Proc. London Math. Soc. (3) 101 (2010) 554–588
doi:10.1112/plms/pdq007

On the Milnor fibres of cyclic quotient singularities

András Némethi and Patrick Popescu-Pampu

Abstract

The oriented link of the cyclic quotient singularity $\mathcal{X}_{p,q}$ is orientation-preserving diffeomorphic to the lens space $L(p,q)$ and carries the standard contact structure ξ_{st}. Lisca classified the Stein fillings of $(L(p,q), \xi_{st})$ up to diffeomorphisms and conjectured that they correspond bijectively through an *explicit* map to the Milnor fibres associated with the irreducible components (all of them being smoothing components) of the reduced miniversal space of deformations of $\mathcal{X}_{p,q}$. We prove this conjecture using the smoothing equations given by Christophersen and Stevens. Moreover, based on a different description of the Milnor fibres given by de Jong and van Straten, we also canonically identify these fibres with Lisca's fillings. Using these and a newly introduced additional structure (the order) associated with lens spaces, we prove that the above Milnor fibres are pairwise non-diffeomorphic (by diffeomorphisms which preserve the orientation and order). This also implies that de Jong and van Straten parametrize in the same way the components of the reduced miniversal space of deformations as Christophersen and Stevens.

1. *Introduction*

1.1. *Lisca's conjecture*

In [**20**], Lisca announced a classification of the symplectic fillings of the standard contact structure on lens spaces up to orientation-preserving diffeomorphisms. Detailed proofs were given in [**21**].

We recall briefly his classification. Let $L(p,q)$ be an oriented lens space. Lisca provides first by surgery diagrams a list of compact oriented 4-manifolds $W_{p,q}(\underline{k})$ with boundary $L(p,q)$. They are parametrized by a set $K_r(p/(p-q))$ of sequences of integers $\underline{k} \in \mathbb{N}^r$ (for its definition see (4.1.3)). He showed that each manifold $W_{p,q}(\underline{k})$ admits a structure of Stein surface, filling the standard contact structure on $L(p,q)$, and that any symplectic filling of this standard contact structure is orientation-preserving diffeomorphic to a manifold obtained from one of the $W_{p,q}(\underline{k})$ by a composition of blow-ups (that is, in the language of differential topology, by doing connected summing with the complex projective plane endowed with the opposite orientation).

Particular cases of his theorem had been proved before by Eliashberg [**13**] (for $\mathbb{S}^3$) and McDuff [**23**] (for the spaces $L(p,1)$ for all $p \geqslant 2$).

In general, the oriented diffeomorphism type of the boundary and the parameter $\underline{k}$ do not determine uniquely the (orientation-preserving) diffeomorphism type of the fillings: for some pairs the corresponding types might coincide (they are also listed by Lisca).

Lisca, following the works of Christophersen [**12**] and Stevens [**35**], noted that, $K_r(p/(p-q))$ parametrizes also the irreducible components of the reduced miniversal base space of deformations of the cyclic quotient singularity $\mathcal{X}_{p,q}$. The oriented link of this singularity is precisely a lens space $L(p,q)$. Each component of the miniversal space is in this case a smoothing component, that is, the generic local fibre over it is smooth. Its oriented differentiable type is

Received 4 May 2009; revised 27 October 2009; published online 31 March 2010.

2000 *Mathematics Subject Classification* 32S55, 53D10, 32S25, 57R17.

This research was supported by Balaton Project F42/2006 (OMFB-0053/2007). The first author was partially supported by OTKA Grants.

independent of the choice of the generic point, and is called *the Milnor fibre* of that component. By construction, the Milnor fibre is orientation-preserving diffeomorphic to a Stein filling of $(L(p,q),\xi_{st})$. Lisca gave the following conjecture.

CONJECTURE 1.1.1 [**21**, p.768]. The Milnor fibre of the irreducible component of the reduced miniversal base space of the cyclic quotient singularity $\mathcal{X}_{p,q}$, parametrized in [**35**] by $\underline{k} \in K_r(p/(p-q))$, is diffeomorphic to $W_{p,q}(\underline{k})$.

