On the Milnor Fibers of Sandwiched Singularities

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The *sandwiched* singularities are those rational surface singularities that dominate
birationally smooth surface singularities. de Jong and van Straten showed that one
can reduce the study of the deformations of a sandwiched singularity to the study of
deformations of a one-dimensional object, a so-called *decorated* plane curve singularity.
In particular, the Milnor fibers corresponding to their various smoothing components
may be reconstructed up to diffeomorphisms from those deformations of associated
decorated curves that have only ordinary singularities. Part of the topology of such a
deformation is encoded in the *incidence matrix* between the irreducible components of
the deformed curve and the points that decorate it, well defined up to permutations of
columns. Extending a previous theorem of ours, which treated the case of cyclic quotient
singularities, we show that the Milnor fibers that correspond to deformations whose
incidence matrices are different up to permutations of columns are not diffeomorphic in
a strong sense. This gives a lower bound on the number of Stein fillings of the contact
boundary of a sandwiched singularity.

1 Introduction

In our previous paper [12], we proved a conjecture of Lisca [9, 10], establishing a bijective correspondence between the Milnor fibers of the smoothing components of a cyclic

quotient singularity and the Stein fillings of the corresponding contact lens space, the boundary (or link) of the singularity. As a particular case of our results, the Milnor fibers corresponding to distinct smoothing components of the reduced miniversal base space of the cyclic quotient singularity are pairwise nondiffeomorphic. Here one has to understand the diffeomorphisms *in a strong sense*: namely, there are natural identifications of the boundaries of the Milnor fibers up to isotopy, and we showed that there are no diffeomorphisms extending those identifications.

Having in mind the construction of de Jong and van Straten in [7] regarding the Milnor fibers of sandwiched singularities, it is natural to try to extend the above result for such singularities. One of the main obstructions of this program is that at present the precise description of all the smoothing components of sandwiched singularities (or/and the classification of all Stein fillings of the corresponding contact boundaries) is out of hope.

Nevertheless, we will prove a slightly weaker version of the above result. In order to explain it, we need some preparation.

The class of sandwiched singularities received its name in [16] from the fact that the corresponding singularities are analytically isomorphic to the germs of surfaces that may be "sandwiched" birationally between two smooth surfaces (see also Section 3.1). They form a subclass of the rational surface singularities and their study may be reduced in many respects to the study of plane curve singularities.

Indeed, de Jong and van Straten showed that this was the case for their deformation theory as well. Namely, one may encode a sandwiched singularity (X, x) by a *decorated curve* (C, l), which is a germ of reduced plane curve whose components C_i are decorated by sufficiently high positive integers l_i. One of the main theorems of [7] states that the deformation properties of a sandwiched singularity reduce to the deformation properties of any such decorated curve which encodes it.

In particular, the 1-parameter smoothings of (X, x) correspond bijectively to the so-called *picture deformations* of (C, l), which are δ-constant deformations of C with generic fibers having only ordinary singularities, accompanied with flat deformations of l (seen as a subscheme of the normalization of C) which generically are reduced and contain the preimage of the singular locus of the generic fiber. The Milnor fiber corresponding to a smoothing component of (X, x) may be reconstructed up to strong diffeomorphism from the embedding in a 4-ball of the generic fiber of a corresponding picture deformation.

Part of the topological structure of the generic fiber of the deformation of (C, l) may be encoded in an associated *incidence matrix* between the irreducible components

of the deformation of C and the images of the support of l seen as a subscheme of the normalization. This matrix is well defined up to permutation of columns.

As proved in [12], the validity of Lisca's conjecture implies that, *for cyclic quotient singularities*, the Milnor fibers that correspond to distinct incidence matrices (up to permutation of columns) are not strongly diffeomorphic.

The aim of this paper is to show that *the same is true for all sandwiched surface singularities* (see Theorem 4.3.4).

In fact, for any fixed realization of (X, x) by a decorated curve (C, l), we provide a canonical method to reproduce all the incidence matrices (associated with (C, l)) from the Milnor fibers. Hence, each realization (C, l) provides a "test" to separate the Milnor fibers (in fact, precise criteria to recognize them).

Recall also that de Jong and van Straten predict (see [7, (4.3)]) that, for any fixed (C, l), the set of smoothing components of the deformation space of (X, x) injects into the set of incidence matrices associated with (C, l). In particular, for any class of singularities, when this is indeed the case, we obtain via our result that there is no pair of smoothing components with (strongly) diffeomorphic Milnor fibers.

In order to connect our main result with the set of Stein fillings of the contact boundary, we have to face a new aspect of the problem: sandwiched singularities are not necessarily taut (that is, their analytical structure is not determined by their topology). Moreover, fixing a topological type for different analytic types, one might have different structure of the miniversal deformation space (therefore different sets of Milnor fibers). Nevertheless, for a fixed topological type, the induced natural contact structure on the boundary is independent of the analytic realization and is invariant up to isotopy by all orientation-preserving diffeomorphisms (see [1]).

Therefore, in Theorem 4.4.1 we are able to extend the above recognition criterion of the Milnor fibers via incidence matrices by considering all the smoothings associated with the deformations of all the decorated germs of plane curves with a fixed topology. In this way *we get a lower bound for the number of Stein fillings of the contact boundary of a sandwiched surface singularity, considered up to orientation-preserving diffeomorphisms fixed on the boundary*.

1.1   Conventions and notations

All the differentiable manifolds we consider are oriented: any letter, say W, denoting a manifold denotes in fact an oriented manifold. We denote by $\overline{W}$ the manifold obtained by changing the orientation of W, and by ∂W its boundary, canonically

oriented by the rule that the outward normal *followed* by the orientation of ∂W gives the orientation of W.

