = \sigma$. Next, for any non-trivial character $\rho \in \hat{H}\backslash\{1\}$, find a vertex $u_\rho \in \mathscr{V}(\Gamma)$ such that either $\rho([E_{u_\rho}^*]) \neq 1$, or u_ρ has an adjacent vertex u with $\rho([E_u^*]) \neq 1$. Then the Fourier transform of $\mathcal{T}$ is

$$\widehat{\mathcal{T}_\sigma}(\rho) = \rho(h_\sigma)^{-1} \cdot \lim_{t\to 1} P_{\rho,u_\rho}(t) \quad (\rho \neq 1). \tag{3.4.3}$$

In the sequel, this limit will be denoted simply by $P_{\rho,u_\rho}(1)$. Recall that $\widehat{\mathcal{T}_\sigma}(1) = 0$. Therefore:

$$\mathcal{T}_\sigma(\Sigma) = \frac{1}{|H|} \cdot \sum_{\rho\in\hat{H}\backslash\{1\}} \widehat{\mathcal{T}_\sigma}(\rho). \tag{3.4.4}$$

If $|H| = 1$ then $\mathcal{T}_\sigma(\Sigma) = 0$ for the unique Spin^c structure σ, hence $\text{sw}_\sigma(\Sigma) = \lambda(\Sigma)$.

3.5. The rational function $\mathscr{H}_{\sigma,u}(t)$. For any $\sigma \in \text{Spin}^c(\Sigma)$ and $u \in \mathscr{V}(\Gamma)$ one defines

$$\mathscr{H}_{\sigma,u}(t) := \frac{1}{|H|} \cdot \sum_{\rho\in\hat{H}} \rho(h_\sigma)^{-1} \cdot P_{\rho,u}(t), \quad \text{where } h_\sigma * \sigma_{\text{can}} = \sigma.$$

3.6. Invariants associated with the distinguished vertex v. Recall that for a fixed vertex v of Γ, the components of $\Gamma\backslash v$ are the graphs Γ_i. Let v_i denote the unique vertex of Γ_i which is adjacent to v in Γ.

We indicate by a subscript i when we use invariants of Γ_i instead of Γ. For example, we write $d_i = \det\Gamma_i$, $H_i = H_1(\Sigma_i;\mathbb{Z})$, L_i, $a_{uw,i}$ and so on.

We regard L_i as a sublattice of L via the natural inclusion

$$H_2\big(\tilde{X}(\Gamma_i);\mathbb{Z}\big) \hookrightarrow H_2\big(\tilde{X}(\Gamma);\mathbb{Z}\big).$$

Hence, for any $w \in \mathscr{V}(\Gamma_i)$, we have $E_{w,i} = E_w$.

Definition 3.6.1. (1) Consider the setup of §1. For a Spin^c structure σ of Σ, its *restriction* σ_i to Σ_i is defined to be the restriction of any extension $\tilde{\sigma} \in \text{Spin}^c\big(\tilde{X}(\Gamma)\big)$ of σ satisfying (1.0.2) to the submanifold Σ_i. In other words, $\tilde{\sigma} = l' * \widetilde{\sigma_{\text{can}}}$ for some $l' \in L'$ with $[l'] * \sigma_{\text{can}} = \sigma$ and

$$-1 < (l', E_v^*) \leqq 0. \tag{3.6.2}$$

(2) The *restriction* $R_i : L' \to L_i'$ is the homomorphism induced by the inclusion $\tilde{X}(\Gamma_i) \hookrightarrow \tilde{X}(\Gamma)$ on second cohomology groups. In other words, $R_i(E_w^*) = E_{w,i}^*$ if $w \in \mathscr{V}(\Gamma_i)$, and $R_i(E_w^*) = 0$ otherwise. Therefore, for $l' = \sum_w r_w E_w = \sum_w s_w E_w^*$, one has

$$R_i(l') = \sum_{w\in\mathscr{V}(\Gamma_i)} s_w E_{w,i}^* = r_v E_{v_i,i}^* + \sum_{w\in\mathscr{V}(\Gamma_i)} r_w E_w. \tag{3.6.3}$$

Since $R_i(l')$ is characterized by $\big(R_i(l'), E_w\big) = (l', E_w)$ for all $w \in \mathscr{V}(\Gamma_i)$, the last equality in (3.6.3) follows. One can verify that $\sigma_i \in \text{Spin}^c(\Sigma_i)$ is independent of the choice

of $\tilde{\sigma}$ thanks to (3.6.2). Since the canonical Spin^c structure of $\tilde{X}(\Gamma)$ restricts to the canonical Spin^c structure of $\tilde{X}(\Gamma_i)$, the restriction of the canonical Spin^c structure of Σ to Σ_i is the canonical one. Moreover, the restriction of $\sigma = [l'] * \sigma_{\mathrm{can}}$ is $[R_i(l')] * \sigma_{\mathrm{can},i}$ provided that $r_v := (l', E_v^*) \in (-1, 0]$. The number r_v depends only on σ and not on the choice of l'.