On the other hand, in [**16**], de Jong and van Straten studied by an approach completely different from Christophersen and Stevens the deformation theory of cyclic quotient singularities (as a particular case of sandwiched singularities). They also parametrized the Milnor fibres of $\mathcal{X}_{p,q}$ using the elements of the set $K_r(p/(p-q))$. Therefore, one can formulate the previous conjecture for their parametrization as well.

1.2. *The main results and their consequences*

(1) We introduce an additional structure associated with any (non-necessarily oriented) lens space: the 'order'. Its meaning in short is the following: geometrically it is a (total) order of the two solid tori separated by the (unique) splitting torus of the lens space; in plumbing language, it is an order of the two ends of the plumbing graph (provided that this graph has at least two vertices). Then we show that the oriented diffeomorphism type and the order of the boundary, together with the parameter $\underline{k}$ determine uniquely the filling.

(2) We endow in a natural way all the boundaries of the spaces involved (Lisca's fillings $W_{p,q}(\underline{k})$, Christophersen–Stevens' Milnor fibres $F_{p,q}(\underline{k})$ and de Jong–van Straten's Milnor fibres $W(\underline{a},\underline{k})$) with orders; the corresponding spaces with these extra-structure will be distinguished by *. Then we prove that all these spaces are connected by orientation-preserving diffeomorphisms that preserve the order of their boundaries: $W_{p,q}(\underline{k})^* \simeq F_{p,q}(\underline{k})^* \simeq W(\underline{a},\underline{k})^*$. This is an even stronger statement than the result expected by Lisca's conjecture since it eliminates the ambiguities present in Lisca's classification.

(3) In fact, we even provide a fourth description of the Milnor fibres: they are constructed by a minimal sequence of blow-ups of the projective plane which eliminates the indeterminacies of a rational function which depends on $\underline{k}$; see Corollary 8.4.11 and § 8.6(1). This is in the spirit of Balke's work [**3**].

(4) As a byproduct it follows (see § 10) that both Christophersen–Stevens and de Jong–van Straten parametrized the components of the miniversal base space in the same way (a fact not proved before, as far as we know).

(5) Moreover, we obtain that the Milnor fibres corresponding to the various irreducible components of the miniversal space of deformations of $\mathcal{X}_{p,q}$ are pairwise non-diffeomorphic by orientation-preserving diffeomorphisms which have restrictions to the boundaries that preserve the order.

1.3. *Symplectic fillings and singularities*

Our work may be considered as a continuation of the efforts to find all possible Stein or, more generally, symplectic fillings of the contact links of normal surface singularities. As a continuation of [**11**], in [**10**] we showed with Caubel that such contact structures are determined up to contactomorphism by the topology of the link, that is, they depend only on the topological type of the singularity, and not on its analytical type. Therefore, singularity theory gives the following Stein fillings up to diffeomorphism: the minimal resolution of good representatives (which may be made Stein by deformation of the complex structure; see [**7**]) and the Milnor fibres of the smoothings of *all* the analytical realizations of the given topological type.

A natural question is *to determine the topological types of normal surface singularities for which one gets in this way* all *the Stein fillings of the associated contact manifold, up to diffeomorphism.*

Ohta and Ono proved that this is the case for simple elliptic singularities (see [**27**]) and for simple singularities, that is, rational double points (see [**28**]). The above positive answer to Lisca's conjecture shows that this is also the case for cyclic quotient singularities.

We would like to stress some points regarding the previous classes of singularities. Both simple and cyclic quotient singularities are *taut* singularities, that is, their analytical type is determined by their topological type. Moreover, they are also *rational* singularities, hence their minimal resolution is diffeomorphic to the Milnor fibre of one of the smoothing components, the so-called *Artin component* (see [**6**, pp. 33–34]). By contrast, simple elliptic singularities are neither rational, nor taut, and their minimal resolution is not diffeomorphic to the Milnor fibre of some smoothing.