We work exclusively with homology groups *with integral coefficients*. If W is four-dimensional, the intersection number in $H_2(W)$ is denoted by $S_1 \cdot S_2$. An element of $H_2(W)$ is called *a (−1)-class* if its self-intersection is equal to −1.

If W_1 and W_2 are two oriented manifolds with boundary, endowed with a fixed isotopy class of orientation-preserving diffeomorphisms $\partial W_1 \to \partial W_2$, we say that W_1 and W_2 are *strongly diffeomorphic* (with respect to that class) if there exists an orientation-preserving diffeomorphism $W_1 \to W_2$ whose restriction to the boundary belongs to the given class.

2   Generalities

We recall briefly the needed facts about deformation theory, δ-constant deformations, and the topology of surface singularities. In what follows, a *singularity* will design a germ of reduced complex analytic space.

2.1   Deformations and smoothings

Let (X, x) be an isolated singularity. Choose one of its representatives embedded in some affine space: $(X, x) \hookrightarrow (\mathbb{C}^n, 0)$. Denote by $\mathbb{B}_r \subset \mathbb{C}^n$ the compact ball of radius r centered at the origin and by $\mathbb{S}_r$ its boundary. By a general theorem (see [11]), there exists $r_0 > 0$ such that X is transversal to the euclidean spheres $\mathbb{S}_r$ for any $r \in (0, r_0]$. $\mathbb{B}_r$ is then called a *Milnor ball* for the chosen embedding and such a representative $X \cap \mathbb{B}_r$ is called a *Milnor representative* of (X, x). The manifold $X \cap \mathbb{S}_r$, oriented as boundary of the complex manifold $(X \cap \mathbb{B}_r) \setminus \{x\}$ is well defined up to orientation-preserving diffeomorphisms (see [4] or [11]) and is called the *(abstract) boundary* of (X, x). We denote it by $\partial(X, x)$.

Definition 2.1.1.   Let (X, x) be a singularity. A *deformation* of (X, x) is a germ of flat morphism $\pi : (Y, y) \to (S, 0)$ together with an isomorphism between (X, x) and the special fiber $\pi^{-1}(0)$. A *1-parameter smoothing* is a deformation over a germ of smooth curve such that the generic fibers are smooth.

A deformation $\pi : (Y, y) \to (S, 0)$ of (X, x) is *versal* if any other deformation is obtainable from it by a base-change. A versal deformation is *miniversal* if the Zariski tangent space of its base $(S, 0)$ has the smallest possible dimension. A *smoothing component* is an irreducible component of the miniversal base space over which the generic fibers are smooth.

If (X, x) is a germ of reduced complex analytic space *with an isolated singularity*, then the following well-known facts hold:

(i) (Schlessinger [15] and Grauert [5]) The miniversal deformation π exists and is unique up to (nonunique) isomorphism.

(ii) There exist (Milnor) representatives Y_{red} and S_{red} of the reduced total and base spaces of π such that the restriction $\pi : \partial Y_{red} \cap \pi^{-1}(S_{red}) \to S_{red}$ is a trivial C^∞-fibration.

Hence, for each smoothing component of (X, x), the oriented diffeomorphism type of the oriented manifold with boundary $(\pi^{-1}(s) \cap Y_{red}, \pi^{-1}(s) \cap \partial Y_{red})$ does not depend on the choice of the generic element s: it is called *the Milnor fiber* of that component. Moreover, its boundary is canonically identified with the boundary $\partial(X, x)$ by an orientation-preserving diffeomorphism, *up to isotopy*.

Therefore, if one takes the Milnor fibers corresponding to two distinct components of S_{red}, one may ask whether they are *strongly diffeomorphic* (a notion introduced in Section 1.1) with respect to the previous natural identification of their boundaries.

2.2   The δ-invariant and δ-constant deformations

Let $(C, 0)$ be a (not necessarily plane) curve singularity. Its δ-*invariant* is by definition

$$\delta(C, 0) := \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\tilde{C}, \tilde{0}} / \mathcal{O}_{C, 0}),$$

where $(\tilde{C}, \tilde{0}) \overset{\nu}{\to} (C, 0)$ is a normalization morphism (in particular, $(\tilde{C}, \tilde{0})$ denotes a multigerm). If C is a reduced curve, by definition, its δ-*invariant* is the sum of the δ-invariants of all its singular points.

As an example, consider an *ordinary m-tuple point* of a reduced curve embedded in a smooth surface, that is, a point where the curve has exactly m local irreducible components, all smooth and meeting pairwise transversally. At such a point, the δ-invariant of the curve is equal to $\frac{m(m-1)}{2}$.

More generally, if $(C, 0)$ is a germ of plane curve, one has the following classical formula:

$$\delta(C, 0) = \sum_{P} \frac{m(C, P)(m(C, P) - 1)}{2}, \tag{2.2.1}$$

where P varies among all the points infinitely near 0 on the ambient surface and $m(C, P)$ denotes the multiplicity of the strict transform of C at such a point P (see [6, pp. 298 and 393]).

Denote by $(\Sigma, 0) \xrightarrow{\pi} (S, 0)$ a deformation of $(C, 0)$ over a smooth germ of curve. It is called *δ-constant* if there exists a Milnor representative of π such that the δ-invariant of its fibers is constant. By a characterization of Teissier [17] (see also [3]), a deformation π is δ-constant if and only if the normalization morphism of Σ has the form $\tilde{C} \times S \to \Sigma$, and it is the *simultaneous normalization* of the fibers of π. In particular, if one has a δ-constant embedded deformation of a germ of *plane* curve $(C, 0)$ such that the general fiber C_s has only ordinary multiple points, then after a choice of a Milnor representative of the morphism, the general fibers are immersed *disks* in a 4-ball.