3.7. Pseudo-characters. We will need to extend the expression (3.4.2) for an arbitrary map $\psi : \mathscr{V}(\Gamma) \to \mathbb{C}^*$ by

$$P_{\psi,v}(t) := \prod_{w \in \mathscr{V}(\Gamma)} \big(1 - \psi(w) t^{a_{wv}}\big)^{\delta_w - 2}. \tag{3.7.1}$$

For such a map ψ and vertex $w \in \mathscr{V}(\Gamma)$, we define

$$\mathrm{def}_w(\psi) := \psi(w)^{b_w} \prod_{j=1}^{\delta_w} \psi\big(w(j)\big),$$

where $\{w(j)\}_j$ are the vertices of Γ adjacent to w. The map ψ is called a *pseudo-character* (associated with the vertex v) if $\mathrm{def}_w(\psi) = 1$ for all $w \neq v$. Their collection will be denoted by $\tilde{H}$. We set $\mathrm{def}(\psi) := \mathrm{def}_v(\psi)$. Notice that pseudo-characters ψ with $\mathrm{def}(\psi) = 1$ are exactly the characters of H via the correspondence $\psi(w) = \psi([E_w^*])$. In fact, ψ can be regarded as a character on L' (which does not necessarily descend to H): any $\psi \in \tilde{H}$ gives a morphism $L' \to \mathbb{C}^*$ defined by

$$\psi\left(\sum_w m_w E_w^*\right) := \prod_w \psi(w)^{m_w}.$$

3.8. Notations regarding rational functions. (1) We write any rational function R as $R^{\mathrm{pol}} + R^{<0}$, where R^{pol} is a polynomial and $R^{<0}$ is a rational function with negative degree. For R without pole at 0 we shall refine it further: one writes $R^{<0}$ in a unique way as a finite sum

$$R^{<0}(t) = \sum_{\alpha \neq 0} (\mathrm{L}_\alpha R)(t),$$

$$\text{where } (\mathrm{L}_\alpha R)(t) = \sum_{k>0} \frac{a_{\alpha,k}}{(1-\alpha t)^k} \quad (\alpha \in \mathbb{C}^*, a_{\alpha,k} \in \mathbb{C}).$$

(2) For any rational function $R(t)$ with Laurent expansion $\sum_{k \geqq k_0} a_k (t-1)^k$ at $t = 1$, we write $\mathrm{coef}_1^0 R(t)$ for the coefficient a_0. Notice that if 1 is not a pole of R then $\mathrm{coef}_1^0 R(t) = R(1)$.

The next identities are elementary and their proofs are left to the reader.

Lemma 3.8.1. *For any $0 \leqq q < d$ one has*

$$\frac{1}{d} \sum_{\alpha^d = \varepsilon} \frac{\alpha^{-q}}{1 - \alpha t} = \frac{t^q}{1 - \varepsilon t^d}, \tag{3.8.2}$$

$$\mathrm{coef}_1^0 \left(\frac{1}{d} \sum_{\alpha^d = 1} \frac{\alpha^{-q}}{1 - \alpha t} \right) = \frac{d - 1 - 2q}{2d}, \tag{3.8.3}$$

$$\frac{1}{d}\sum_{\alpha^d=1}\frac{\alpha^{-q}}{(1-\alpha t)^2}=\frac{dt^q}{(1-t^d)^2}-\frac{(d-q-1)t^q}{1-t^d}, \tag{3.8.4}$$

$$\mathrm{coef}_1^0\left(\frac{1}{d}\sum_{\alpha^d=1}\frac{\alpha^{-q}}{(1-\alpha t)^2}\right)=-\frac{(d-1)(d-5)}{12d}-\frac{q^2+2q-qd}{2d}. \tag{3.8.5}$$

4. Identities about determinants and restrictions

Our calculation will extensively use the following general properties of graph-determinants.

Lemma 4.0.1. (a) *Consider two vertices $u, w \in \mathscr{V}(\Gamma)$ of Γ. Let $\Gamma\backslash\overline{uw}$ be the subgraph of Γ obtained by deleting the path connecting u and w (including u and w). Then*

$$a_{uw}=\det(\Gamma\backslash\overline{uw}). \tag{4.0.2}$$

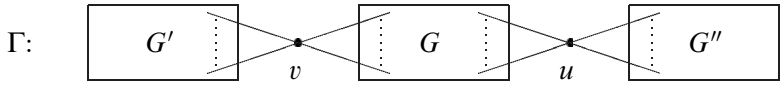
(b) *For every component Γ_i of $\Gamma\backslash v$ and vertex w of Γ_i*

$$a_{vw}=a_{v_iw,i}\cdot\det(\Gamma\backslash v\backslash\Gamma_i). \tag{4.0.3}$$

(c) *For any $u \in \mathscr{V}(\Gamma)$ one has*

$$a_{uu}\cdot\prod_{w\in\mathscr{V}(\Gamma)}a_{uw}^{\delta_w-2}=1. \tag{4.0.4}$$

(d) *Consider a decomposition of Γ as follows*:



Above, the subgraphs G', G and G'' can be empty. If G is empty then v and u is connected by a single edge. The vertices v and u are not allowed to be the same.

Then (*with the convention* $\det(\emptyset)=1$), *one has*:

$$\begin{aligned}\det(\Gamma)\cdot\det(G)=&\det(G\cup G'\cup v)\cdot\det(G\cup G''\cup u)\\&-\det(G')\cdot\det(G'')\cdot\det(G\backslash\overline{uv})^2.\end{aligned} \tag{4.0.5}$$

(*Here $G\cup G'\cup v$ and $G\cup G''\cup u$ also contain the edges adjacent to v and u, respectively.*)

Proof. Equation (4.0.2) is proved in [5], (20.2). Equation (4.0.3) follows from (4.0.2) and by noting that the determinant of graphs is multiplicative over disjoint union of graphs.

Statement (c) immediately follows from (4.0.2) and (4.0.4) by an easy induction on the number of vertices of the graph.