1.4. *Organization of the paper*

In § 2 we recall necessary facts about Hirzebruch–Jung continued fractions and their geometric interpretation, while § 3 contains some basic properties of cyclic quotient singularities and lens spaces, expressed in terms of the geometry of continued fractions. This section introduces the 'order' of the lens spaces too. Lisca's classification of the Stein fillings of lens spaces is presented in § 4. In § 4.4 we reformulate his result using the notion of order. Section 5 contains the results of Christophersen and Stevens regarding the structure of the reduced base of the miniversal deformation of cyclic quotient singularities. In § 6 we recall de Jong and van Straten's theory of deformations of sandwiched surface singularities using decorated curves, which is specialized to cyclic quotient singularities in § 7.

In these preliminary six sections we provide several details on the objects manipulated in order to try to make the paper readable both by singularity theorists and topologists interested in contact/symplectic topology. A considerable part of the preliminary material is used in the proofs (and the remaining part is conducive to the proper understanding of the main ideas/statements). On the other hand, even in these preliminary sections, most of the 'known' results are harmonized with the newly introduced notion of order.

The main new results are contained in the last three sections. In § 8 we prove the 'strong' version (cf. § 1.2) of Lisca's conjecture using the equations of Christophersen and Stevens describing the deformations of cyclic quotient singularities.

The identification of the Milnor fibres provided by the construction of de Jong and van Straten with the Stein fillings is done in § 10. The proof needs a generalization of Lisca's criterion for the recognition of each filling to a more homological criterion, which is in turn proved in § 9. The two most important consequences are listed in § 11.

1.5. *Conventions and notations*

All the manifolds we consider are oriented: any letter, say W, denoting a manifold will denote in fact an oriented manifold. We denote by $\overline{W}$ the manifold obtained by changing the orientation of W, and by ∂W its boundary, canonically oriented by the rule that the outward normal *followed* by the orientation of ∂W gives the orientation of W.

We work exclusively with (co)homology groups with integral coefficients.

If W is a 4-manifold, we denote by $Q_W : H_2(W) \times H_2(W) \to \mathbb{Z}$ its intersection form and by $\partial_W : H_2(W, \partial W) \to H_1(W)$ the boundary homomorphism. Additionally, if W has a non-empty boundary, and S_1 and S_2 are two 2-dimensional compact chains in W with disjoint boundaries which are contained in ∂W, then their intersection number is also well defined and is denoted by $S_1 \cdot S_2$ or $S_1 \cdot_W S_2$.

If $a, b \in \mathbb{N}^*$ and A is a commutative ring, we denote by $\mathrm{Mat}_{a,b}(A)$ the set of matrices with a rows and b columns with coefficients in A.

If M is an abelian group and $\mathbb{K}$ is a field, we write $M_{\mathbb{K}} := M \otimes_{\mathbb{Z}} \mathbb{K}$.

2. *Generalities on continued fractions and duality of supplementary cones*

2.1. *Hirzebruch–Jung continued fractions*

If $\underline{x} = (x_1, \dots, x_n)$ are variables, then the *Hirzebruch–Jung continued fraction*

$$[x_1, \dots, x_n] := x_1 - \cfrac{1}{x_2 - \cfrac{1}{\dots - \cfrac{1}{x_n}}} \tag{2.1.1}$$

can also be defined by induction on n through the formulae $[x_1] = x_1$ and $[x_1, \dots, x_n] = x_1 - 1/[x_2, \dots, x_n]$ for $n \geqslant 2$. One shows that

$$[x_1, \dots, x_n] = \frac{Z_n(x_1, \dots, x_n)}{Z_{n-1}(x_2, \dots, x_n)}, \tag{2.1.2}$$

where the polynomials $Z_n \in \mathbb{Z}[x_1, \dots, x_n]$ satisfy the inductive formulae