2.3 The topology of normal surface singularities

Consider a normal surface singularity (X, x). It has a unique minimal normal crossings resolution, that is, a resolution whose exceptional set E is a divisor with normal crossings and which loses this property if one contracts any (-1)-curve of it. Consider also the *weighted dual graph* of this resolution. Each vertex is weighted by the genus and by the self-intersection number of the corresponding irreducible component of the divisor E.

From the resolution, the abstract boundary $\partial(X, x)$ inherits a *plumbing structure*, that is, a family of pairwise disjoint embedded tori whose complement is fibered by circles and such that on each torus, the intersection number of the fibers from each side is ± 1. The tori correspond to the edges of the dual graph and the connected components of their complement correspond to the vertices; for more details, see [13] or [14].

Neumann proved in [13] that the knowledge of the boundary of (X, x), up to orientation-preserving homeomorphisms, is equivalent to the knowledge of the weighted dual graph of the minimal normal crossings resolution, up to isomorphisms. In [14, Theorems 9.1 and 9.7], Popescu-Pampu proved the following enhancement of Neumann's result.

Theorem 2.3.1. The plumbing structure corresponding to the minimal normal crossings resolution is determined by the oriented boundary (as abstract manifold); in particular, it is invariant, up to isotopy, by all orientation-preserving diffeomorphisms of the boundary.

It is important to note that in the definition of a plumbing structure, the circles of the fibration of the complement of the tori are considered *unoriented* (otherwise the previous theorem would not be true). But in the case of the boundary of a surface singularity, once a fiber is oriented, there is a canonical way to orient all of them (see [14, Corollary 8.6]).

Note also that any fiber has a natural framing given by the nearby fibers. We say that it is *the framing coming from the plumbing structure*.

3 The Smoothings of Sandwiched Surface Singularities, Following de Jong and van Straten

de Jong and van Straten related in [7] the deformation theory of sandwiched surface singularities to the deformation theory of so-called *decorated plane curve singularities*. They showed that 1-parameter deformations of particular decorated curves provide 1-parameter deformations for sandwiched singularities, and that all of these latter ones can be obtained in this way. Moreover, the Milnor fibers of the deformations which are smoothings can be combinatorially described by the so-called *picture deformations*. This section briefly recalls this theory.

3.1 Basic facts about sandwiched singularities

A normal surface singularity (X, x) is called *sandwiched* (see [16]) if it is analytically isomorphic to a germ of algebraic surface that admits a birational map $X \to \mathbb{C}^2$; see also [7, 8] for other viewpoints.

Sandwiched singularities are rational; therefore their minimal normal crossings resolution coincides with their minimal resolution and the associated weighted graph is a negative-definite tree of rational curves. They are characterized (like the rational singularities) by their dual resolution graphs, which allows to speak about *sandwiched graphs*:

Proposition 3.1.1. A graph Γ is sandwiched if by gluing (via new edges) to some of the vertices of Γ new rational vertices with self-intersections -1, one may obtain a "smooth graph," i.e., the dual tree of a configuration of $\mathbb{P}^1$'s which blows down to a smooth point.

The way to add such (-1)-vertices is not unique. But once such a choice is done, for each sandwiched singularity with that topology, one may embed a tubular neighborhood of the exceptional set E of the minimal resolution in a larger surface that also contains some (-1)-curves, whose union with E can be contracted and has as dual graph the chosen "smooth graph." This procedure will be used in the next subsection.

3.2   Decorated curves and their deformations

Any sandwiched singularity can be obtained from a weighted curve (C, l). Here $(C, 0) \subset (\mathbb{C}^2, 0)$ denotes a reduced germ of plane curve *with numbered branches* (irreducible components) $\{C_i\}_{1 \leq i \leq r}$ and $(l_i)_{1 \leq i \leq r} \in (\mathbb{N}^*)^r$. Let us explain this.

Consider the minimal abstract resolution of C obtained by a sequence of blowing-ups of (closed) points. If one realizes those blowing-ups inside the smooth ambient surface, one does not necessarily obtain an embedded resolution of C, that is, the total transform of C is not necessarily a divisor with normal crossings.

The *multiplicity sequence* associated with C_i is the sequence of multiplicities on the successive strict transforms of C_i, starting from C_i itself and not counting the last strict transform. The *total multiplicity $m(i)$* of C_i with respect to C is the sum of the sequence of multiplicities of C_i defined before (see the example which follows Definition 3.2.2).

Definition 3.2.1 [7, (1.3)]. A *decorated germ* of plane curve is a weighted germ (C, l) such that $l_i \geq m(i)$ for all $i \in \{1, \ldots, r\}$.

The point is that, starting from a decorated germ, one can blow up iteratively points infinitely near 0 on the strict transforms of C, such that the sum of multiplicities of the strict transform of C_i at such points is exactly l_i. Such a composition of blow-ups is determined canonically by (C, l). If l_i is sufficiently large (in general, this bound is larger than $m(i)$, see below), then the union of the exceptional components that do not meet the strict transform of C form a *connected* configuration of (compact) curves $E(C, l)$. After the contraction of $E(C, l)$, one gets a sandwiched singularity $X(C, l)$, determined uniquely by (C, l); for details see [7].

Using Proposition 3.1.1, for any sandwiched singularity (X, x) one can find (C, l) such that (X, x) can be represented as $X(C, l)$. Indeed, choose an extension of the sandwiched graph to a smooth graph by adding (-1)-vertices (different choices provide different realizations $X(C, l)$). Fix a geometrical realization of this extension starting from the minimal resolution of the given sandwiched singularity (by gluing tubular neighborhoods of the new (-1)-curves). Consider then a *curvetta* (that is, a smooth transversal slice through a general point) for each (-1)-component and blow down the union of these curvettas to the smooth surface obtained by contracting the configuration with smooth graph. Then number the components of the plane germ C obtained in this way and consider the function l describing the reverse process of reconstruction of the initial
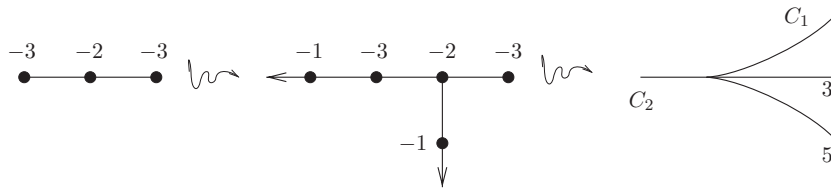
Fig. 1.   A construction of standard decorated germ.

configuration of compact curves and curvettas. One obtains a decorated germ (C, l) such that $X(C, l)$ is the original singularity.