The claim (d) is an exercise on graph determinants. For example, let us consider the components of G, which are connected only to v and not to u. By moving these components from G to G', we reduce to the case that v and G are connected by a single edge. Similarly, we reduce to the case when u and G are also connected by a single edge. Then (4.0.5) follows from [24], Lemma 12.7. □

Corollary 4.0.6. *Using the decomposition of* 4.0.1(d), *for any* $S \subseteqq \mathscr{V}(G'')$, *one has*

$$\left(\prod_{w\notin S} a_{wv}^{\delta_w-2}\right)^{-1} = \det G' \cdot \det(G\cup G''\cup u) \cdot \left(\det G'\cdot \det(G\backslash \overline{uv})\right)^{\sum_{w\in S}\delta_w-2} \cdot \prod_{w\in S} \det(G''\backslash \overline{uw})^{\delta_w-2}. \tag{4.0.7}$$

The subgraph G is allowed to be empty. Furthermore, v and u are allowed to be the same, and in this case G is empty and one should write G'' instead of $G\cup G''\cup u$ in the formula. In particular,

$$\prod_{w\in \mathscr{V}(\Gamma)\backslash \mathscr{V}(\Gamma_i)} a_{wv}^{\delta_w-2} = \frac{1}{d_i}. \tag{4.0.8}$$

Proof. The left-hand side of the first equation, by (4.0.4), is $a_{vv}\prod_{w\in S} a_{wv}^{\delta_w-2}$, which equals the right-hand side by (4.0.2). The second equation follows from the first one by the choices $u:=v$, $G'':=\Gamma_i$, $S:=\mathscr{V}(G'')$ and $G'=\bigcup_{j\neq i}\Gamma_j$. Note that $\sum_{w\in S}(\delta_w-2)=-1$ and $\prod_{w\in S}\det(G''\backslash\overline{uw})^{\delta_w-2}=1$ (the latter is (4.0.4) applied to $G''\cup u$). □

Lemma 4.0.9. *For any $x\in L'$ and its restrictions $x_i:=R_i(x)$ (see* 3.6.1(2)),

$$x-\sum_i x_i = -\frac{d(x,E_v^*)}{a_{vv}}E_v^*, \tag{4.0.10}$$

$$x^2-\sum_i x_i^2 = -\frac{d(x,E_v^*)^2}{a_{vv}}. \tag{4.0.11}$$

Proof. The main idea of the proof of (4.0.10) is that since the scalar product is definite, it is enough to verify that the scalar product with either side of the equation agrees, at least on a basis of L' over $\mathbb{Q}$. We choose the basis consisting of the E_w for $w\neq v$ and E_v^*. It is easy to verify that the scalar product of either side of (4.0.10) with E_w is 0 for $w\neq v$, and the scalar product of either side with E_v^* is (x,E_v^*).

Equation (4.0.11) is the scalar product of (4.0.10) with x. Here we use the identity $(x,x_i)=x_i^2$, which is true, since x_i is the restriction of x. □

5. Additivity formulas. Proof of Theorem 1.0.1

We break the main identity (1.0.2) into the additivity formulas (5.0.2) and (5.0.3), and we also break the latter one into (5.0.4) and (5.0.5).

Proposition 5.0.1. *With the notations of* §3 (*especially of* 3.6.1), *one has*:

$$c_1(\tilde{\sigma})^2 + |\mathscr{V}(\Gamma)| - \sum_i \big(c_1(\tilde{\sigma}_i)^2 + |\mathscr{V}(\Gamma_i)|\big) = 1 - \frac{(\alpha_v + d + 2dr_v)^2}{da_{vv}}, \tag{5.0.2}$$

$$\mathrm{sw}_\sigma(\Sigma) - \sum_i \mathrm{sw}_{\sigma_i}(\Sigma_i) = -\mathscr{H}^{\mathrm{pol}}_{\sigma,v}(1) - \frac{1}{8} + \frac{(\alpha_v + d + 2dr_v)^2}{8da_{vv}}, \tag{5.0.3}$$

$$24\frac{\lambda}{|H|} - \sum_i 24\frac{\lambda_i}{|H_i|} = -3 + \frac{d^2 - \beta_v}{da_{vv}}, \tag{5.0.4}$$

$$\mathcal{T}_\sigma(\Sigma) - \sum_i \mathcal{T}_{\sigma_i}(\Sigma_i) = \mathscr{H}^{\mathrm{pol}}_{\sigma,v}(1) + \frac{d^2 - \beta_v}{24da_{vv}} - \frac{(\alpha_v + d + 2dr_v)^2}{8da_{vv}}. \tag{5.0.5}$$

Equation (5.0.3) is a combination of (5.0.4), (5.0.5) and (3.4.1). The proof of (5.0.5) is given in §6. Here we prove (5.0.2) and (5.0.4) as applications of (4.0.11).

Proof of (5.0.2). We apply (4.0.11) to $x := c_1(\tilde{\sigma})$. Then $x_i = c_1(\tilde{\sigma}_i)$, and

$$c_1(\tilde{\sigma})^2 - \sum_i c_1(\tilde{\sigma}_i)^2 = -\frac{d\big(c_1(\tilde{\sigma}), E_v^*\big)^2}{a_{vv}}. \tag{5.0.6}$$

By the definition of r_v:

$$2r_v = \big(c_1(\tilde{\sigma}) - c_1(\widetilde{\sigma_{\mathrm{can}}}), E_v^*\big). \tag{5.0.7}$$

Next, we compute $\big(c_1(\widetilde{\sigma_{\mathrm{can}}}), E_v^*\big)$. Expressing the Chern class from (3.3.1) as

$$c_1(\widetilde{\sigma_{\mathrm{can}}}) = \sum_w E_w - \sum_w (\delta_w - 2)E_w^*,$$

and then using (3.1.2) we get

$$\big(c_1(\widetilde{\sigma_{\mathrm{can}}}), E_v^*\big) = 1 + \frac{\alpha_v}{d}. \tag{5.0.8}$$