We introduce the following definition.

Definition 3.2.2.   Note that (C, l) is called *a standard decorated germ* if it was obtained in the previous way.

An example of the previous process applied to a cyclic quotient singularity is indicated in Figure 1. A representative of the standard decorated germ obtained like this is $C = C_1 \cup C_2$, where C_1 is defined by $y^2 - x^3 = 0$, C_2 is defined by $y = 0$, and $l_1 = 5, l_2 = 3$. One has $m(1) = 3, m(2) = 2$.

We wish to emphasize again that for standard decorated germs the integers l_i usually are larger than $m(i)$. The "right" bound is the following. Consider the *minimal embedded resolution* of $(C, 0) \hookrightarrow (\mathbb{C}^2, 0)$ by blowing-ups of closed points. That is, one does not only ask that the strict transform of C be smooth, but also that its total transform be a normal crossings divisor. For each $i \in \{1, \dots, r\}$, denote by $M(i)$ (see [7, p. 456]) the sum of multiplicities of the strict transforms of C_i, before arriving at this embedded resolution (for instance, for the germ represented in Figure 1, one gets $M(1) = 4$ and $M(2) = 2$). One may see easily that (C, l) is a standard decorated germ if and only if $l_i \geq M(i) + 1$, for all $i \in \{1, \dots, r\}$.

Consider again an *arbitrary* decorated germ (C, l). The total multiplicity of C_i with respect to C may be encoded also as the unique subscheme of length $m(i)$ supported on the preimage of 0 on the normalization of C_i. This allows us to define the *total multiplicity scheme $m(C)$* of any reduced curve contained in a smooth complex surface, to be the union of the total multiplicity schemes of all its germs. The deformations of (C, l) considered by de Jong and van Straten are:

Definition 3.2.3.   (i) [7, (4.1)] Given a smooth complex analytic surface Σ, a pair (C, l) consisting of a reduced curve $C \hookrightarrow \Sigma$ and a subscheme l of the normalization $\tilde{C}$ of C is called *a decorated curve* if $m(C)$ is a subscheme of l.
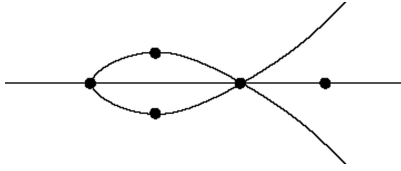
Fig. 2.   A picture deformation of a standard decorated germ.

(ii) [7, p. 476] A 1-*parameter deformation* of a decorated curve (C, l) over a germ of a smooth curve $(S, 0)$ consists of:

(1) a δ-constant deformation $C_S \to S$ of C;

(2) a flat deformation $l_S \subset \tilde{C}_S = \tilde{C} \times S$ of the scheme l, such that;

(3) $m_S \subset l_S$, where the *relative total multiplicity scheme m_S of $\tilde{C}_S \to C_S$* is defined as the closure $\overline{\bigcup_{s \in S \setminus 0} m(C_s)}$.

(iii) A 1-parameter deformation (C_S, l_S) is called a *picture deformation* if for generic $s \neq 0$ the divisor l_s is reduced.

It is an immediate consequence of the definitions that for a picture deformation C_S, the singularities of $C_{s \neq 0}$ are only ordinary multiple points. Therefore, it is easy to draw a real picture of such a deformed curve, which motivates the name. As an example, in Figure 2 we represented a generic fiber $C_{s \neq 0}$ and the image of the support of the subscheme l_s of its normalization for one of the picture deformations of the standard decorated curve obtained in Figure 1.

de Jong and van Straten proved that part of the deformation theory of sandwiched surface singularities may be reduced to that of 1-parameter deformations of decorated germs.

Theorem 3.2.4 [7, (4.4)].   All the 1-parameter deformations of $X(C, l)$ are obtained by 1-parameter deformations of the decorated germ (C, l). Moreover, *picture deformations* provide all the smoothings of $X(C, l)$.

3.3   Picture deformations and the associated Milnor fibers

Consider a decorated germ (C, l) and one of its picture deformations (C_S, l_S). Fix a closed Milnor ball B for the germ $(C, 0)$. For $s \neq 0$ sufficiently small, C_s will have a representative in B, denoted by D, which meets ∂B transversally. As explained in Section 2.2, it is a union of *immersed disks* $\{D_i\}_{1 \leq i \leq r}$ canonically oriented by their complex structures (and whose

set of indices correspond canonically to those of $\{C_i\}_{1\leq i\leq r}$). The singularities of D consist of ordinary m-tuple points, for various $m \in \mathbb{N}^*$.

Denote by $\{P_j\}_{1\leq j\leq n}$ the set of images in B of the points in the support of l_s. It is a finite set of points that contains the singular set of D (because $m_s \subset l_s$ for $s \neq 0$), but it might contain some other "free" points as well. There is a priori no preferred choice of their ordering. Hence, the matrix introduced next is well defined only *up to permutations of columns*. By contrast, the ordering of lines is determined by the fixed order of the components of C.

Definition 3.3.1 [7, p. 483].   The *incidence matrix* of a picture deformation (C_S, l_S) is the matrix $\mathcal{I}(C_S, l_S) \in \mathrm{Mat}_{r,n}(\mathbb{Z})$ (with r rows and n columns) whose entry $m_{ij} \in \mathbb{N}$ at the intersection of the ith row and the jth column is equal to the multiplicity of P_j as a point of D_i.