Finally, combining (5.0.6), (5.0.7) and (5.0.8) gives the desired formula. □

Proof of (5.0.4). This time, we apply (4.0.11) first to $x := E_w^*$ for some $w \neq v$. Then $x_i = E_{w,i}^*$ if $w \in \mathscr{V}(\Gamma_i)$, and $x_i = 0$ otherwise. Hence, (4.0.11) reads as

$$-\frac{a_{ww}}{d} + \frac{a_{ww,i}}{d_i} = -\frac{a_{vw}^2}{da_{vv}}, \quad w \in \mathscr{V}(\Gamma_i). \tag{5.0.9}$$

Next, we apply (4.0.11) to $x := E_v$. Then $x_i = E_{v_i,i}^*$ and we get

$$b_v + \sum_i \frac{a_{v_i v_i,i}}{d_i} = -\frac{d}{a_{vv}}. \tag{5.0.10}$$

The claimed equality is a linear combination of (3.1.3), (5.0.9), (5.0.10) and (3.2.1), where the latter is applied to Σ and all the Σ_i. □

6. Proof of (5.0.5)

6.1. Breaking up the torsion. We start with some preparations. For an arbitrary map $\psi : \mathscr{V}(\Gamma) \to \mathbb{C}^*$ we define

$$\mathscr{V}(\Gamma)_\psi := \{w \in \mathscr{V}(\Gamma) : \psi(w) = 1\},$$
$$\mathrm{supp}(\psi) := \mathscr{V}(\Gamma)\backslash\mathscr{V}(\Gamma)_\psi,$$
$$\psi_i := \psi|_{\mathscr{V}(\Gamma_i)} : \mathscr{V}(\Gamma_i) \to \mathbb{C}^*.$$

Lemma 6.1.1. *Let Γ be a negative definite tree and $\psi : \mathscr{V}(\Gamma) \to \mathbb{C}^*$ a function on it. Then the least degree term of the Laurent series of $P_{\psi,v}$ (see* (3.7.1)*) at* 1 *is*

$$P_{\psi,v}(t) = \prod_{w \notin \mathscr{V}(\Gamma)_\psi} \big(1 - \psi(w)\big)^{\delta_w - 2} \cdot \prod_{w \in \mathscr{V}(\Gamma)_\psi} a_{vw}^{\delta_w - 2} \cdot (1-t)^n + O\big((1-t)^{n+1}\big), \tag{6.1.2}$$

where

$$\begin{aligned} n &:= \sum_{w \in \mathscr{V}(\Gamma)_\psi} (\delta_w - 2) \\ &= -2|\{\textit{components of } \mathscr{V}(\Gamma)_\psi\}| + |\{\textit{edges going out of } \mathscr{V}(\Gamma)_\psi\}|. \end{aligned} \tag{6.1.3}$$

In particular, if every component of $\mathscr{V}(\Gamma)_\psi$ has a vertex with at least two outgoing edges (e.g. ψ is a non-trivial character) then $n \geqq 0$ with equality if and only if all components have exactly two outgoing edges.

Proof. This is mainly a repetition of [18], A.7. The first formula obviously follows from (3.7.1) by taking the least degree term in $t-1$ of every factor of the product. This gives $\sum_{w \in \mathscr{V}(\Gamma)_\psi} (\delta_w - 2)$ for the degree n of the least degree term. The second equality of (6.1.3) is a well-known identity for circuit-free graphs. □

Proposition 6.1.4. *For all non-trivial characters $\rho \in \hat{H}$ and* Spin^c *structures $\sigma = h * \sigma_{\mathrm{can}}$ of Σ with $h \in H$,*

$$\frac{1}{d}\widehat{\mathcal{T}_\sigma}(\rho) = \frac{1}{d}\rho(h)^{-1} \cdot P_{\rho,v}(1) + \begin{cases} \dfrac{1}{d_i}\widehat{\mathcal{T}_{\sigma_i}}(\rho_i) & \textit{if } \rho|_{(\mathscr{V}(\Gamma)\backslash\mathscr{V}(\Gamma_i))\cup\{v_i\}} = 1, \\ 0 & \textit{otherwise}, \end{cases} \tag{6.1.5}$$

where σ_i is the restriction of σ defined in Definition 3.6.1(1).

Proof. Obviously, if $\rho(v) = 1$ then $\rho_i := \rho|_{\mathscr{V}(\Gamma_i)}$ is a character of H_i.

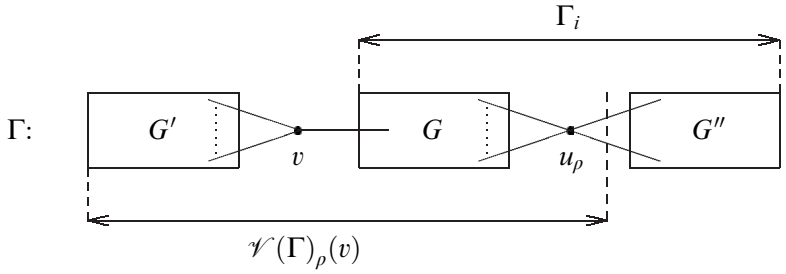
The proof of the proposition is a case-by-case verification.

First, let us consider the case when ρ is non-trivial at v or one of its neighbours. Then we can choose $u_\rho := v$ in (3.4.3), so (6.1.5) immediately follows.

In the remaining cases, ρ is trivial on v and its neighbours v_i. By the second part of Lemma 6.1.1, all three terms of (6.1.5) are 0 (because $n > 0$) unless every component of $\mathscr{V}(\Gamma)_\rho$ has exactly two outgoing edges. Hence the only remaining case is when every component of $\mathscr{V}(\Gamma)_\rho$ has exactly two outgoing edges. Therefore $\sum_{w \notin \mathscr{V}(\Gamma)_\rho} (\delta_w - 2) = -2$, and there exists an index i with $\mathrm{supp}(\psi) \subset \mathscr{V}(\Gamma_i)$. Hence, the upper case of equation (6.1.5) should hold.

Let u_ρ be the vertex of the component $\mathscr{V}(\Gamma)_\rho(v)$ of $\mathscr{V}(\Gamma)_\rho$ containing v where its two outgoing edges start.

We decompose Γ into subgraphs as shown in the next picture.



We express the terms of (6.1.5) in terms of determinants of subgraphs using (3.4.2), (3.4.3) and (4.0.7):

$$P_{\rho,v}(1) = \frac{\det(G') \cdot \big(\det(G \backslash \overline{vu_\rho})\big)^2}{\det(G \cup G'' \cup u_\rho)} \prod_{w \notin \mathscr{V}(\Gamma)_\rho} \left(\frac{\big(1 - \rho([E_w^*])\big)}{\det(G'' \backslash \overline{u_\rho w})} \right)^{\delta_w - 2},$$

$$\widehat{\mathcal{T}_\sigma}(\rho) = P_{\rho,u_\rho}(1) = \frac{\det(G' \cup G \cup v)}{\det(G'')} \prod_{w \notin \mathscr{V}(\Gamma)_\rho} \left(\frac{\big(1 - \rho([E_w^*])\big)}{\det(G'' \backslash \overline{u_\rho w})} \right)^{\delta_w - 2},$$

$$\widehat{\mathcal{T}_{\sigma_i}}(\rho_i) = P_{\rho_i,u_\rho}(1) = \frac{\det(G)}{\det(G'')} \prod_{w \notin \mathscr{V}(\Gamma)_\rho} \left(\frac{\big(1 - \rho([E_w^*])\big)}{\det(G'' \backslash \overline{u_\rho w})} \right)^{\delta_w - 2}.$$

Note that $\rho(h) = \rho_i(h_i)$ where $\sigma_i = h_i * \sigma_{\mathrm{can},i}$ by Definition 3.6.1, and hence these factor out of (6.1.5). We can also factor out the $\prod_{w \notin \mathscr{V}(\Gamma)_\rho}$ product. Finally, recall that $d_i = \det(G \cup G'' \cup u_\rho)$ and $d = \det \Gamma$. Hence (6.1.5) reduces to 4.0.1(d). □

6.2. Principal part of the Hilbert function. Next, we concentrate on $\mathscr{H}_{\sigma,v}$. We invite the reader to recall the notations from 3.7–3.8.

Lemma 6.2.1. *For every non-trivial pseudo-character ψ associated with v,*

$$\text{(6.2.2)} \quad \mathrm{L}_1 P_{\psi,v}(t) = \begin{cases} \dfrac{1}{d_i} \cdot \dfrac{P_{\psi_i,v_i}(1) \cdot \big(1 - \psi_i(v_i)\big)}{1 - t} & \text{if } \mathrm{supp}\,\psi \subseteqq \mathscr{V}(\Gamma_i) \text{ and } \psi(v_i) \neq 1, \\ 0 & \text{for all other } \psi \neq 1. \end{cases}$$

Proof. We apply Lemma 6.1.1. By the pseudo-character relations, all components of $\mathscr{V}(\Gamma)_\psi$ have at least two outgoing edges except possibly the component containing v, which can have only one outgoing edge, which must start at v. Hence the lower degree of the Laurent expansion of $P_{\psi,v}$ at 1 is at least -1 with equality if and only if all the components have the minimum number of outgoing edges declared above. In particular, if $\mathrm{L}_1 P_{\psi,v} \neq 0$ then $\psi(v_i) \neq 1$ for some i and $\mathrm{supp}(\psi) \subset \mathscr{V}(\Gamma_i)$. This proves the lower case of (6.2.2).

To prove the upper case, note that by (4.0.3) for any $w \in \mathscr{V}(\Gamma_i)$

$$P_{\psi,v}(t) = P_{\psi_i,v_i}(t^{\det(\Gamma\setminus v\setminus\Gamma_i)}) \cdot \big(1-\psi_i(v_i)t^{a_{vv_i}}\big) \cdot \prod_{w\notin\mathscr{V}(\Gamma_i)} (1-t^{a_{wv}})^{\delta_w-2}.$$

Obviously, ψ_i is a non-trivial character of H_i, hence P_{ψ_i,v_i} is regular at 1. Moreover, $\sum_{w\notin\mathscr{V}(\Gamma_i)} (\delta_w-2) = -1$. Thus

$$\mathrm{L}_1 P_{\psi,v}(t) = \frac{1}{1-t} \cdot P_{\psi_i,v_i}(1) \cdot \big(1-\psi_i(v_i)\big) \cdot \prod_{w\notin\mathscr{V}(\Gamma_i)} a_{wv}^{\delta_w-2}.$$

For the last product one can use (4.0.8), and this finishes the proof. □

We fix an $l' \in L'$ with $\sigma = [l'] * \sigma_{\mathrm{can}}$ and $-1 < r_v = (l', E_v^*) \leqq 0$ and $\sigma_i = [l_i'] * \sigma_{\mathrm{can},i}$ for the restriction l_i' of l' to Γ_i. Note that all the poles α of $P_{\rho,v}$ are roots of unity.