Such a matrix satisfies the following necessary conditions:

$$\sum_{j=1}^{n} \frac{m_{ij}(m_{ij}-1)}{2} = \delta_i, \quad \text{for all } i \in \{1, \ldots, r\}, \tag{3.3.2}$$

where δ_i denotes the δ-invariant of the branch C_i;

$$\sum_{j=1}^{n} m_{ij} m_{kj} = C_i \cdot C_k, \quad \text{for all } i, k \in \{1, \ldots, r\}, i \neq k, \tag{3.3.3}$$

where $C_i \cdot C_k$ stands for the intersection multiplicity at 0 of the branches C_i and C_k;

$$\sum_{j=1}^{n} m_{ij} = l_i, \quad \text{for all } i \in \{1, \ldots, r\}. \tag{3.3.4}$$

The above equations are consequences of Definition 3.2.3(ii) and of the fact that $C_{s\neq 0}$ has only ordinary singularities.

It is an open problem *to determine, in general, the set of incidence matrices associated with a given decorated germ, among the matrices that satisfy simultaneously* (3.3.2)–(3.3.4). It is also an open problem *to determine the union of the sets of incidence matrices associated with all decorated germs with a given topology.*

The Milnor fiber of a smoothing associated to a picture deformation of a standard decorated germ (C, l) is recovered as follows. Let

$$(\tilde{B}, \tilde{D}) \xrightarrow{\ \beta\ } (B, D) \tag{3.3.5}$$

be the simultaneous blow-up of the points P_j of D. Here, $\tilde{D} := \cup_{1 \le i \le r} \tilde{D}_i$, where $\tilde{D}_i$ is the strict transform of the disk D_i by the modification β. Consider pairwise disjoint compact tubular neighborhoods of the disks $\tilde{D}_i$ in $\tilde{B}$ and denote their interiors by T_i.

Proposition 3.3.6 [7, (5.1)].   Let (C, l) be a standard decorated germ. Then the Milnor fiber of the smoothing of $X(C, l)$ corresponding to the picture deformation (C_S, l_S) is orientation-preserving diffeomorphic to the compact oriented manifold with boundary $W := \tilde{B} \setminus (\bigcup_{1 \le i \le r} T_i)$ (whose corners are smoothed).

Moreover, one can show that in this realization, the canonical identification of the boundaries of the Milnor fibers corresponding to different smoothing components is the unique identification (up to isotopy) which extends the identity morphism on the complement in $\partial \tilde{B}$ of some tubular neighborhood of ∂C. Furthermore, by this presentation (for a fixed (C, l)) one can also see the canonical identification of all the boundaries of the Milnor fibers with the boundary $\partial(X, x)$; see also Lemma 4.2.4 and the paragraphs following it.

We mention that although many results regarding decorated germs are valid for any $l_i \ge m(i)$, in Proposition 3.3.6, one needs $l_i \ge M(i) + 1$, i.e., standard decorated germs (a fact omitted in [7, (5.1)]). Without this hypothesis, Proposition 3.3.6 is false, as one may see by considering more than one curvetta attached to a (-1)-curve: by varying the number of such curvettas, one gets an infinite number of manifolds W up to diffeomorphisms, the associated sandwiched singularities having, by contrast, a fixed topology.

4   From a Milnor Fiber to the Incidence Matrix

In this section, we prove our main results: Theorems 4.3.4 and 4.4.1. The main idea of the proofs, inspired by the article [10] of Lisca, is to close all the Milnor fibers with the same "cap" (a 4-manifold with boundary) that is glued each time in the "same way" (this makes sense thanks to the canonical identifications up to isotopy of the boundaries of the Milnor fibers). The cap we use emerges naturally from the viewpoint of de Jong and van Straten.

In this section, we suppose that (C, l) is a *standard* decorated germ. We keep the notations of Section 3.

4.1 Markings of oriented sandwiched 3-manifolds

In this subsection, we reinterpret the ambiguity of the choice of the embedding of the fixed sandwiched graph into a smooth one as a *marking*.

Definition 4.1.1. An oriented 3-manifold is called a *sandwiched manifold* if it is orientation-preserving diffeomorphic to the oriented boundary $\partial(X, x)$ of a sandwiched surface singularity (X, x). Any such singularity is called a *defining singularity* of M.

By Neumann's theorem recalled in Section 2.3, sandwiched 3-manifolds determine the sandwiched graph associated with any defining singularity. By Theorem 2.3.1, they may also be endowed canonically, up to an isotopy, with a plumbing structure $\mathcal{T}_{min}$ corresponding to the minimal resolution of a defining singularity. We say that the connected components of the complement in M of the tori of $\mathcal{T}_{min}$ are the *pieces* of M.

Suppose now that $(X, x) = X(C, l)$, where (C, l) is a standard decorated germ with $(C, 0) \subset (\mathbb{C}^2, 0)$. Let $(\Sigma, E(\pi)) \xrightarrow{\pi} (\mathbb{C}^2, 0)$ be the composition of blow-ups of points infinitely near 0 determined by (C, l). One may write

$$E(\pi) = E(C, l) + \sum_{i=1}^{r} E_i,$$

where $E(C, l)$ is the exceptional divisor of the minimal resolution of (X, x) and $(E_i)_{1 \le i \le r}$ are all the (-1)-curves contained in the exceptional divisor $E(\pi)$, numbered such that E_i is the unique irreducible component of $E(\pi)$ that intersects the strict transform of C_i. Denote by F_i the unique irreducible component of $E(C, l)$ that intersects E_i. To F_i corresponds a well-defined piece of M. In this way, one gets a map from the set $\{1, \ldots, r\}$ to the set of pieces of M.