$$d \cdot \mathscr{H}^{<0}_{\sigma,v}(t) = \sum_\alpha \sum_{\rho\in\tilde{\boldsymbol{H}}} \rho([l'])^{-1}\, \mathrm{L}_\alpha P_{\rho,v}(t) = \sum_\alpha \sum_{\substack{\psi\in\tilde{\boldsymbol{H}}\\ \mathrm{def}(\psi)=\alpha^d}} \psi(l')^{-1}\alpha^{dr_v}(\mathrm{L}_1 P_{\psi,v})(\alpha t),$$

where the last equality is obtained via the substitutions $\psi(w) := \rho(w)\alpha^{-a_{vw}}$ implying $\psi(x) = \rho([x])\alpha^{d(x,E_v^*)}$ for all $x \in L'$. To compute $\mathrm{def}_w(\psi)$, we have used the identity $I \cdot I^{-1} = 1$ in the form

$$b_w a_{wv} + \sum_i a_{wv_i} = \begin{cases} -d & \text{if } w = v,\\ 0 & \text{if } w \neq v. \end{cases}$$

To compute further, we apply Lemma 6.2.1 to index the pseudo-characters ψ for which the summand maybe non-zero by characters ψ_i of H_i with $\psi_i(v_i) \neq 1$:

$$\begin{aligned} d \cdot \mathscr{H}^{<0}_{\sigma,v}(t) = &\sum_{\alpha^d=1} \alpha^{dr_v}(\mathrm{L}_1 P_{1,v})(\alpha t)\\ &+ \sum_\alpha \sum_i \frac{1}{d_i} \sum_{\substack{\psi_i\in\widehat{H}_i\\ \psi_i(v_i)=\alpha^d\neq 1}} \psi_i([l_i'])^{-1}\alpha^{dr_v} P_{\psi_i,v_i}(1)\big(1-\psi_i(v_i)\big)\frac{1}{1-\alpha t}. \end{aligned}$$

Using (3.4.3) in the form $\widehat{\mathcal{T}_{\sigma_i}}(\psi_i) = \psi_i([l_i'])^{-1} P_{\psi_i,v_i}(1)$, and summing in the variable α by (3.8.2) (recall that $-d < dr_v \leqq 0$):

$$\text{(6.2.3)} \qquad \mathrm{coef}_1^0\, \mathscr{H}^{<0}_{\sigma,v}(t) = \mathrm{coef}_1^0\left(\frac{1}{d}\sum_{\alpha^d=1} \alpha^{dr_v}(\mathrm{L}_1 P_{1,v})(\alpha t)\right) + \sum_i \frac{1}{d_i} \sum_{\substack{\psi_i\in\widehat{H}_i\\ \psi_i(v_i)\neq 1}} \widehat{\mathcal{T}_{\sigma_i}}(\psi_i).$$

6.3. Additivity formula for torsion. Now, we are ready to establish an additivity formula for the torsion. By 6.1.4 and (3.4.4)

$$\mathcal{T}_\sigma(\Sigma) = \mathrm{coef}_1^0\, \mathscr{H}_{\sigma,v}(t) - \frac{1}{d}\,\mathrm{coef}_1^0\, P_{1,v}(t) + \sum_i \frac{1}{d_i} \sum_{\substack{\psi_i \in \widehat{H}_i\setminus 1 \\ \psi_i(v_i)=1}} \widehat{\mathcal{T}_{\sigma_i}}(\psi_i).$$

Then, using $\mathscr{H}_{\sigma,v} = \mathscr{H}^{\mathrm{pol}}_{\sigma,v} + \mathscr{H}^{<0}_{\sigma,v}$ and (6.2.3) we get the next identity. We highlight it, since it shows the more conceptual source of the correction constant in (5.0.5):

$$\mathcal{T}_\sigma(\Sigma) - \sum_i \mathcal{T}_{\sigma_i}(\Sigma_i) = \mathscr{H}^{\mathrm{pol}}_{\sigma,v}(1) + \frac{1}{d}\,\mathrm{coef}_1^0\left(\sum_{\alpha^d=1} \alpha^{dr_v}(\mathrm{L}_1 P_{1,v})(\alpha t) - P_{1,v}(t)\right). \tag{6.3.1}$$

The last two terms depend only on the coefficients of terms with non-positive degree of the Laurent expansion of $P_{1,v}$ at 1. These terms can be computed elementarily:

$$\begin{aligned} P_{1,v}(t) &= \prod_w (1 - t^{a_{vw}})^{\delta_w - 2} \\ &= \frac{1}{a_{vv}}\left(\frac{1}{(t-1)^2} + \frac{1+\alpha_v/2}{t-1} + \frac{(\alpha_v+1)^2}{8} + \frac{\beta_v - 1}{24} + O(t-1)\right). \end{aligned}$$

Hence 3.8.1 and a simple computation provides (5.0.5).

7. Proof of Theorem 2.2.1 and Corollary 2.2.4

In this section we combine our surgery formula with the main result of Okuma from [27] to derive the results of §2.

Okuma's article [27] uses a constant invariant of the Taylor expansion at the origin of R in place of our $R^{\mathrm{pol}}(1)$. This constant invariant was later called the periodic constant, which terminology we adopt.

In the first paragraphs we prove that they are equal. After the proof appeared in a public preprint of this article, the result (Lemma 7.0.2) was also incorporated into Okuma's article as Proposition 4.8.

Definition 7.0.1 (Periodic constant [21], 3.9, [27], just before Proposition 4.8). Let $F(t) = \sum_{i \geqq 0} a_i t^i$ be a formal power series. Suppose that for some positive integer p, the expression $\sum_{i=0}^{pn-1} a_i$ is a polynomial $P_p(n)$ in the variable n. Then the constant term of $P_p(n)$ is independent of p. We call this constant term the *periodic constant* of F and denote it by $\mathrm{pc}\, F$.

For rational functions, one has the following equivalent description of the periodic constant. Here, we identify the rational function R with its Taylor expansion at the origin.

Lemma 7.0.2. *Let R be a rational function having poles only at infinity and roots of unity. Then R has a periodic constant and* $\operatorname{pc} R = R^{\mathrm{pol}}(1)$, *where* R^{pol} *is the polynomial part of R as in* 3.8(1).

Proof. Write

$$R(t) = R^{\mathrm{pol}}(t) + \sum_{\substack{k\geqq 0\\ 0\leqq j<p}} a_{kj}\frac{t^j}{(1-t^p)^{k+1}} \quad (a_{kj}\in\mathbb{C}),$$

where the sum is finite. Note that if two formal power series F_1 and F_2 have periodic constants then $\operatorname{pc}(F_1+F_2) = \operatorname{pc} F_1 + \operatorname{pc} F_2$. Also, every polynomial A has a periodic constant, namely, $\operatorname{pc} A = A(1)$. Hence it is enough to prove that $t^j(1-t^p)^{-(k+1)} = \sum_{l\geqq 0}\binom{k+l}{k}t^{lp+j}$ admits a periodic constant, which is 0. Indeed, the constant term of

$$\sum_{l=0}^{n-1}\binom{k+l}{k} = \binom{k+n}{k+1}$$

is 0 as a polynomial in n. □

Proof of Theorem 2.2.1. We prove the statement by induction on the number of vertices in the dual resolution graph of the singularity (X,o).

First, let us suppose that the class satisfies the Seiberg–Witten invariant conjecture 2.1.1. Then expressing the Seiberg–Witten invariants from (2.1.2) and substituting the result into (1.0.3), we obtain (2.2.2) (for all singularity (X,o) in the class and all splitting vertex v).

To prove the converse, let us assume that the class satisfies (2.2.2). We prove the Seiberg–Witten invariant conjecture 2.1.1 for every member of the family by induction on the number of vertices of the dual resolution graph. For rational singularities, 2.1.1 is true by [11], Theorem 6.2. This starts the induction.

For a non-rational (X,o) in the class, let us choose a vertex v of the dual resolution graph satisfying (2.2.2). Let $l'\in R$. Then $R_i(l')\in R(\Gamma_i)+L(\Gamma_i)_e$ by (3.6.3), since the first term $r_vE^*_{v_i,i}$ of its right-hand side is a non-negative rational cycle, and its second term is contained in $R(\Gamma_i)$. So, by the induction hypothesis and Remark 2.2.3, equation (2.1.2) applies to (X_i,o) and $R_i(l')$. Combining these with (1.0.3) and (2.2.2) for (X,o) and v, we obtain (2.1.2) for (X,o). □

Proof of Corollary 2.2.4. The corollary follows from Theorem 2.2.1 by Okuma's results from [27], which show that the class of splice-quotient singularities satisfy all the necessary conditions.

Specifically, for every splice-quotient singularity (X,o) and vertex v of the dual resolution graph, the singularities (X_i,o) are also splice-quotient by [27], 2.16.

Moreover, the additivity formula (2.2.2) for all $l'\in R$ and v with degree at least 3 is a combination of [27], Theorem 4.5 and Lemma 4.2(3), and Lemma 7.0.2. □

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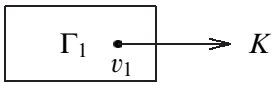
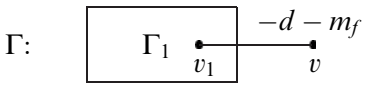
8. Examples

8.1. $\Sigma = S^3_{-d}(K)$. Let $K \subset S^3$ be an algebraic knot, i.e. the link of an analytic irreducible plane curve singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Let μ and $\Delta(t)$ be its Milnor number and Alexander polynomial, respectively. Let $\Sigma := S^3_{-d}(K)$ be obtained by $(-d)$-surgery ($d \in \mathbb{N}^+$) along $K \subset S^3$. The Heegaard Floer homology of Σ was computed in [15] in terms of Δ (see also [31], Theorem 4.1). Here we recover the formula [15], 4.3, for $\mathrm{sw}_*(\Sigma)$ from our results.

Let the (minimal) good resolution of $\big(\mathbb{C}^2, f^{-1}(0)\big)$ be given by the schematic diagram



Write m_f for the vanishing order of the lifting of f along the exceptional divisor E_{v_1}. Then (see [15]), a possible plumbing graph of Σ is



Let v be the 'new' vertex. Then $\Gamma \backslash v$ has only one component, namely Γ_1, which can be blown down completely, hence $\Sigma_1 = S^3$. One can verify that $H = \mathbb{Z}_d$ and it is generated by $[E_v^*]$. Hence $\hat{H}$ consists of the maps ρ given by $\rho([kE_v^*]) = \xi^k$ for all dth roots of unity ξ. Moreover, the Spin^c structures of Σ are $[qE_v^*] * \sigma_{\mathrm{can}}$ for $0 \leqq q < d$. Then, using e.g. the formula [5], 11.3, for Δ, one has

$$\mathscr{H}_{[qE_v^*]*\sigma_{\mathrm{can}},v}(t) = \frac{1}{d} \sum_{\xi^d=1} \xi^{-q} \frac{\Delta(\xi t)}{(1-\xi t)^2}.$$

One can write $\Delta(t) = 1 + (t-1)\mu/2 + (t-1)^2 \sum_l a_l t^l$. Hence

$$\mathscr{H}^{\mathrm{pol}}_{[qE_v^*]*\sigma_{\mathrm{can}},v}(t) = \frac{1}{d} \sum_{\xi} \xi^{-q} \sum_l a_l \xi^l t^l = \sum_l a_{q+ld} t^{q+ld}.$$

Note that $a_{vv} = 1$, hence $(qE_v^*, E_v^*) = -q/d \in (-1, 0]$ and so $r_v = -q/d$. Recall e.g. from [5], 11.1, that $\mu - 2 = \alpha_v$. Thus, using (5.0.3), we recover [15], 4.3, as promised:

$$\mathrm{sw}_{[qE_v^*]*\sigma_{\mathrm{can}}}\big(S^3_{-d}(K)\big) = -\sum_l a_{q+ld} + \frac{(\mu - 2 + d - 2q)^2}{8d} - \frac{1}{8}.$$

Similarly (with slightly more computations) one can recover the Seiberg–Witten invariant of $S^3_{-p/q}(K)$, too (here $p/q \in \mathbb{Q}$, $p/q > 0$); for a possible formula see [12], 4.5.