Definition 4.1.2. A map from $\{1, \ldots, r\}$ to the set of pieces of M obtained as above is called *a marking* of M.

We see that each choice of defining singularity of M of the form $X(C, l)$, where (C, l) is a standard decorated germ, determines a well-defined marking of M. The problem of finding the markings of M is equivalent to the problem of finding the extensions of a sandwiched graph to a smooth graph (see Section 3.1). As far as we know, there is no closed formula computing the number of such extensions in terms of the sandwiched graph.
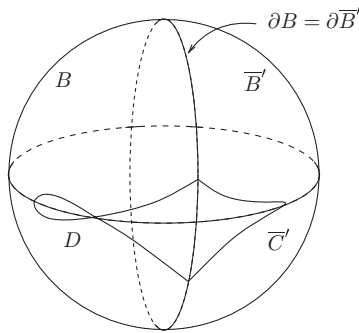
Fig. 3.   Gluing a curve and one of its picture deformations.

4.2 Closing the boundary of the Milnor fiber

Let (C_S, l_S) be a picture deformation of the decorated germ (C, l).

As the disk-configuration D is obtained by deforming C, its boundary $\partial D := \cup_{1 \le i \le r} D_i \hookrightarrow \partial B$ is isotopic as an oriented link to $\partial C \hookrightarrow \partial B$. Therefore, we can isotope D outside a compact ball containing all the points P_j till its boundary coincides with the boundary of C. From now on, D will denote the result of this isotopy. Let (B', C') be a second copy of (B, C) and define (see Figure 3):

$$(V, \Sigma) := (B, D) \cup_{id} (\overline{B}', \overline{C}').$$

Here, V is the oriented 4-sphere obtained by gluing the boundaries of B and $\overline{B}'$ by the tautological identification, and $\Sigma := \cup_{i=1}^r \Sigma_i$, where Σ_i is obtained by gluing D_i (perturbed by the above isotopy) and $\overline{C}'_i$ along their common boundaries. Recall that $\overline{C}'_i$ is a topological disk, while D_i is an immersed disk. Moreover, one can also glue $(\overline{B}', \overline{C}')$ with $(\tilde{B}, \tilde{D})$ in such a way that the blow-up morphism β of (3.3.5) may be extended by the identity on $\overline{B}'$, yielding

$$(\tilde{V}, \tilde{\Sigma}) \xrightarrow{\beta} (V, \Sigma).$$

Here, $\tilde{\Sigma} := \cup_{i=1}^r \tilde{\Sigma}_i$, where $\tilde{\Sigma}_i$ denotes the strict transform of Σ_i, i.e., $\tilde{\Sigma}_i = \tilde{D}_i \cup \overline{C}'_i$. In particular, $\tilde{\Sigma}_i$ is a topologically *embedded* 2-sphere in $\tilde{V}$, which is smoothly embedded in $\tilde{V} \setminus 0'$ (it is smoothly embedded everywhere if and only if the branch C_i is smooth).

Lemma 4.2.1. The intersection numbers of the oriented spheres $(\tilde{\Sigma}_i)_{1 \le i \le r}$ inside the oriented 4-manifold $\tilde{V}$ are the following:

$$\begin{cases} \tilde{\Sigma}_i^2 = -l_i - 2\delta_i, & \text{for all } i \in \{1, \dots, r\}, \\ \tilde{\Sigma}_i \cdot \tilde{\Sigma}_j = -C_i \cdot C_j, & \text{for all } i < j. \end{cases}$$

Proof.   Fix $i \in \{1, \ldots, r\}$ and, as in Definition 3.3.1, denote by m_{ij} the multiplicity of P_j as a point of the curve D_i. By combining (3.3.2) and (3.3.4), we get

$$\sum_{j=1}^{n} m_{ij}^2 = l_i + 2\delta_i. \tag{4.2.2}$$

By blowing-up a point of multiplicity $m \in \mathbb{N}^*$ of a compact complex curve on a smooth complex surface, its self-intersection drops by m^2. This is also true if one looks simply at a 2-cycle on an oriented 4-manifold that becomes holomorphic in an adequate chart containing the point to be blown-up. Applying this last fact to $\Sigma_i \hookrightarrow V$, which is holomorphic as seen inside a neighborhood of the points $P_j \in B \hookrightarrow \mathbb{C}^2$, we get

$$\tilde{\Sigma}_i^2 = \Sigma_i^2 - \sum_{j=1}^{n} m_{ij}^2 \stackrel{(4.2.2)}{=} -l_i - 2\delta_i,$$

since $\Sigma_i^2 = 0$ by the fact that $H_2(V) = 0$. The first family of formulae is proved.

The second family of formulae results simply from the fact that $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_j$ intersect only at $0'$, and that $\overline{B}'$ is a copy of B with changed orientation.   ∎

Write $T := \bigcup_{1 \leq i \leq r} T_i$ and also set

$$U := \overline{B}' \cup T. \tag{4.2.3}$$

Since $W = \tilde{B} \setminus T$ (see Proposition 3.3.6), the closed oriented 4-manifold $\tilde{V}$ is obtained by closing the boundary of W by the cap U. This cap is independent of the chosen picture deformation.

Lemma 4.2.4. U depends only on (C, l) and is independent of the chosen picture deformation (therefore one may close all the different Milnor fibers using the same U). In fact, each T_i in U is a four-dimensional handle of index 2 glued to $\overline{B}'$ along the knot $\partial C_i \hookrightarrow \partial \overline{B}'$ endowed with the $(-l_i - 2\delta_i)$-framing.

Proof.   The chosen framing index is equal to the self-intersection number of a surface obtained by gluing the core of the handle to a 2-chain contained in the 4-ball $\overline{B}'$ and whose boundary is ∂C_i. By taking $\tilde{D}_i$ as core disk and $\overline{C}'_i$ as 2-chain, we get the sphere $\tilde{\Sigma}_i$. We apply then the first family of formulae stated in Lemma 4.2.1.   ∎

Moreover, the cap U is always glued in the same way, up to isotopy, to the boundaries of the Milnor fibers corresponding to the various picture deformations identified by the natural diffeomorphisms. To see this, note first that T_i may be seen as a 2-handle

attached to $\overline{B}'$, with core disk $\tilde{D}_i$. Take a co-core K_i. It is a core of T_i seen as a 2-handle attached to W. In order to describe the attaching of U to W, it is enough to describe the link $\cup_{i=1}^r \partial K_i$ in W and its associated framing up to isotopy.

In the following lemma, we also use the notations of Section 4.1.

Lemma 4.2.5. Let (C, l) be a standard decorated germ. Consider on ∂W the plumbing structure corresponding to the minimal resolution of $X(C, l)$. Consider then an ordered collection of r regular fibers $f_1, \ldots, f_r$, where f_i is the generic fiber of the piece of W marked by i. Each knot f_i is endowed with the framing coming from the plumbing structure, increased by 1. Then one obtains a framed link isotopic to the attaching framing of T.

Proof. Note that the orientation of ∂W (as induced from W) coincides with the orientation induced from $\overline{U}$. In what follows, we will reason inside $\overline{U} = B' \cup \overline{T}$. Consider on B' the natural complex structure, as a copy of B. Then $C' \hookrightarrow \tilde{\tilde{\Sigma}}_i$ is holomorphic with respect to this complex structure and we may consider the sequence of blow-ups above $0' \in B'$ dictated by the copy (C', l') of (C, l). Let $\overline{U}_\alpha \xrightarrow{\alpha} \overline{U}$ be this total blow-up morphism and Σ_i^α be the strict transform of $\tilde{\tilde{\Sigma}}_i$ by the morphism α.

Denote by $(m_0(i), \ldots, m_{I(i)}(i))$ the multiplicity sequence associated to $C_i \hookrightarrow C$. The total multiplicity of C_i with respect to C is given by $m(i) = \sum_{k=0}^{I(i)} m_k(i)$. One has $l_i = m(i) + h_i$, where $h_i \in \mathbb{N}$, which implies that

$$(\Sigma^\alpha)^2 = (\tilde{\tilde{\Sigma}}_i)^2 - \sum_{k=0}^{I(i)} m_k(i)^2 - h_i. \tag{4.2.6}$$

But $\delta_i = \sum_{k=0}^{I(i)} \frac{m_k(i)(m_k(i)-1)}{2}$ by formula (2.2.1), which implies that $\sum_{k=0}^{I(i)} m_k(i)^2 = 2\delta_i + m(i)$. Therefore,

$$\sum_{k=0}^{I(i)} m_k(i)^2 + h_i = 2\delta_i + m(i) + h_i = 2\delta_i + l_i.$$

From formula (4.2.6) and Lemma 4.2.1, we deduce that $(\Sigma^\alpha)^2 = 0$.

The total space $\overline{U}_\alpha$ appears as a tubular neighborhood of a configuration of spheres whose dual graph is obtained from the dual graph of the exceptional divisor of α by attaching a (0)-vertex to each (−1)-vertex. The co-cores of the handles $\overline{T}_i$ appear as curvettas for Σ_i^α. An easy application of plumbing calculus (see [13]) finishes the proof.   ∎

4.3 The reconstruction of the incidence matrix from the Milnor fiber

Fix a picture deformation (C_S, l_S) of (C, l).

Lemma 4.3.1. Up to permutations, there exists a unique basis $(e_j)_{1 \le j \le n}$ formed by (-1)-classes of $H_2(\tilde{V})$ such that the matrix

$$N(C_S, l_S) := ([\tilde{\Sigma}_i] \cdot e_j)_{1 \le i \le r, 1 \le j \le n}$$

has only nonnegative entries.

Proof. Denote by $\sigma_j := \beta^{-1}(P_j)$ the 2-spheres embedded in $\tilde{V}$ obtained by blowing-up the points P_j. We suppose them oriented by the complex structure existing on $\tilde{B}$. Their homology classes are (-1)-classes in $H_2(\tilde{V})$. As V is diffeomorphic to a 4-sphere, we deduce that $([\sigma_j])_{1 \le j \le n}$ is a basis of the free group $H_2(\tilde{V})$. Moreover, the spheres being disjoint, the intersection form on $H_2(V)$ is in canonical form (the opposite of a sum of squares) in this basis. This shows that any (-1)-class is equal to some $\pm [\sigma_j]$.

By construction, if we take the basis $([\sigma_j])_{1 \le j \le n}$, the matrix $([\tilde{\Sigma}_i] \cdot [\sigma_j])_{1 \le i \le r, 1 \le j \le n}$ has only nonnegative entries (in fact, elements of $\{0, +1\}$). Suppose that we change the sign of one of the classes $[\sigma_j]$. As the corresponding column contains at least one nonzero entry (because the point P_j lies on at least one of the curves D_i), we deduce that the new matrix has some negative entries in that column. This proves the lemma. ∎

Proposition 4.3.2. Let $\mathcal{I}(C_S, l_S)$ be the incidence matrix of the picture deformation (C_S, l_S). Then the matrices $\mathcal{I}(C_S, l_S)$ and $N(C_S, l_S)$ are equal, up to permutations of columns.

Proof. The previous proof shows that, up to permutation of columns,

$$N(C_S, l_S) = ([\tilde{\Sigma}_i] \cdot [\sigma_j])_{1 \le i \le r, 1 \le j \le n}.$$

But $[\tilde{\Sigma}_i] \cdot [\sigma_j]$ is equal to the incidence number between the curve D_i and the point P_j, that is, to the (i, j)-entry m_{ij} of the matrix $\mathcal{I}(C_S, l_S)$. The proposition is proved. ∎

The next theorem explains that one may reconstruct the incidence matrix of a picture deformation of (C, l) from the associated Milnor fiber with marked boundary.