8.2. Seifert manifolds. Let Σ be a Seifert manifold. Recall that either Σ or $-\Sigma$ can be realized as a negative definite plumbing (and $\mathrm{sw}(-\Sigma) = -\mathrm{sw}(\Sigma)$), hence we may assume without loss of generality that $\Sigma = \Sigma(\Gamma)$ for a (minimal) negative definite graph Γ. We will assume that Γ is not a string (i.e. Σ is not a lens space). Then Γ is star-shaped; let v be

its central vertex. There exists an affine complex surface singularity X whose link at the origin is Σ, and which admits a good $\mathbb{C}^*$ action. In particular, its affine coordinate ring A is graded.

First we show how $\mathscr{H}_{\sigma,v}(t)$ and its periodic constant can be expressed from the Seifert invariants of Σ.

Let $(\alpha_i,\omega_i)_{i=1}^r$ denote the normalized Seifert invariants of Σ (for more details, see [19]). Set $\alpha=\mathrm{lcm}(\alpha_i : i=1,\dots,r)$ and $o=\alpha\cdot|H|/\prod_i \alpha_i$. We denote the end-vertices (i.e. vertices of degree 1) by $\{w_i\}_i$. Then $[E_v^*]$ and $\{[E_{w_i}^*]\}_i$ generate H, hence $l'\in L'$ can be written as $l'=aE_v^*+\sum_i a_iE_{w_i}^*$ modulo L. Set $\tilde{a}:=\alpha\left(a+\sum_i a_i/\alpha_i\right)$. Then, by [19], Theorem (3.1), for $\sigma=[l']*\sigma_{\mathrm{can}}$ one has

$$\mathscr{H}_{\sigma,v}(t)=\sum_{l\geqq -\tilde{a}/o}\max\left(0,1+a-lb_v+\sum_{i=1}^r\left\lfloor\frac{-l\omega_i+a_i}{\alpha_i}\right\rfloor\right)t^{ol+\tilde{a}}. \tag{8.2.1}$$

In the case $\sigma=\sigma_{\mathrm{can}}$, one has $a=a_i=\tilde{a}=0$. Moreover, we claim that

$$\mathrm{pc}\,\mathscr{H}_{\mathrm{can},v}=\sum_{l\geqq 0}\max\left(0,-1+lb_v-\sum_{i=1}^r\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right). \tag{8.2.2}$$

The idea of the proof is the following: let us define the polynomial

$$P(t):=\sum_{l\geqq 0}\max\left(0,-1+lb_v-\sum_{i=1}^r\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)t^{ol}.$$

By the identity $\max(0,x)-\max(0,-x)=x$ we get that

$$\mathscr{H}_{\mathrm{can},v}(t)-P(t)=\sum_{l\geqq 0}\left(1-lb_v+\sum_{i=1}^r\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)t^{ol}.$$

Then a computation shows that the periodic constant of the last expression is zero. Hence $\mathrm{pc}\,\mathscr{H}_{\mathrm{can},v}=P(1)$, which is exactly (8.2.2).

Note that by [27], [32] the right hand side of (8.2.1) is the Hilbert (Poincaré) series of a graded A-module. If $\sigma=\sigma_{\mathrm{can}}$ then this module is exactly A. On the other hand, by [32], [4], the expression from the right-hand side of (8.2.2) is exactly the *geometric genus* p_g of (X,o). In particular, we have also proved that the periodic constant of the Poincaré series of the graded algebra A is exactly the geometric genus of the singularity.

Now, let us apply 1.0.1 for $\sigma=\sigma_{\mathrm{can}}$. Since all the components of $\Gamma\backslash v$ are strings, they support rational singularities. Therefore, by [11], 4.1.1,

$$\mathrm{sw}_{\mathrm{can}}(\Sigma_i)+\frac{c_1(\widetilde{\sigma_{\mathrm{can},i}})^2+|\mathscr{V}(\Gamma_i)|}{8}=0.$$

Hence 1.0.1 reads as $\mathrm{sw}_{\mathrm{can}}(\Sigma)+\left(c_1(\widetilde{\sigma_{\mathrm{can}}})^2+|\mathscr{V}(\Gamma)|\right)/8=-p_g$.

Notice that this is exactly the claim of the Seiberg–Witten invariant conjecture 2.1.1 for weighted homogeneous singularities and for the canonical Spin^c structure. Its original proof from [19] is based on completely different combinatorial identities.

We would like to emphasize that, in general, pc $\mathscr{H}$ can be a rather complicated arithmetical expression. E.g., when Σ is the Seifert 3-manifold $\Sigma(a,b,c)$ (the link of $x^a+y^b+z^c$ with a, b, c pairwise relative prime numbers), then pc $\mathscr{H}_{\mathrm{can},v}$ is the number of interior lattice points in the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$. (This can be expressed by Dedekind sums by a result of Mordell.)

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