Theorem 4.3.3. The incidence matrix $\mathcal{I}(C_S, l_S)$ associated to a picture deformation of a standard decorated germ (C, l) is determined (up to a permutation of its columns) by the associated Milnor fiber and the marking of $\partial(X(C, l))$.

Proof.   Consider the compact 4-manifold with boundary U described in Lemma 4.2.4. Attach it to the Milnor fiber $W(C_S, l_S)$ as described in Lemma 4.2.5. This attaching is determined by the marking of $\partial(X(C, l))$. Then use Proposition 4.3.2.   ∎

Consider now two decorated germs of plane curves. One says that they are *topologically equivalent* if one chooses Milnor representatives of them inside euclidean balls centered at the origin of $\mathbb{C}^2$ and homeomorphisms between the balls which restrict to homeomorphisms between the representatives and which additionally respect the numberings of their branches and the decoration functions l.

One may show easily that topologically equivalent decorated germs give rise to topologically equivalent sandwiched singularities with induced diffeomorphisms of their boundaries preserving the associated markings.

Look at the diffeomorphisms between the boundaries of the two singularities that preserve the orientations and the markings. One may consider those diffeomorphisms between fillings of their boundaries that restrict to such special diffeomorphisms on the boundaries.

Theorem 4.3.4.   Consider two topologically equivalent standard decorated germs of plane curves, and for each one of them a picture deformation. If their incidence matrices are different up to permutation of columns, then their associated Milnor fibers are not diffeomorphic by an orientation-preserving diffeomorphism that preserves the markings of the boundaries.

Proof.   This is an immediate consequence of Theorem 4.3.3.   ∎

4.4   Consequences for the existence of Stein fillings

In Section 2, we recalled the notion of abstract boundary $\partial(X, x)$ of an isolated singularity (X, x) of arbitrary dimension. In fact, by considering the field of maximal complex subspaces of the tangent space to a representative of the boundary, one gets a contact structure. We denote it by $(\partial(X, x), \xi(X, x))$ and call it the *contact boundary* of (X, x). It is well defined up to contactomorphisms that are unique up to isotopy (see [18] and [2]).

Any Milnor fiber of an isolated equidimensional singularity (X, x) may be endowed with the structure of Stein manifold compatible with the contact structure $\xi(X, x)$ on $\partial(X, x)$.

By the main theorem of [2], for normal *surface* singularities the contact structure $\xi(X, x)$ is a topological invariant of the germ, that is, the contact boundaries of
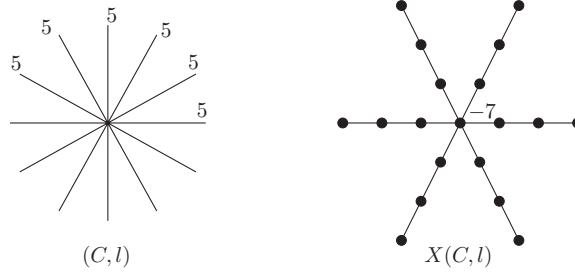
Fig. 4.   A decorated germ and the associated sandwiched graph.

two singularities with homeomorphic oriented boundaries are contactomorphic by an orientation-preserving diffeomorphism. Moreover, as proved in [1], this (unoriented) contact structure is left invariant by all orientation-preserving diffeomorphisms of the boundary $\partial(X, x)$, when this boundary is a rational homology sphere.

This happens in particular for sandwiched singularities. Therefore, we can speak about the *standard contact structure* of a sandwiched 3-manifold, well defined *up to isotopy*. By taking Milnor fibers of possibly nonanalytically equivalent but topologically equivalent sandwiched singularities, one gets Stein fillings of the same contact manifold (the boundaries being identified by any orientation-preserving diffeomorphism). We get the following result.

Theorem 4.4.1.   Let (M, ξ) be a sandwiched 3-manifold endowed with its standard contact structure. Fix the topological type of a defining standard decorated germ. Then, there are at least as many Stein fillings (up to diffeomorphisms fixed on the boundary) of (M, ξ) as there are incidence matrices (up to permutations of columns) realized by the picture deformations of all the decorated germs with the given topology.

4.5   An example

We consider the sandwiched singularities $X(C, l)$, where C is an ordinary 6-uple singularity and l takes the constant value 5. The total multiplicity of each component of C is equal to 1 with respect to C; therefore, one has to blow up 4 times on the strict transforms of the branches of C, starting from their separation produced by the blow-up of 0. As a consequence, (C, l) is a standard decorated germ and the sandwiched singularity $X(C, l)$ has as dual graph of its minimal resolution a star as in Figure 4, in which all the vertices distinct from the central one have attached self-intersection -2.

In particular, one may choose for C the union of six pairwise distinct lines passing through the origin of $\mathbb{C}^2$. As explained in [7, p. 501], one gets *in general* 323 possibilities
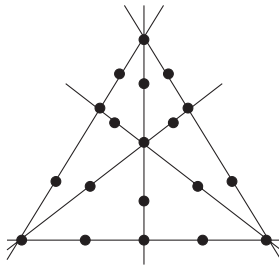
Fig. 5.   The special picture deformation.

for the incidence matrix of a picture deformation obtained by simply moving the lines by parallelism. But for special positions of the lines, one gets *one more* incidence matrix, the configuration generating it being drawn in Figure 5. By Theorem 4.4.1, we see that the contact boundary of such sandwiched singularities has at least 324 Stein fillings, up to orientation-preserving diffeomorphisms fixed on the boundary. We expect that these are *all* the Stein fillings of that contact manifold up to strong diffeomorphisms.

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