$$Z_n(x_1, \dots, x_n) = x_n \cdot Z_{n-1}(x_1, \dots, x_{n-1}) - Z_{n-2}(x_1, \dots, x_{n-2}) \quad \text{for all } n \geqslant 1, \tag{2.1.3}$$

with $Z_{-1} \equiv 0$, $Z_0 \equiv 1$ and $Z_1(x) = x$. In fact, $Z_n(\underline{x})$ equals the determinant of the matrix $M(\underline{x}) \in \mathrm{Mat}_{n,n}(\mathbb{Z})$, which has entries that are $M_{i,i} = x_i$, $M_{i,j} = -1$ if $|i - j| = 1$ and $M_{i,j} = 0$ otherwise. Hence, in addition to (2.1.3), they satisfy many 'determinantal relations', for example,

$$Z_n(x_1, \dots, x_n) = Z_n(x_n, \dots, x_1). \tag{2.1.4}$$

The referee has drawn our attention to the fact that continued fractions with negative signs were known much before the work of Jung and Hirzebruch on surface singularities; for instance, Cayley called them 'improper'.

DEFINITION 2.1.5 [**29**]. We say that $\underline{x} \in \mathbb{N}^n$ is *admissible* if the matrix $M(\underline{x})$ is positive semi-definite of rank at least $n - 1$. Denote by $\mathrm{adm}(\mathbb{N}^n)$ the set of admissible n-tuples.

If $\underline{x}$ is admissible and $n > 1$, then each $x_i > 0$. Moreover, if $[x_1, \dots, x_n]$ is admissible then $[x_n, \dots, x_1]$ is admissible too.

Each rational number $\lambda > 1$ admits a unique *Hirzebruch–Jung continued fraction expansion* (in short, an *HJ-expansion*) of the form

$$\lambda = [x_1, \dots, x_n], \quad \text{where } x_i \in \mathbb{N},\ x_i \geqslant 2 \ \text{ for all } \ i \in \{1, \dots, n\}.$$

2.2. *The geometrical interpretation*

Next we explain an interpretation of the HJ-expansions using affine geometry; for example, see [**26**, **30**].

Consider a free abelian group N of rank 2. An *oriented cone* is a rational strictly convex cone σ in $N_{\mathbb{R}}$ with a choice of an order of its edges. We denote the two primitive elements of N which generate the edges by e_1 and e_2, where (e_1, e_2) is the order of the edges. We denote by σ' the same cone with the opposite choice (e_2, e_1) of order of its edges, and by $\overline{\sigma}$ the *supplementary cone* generated by $(-e_1, e_2)$.

Consider the convex hull of the points of N situated in $\sigma \setminus 0$ and denote by P_σ the union of the compact edges of its boundary. It is a finite polygonal line joining e_1 to e_2. Denote by $v_0, \dots, v_{s+1}$ the lattice points situated on it, in the order in which they are encountered when one travels from $v_0 := e_1$ to $v_{s+1} := e_2$. Then, for some integers $b_i \geqslant 2$, we have

$$v_{i-1} + v_{i+1} = b_i v_i \quad \text{for all } i \in \{1, \dots, s\}. \tag{2.2.1}$$

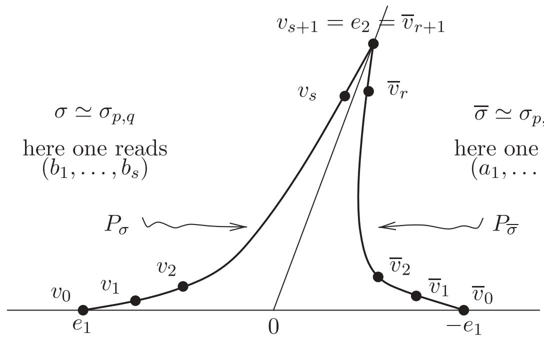


FIGURE 1. *Two supplementary cones.*

Write also $e_2 = -qv_0 + pv_1$. Then p and q are coprime with $p > q > 0$ provided that (e_1, e_2) is not a basis of N, and $p = 1$, $q = 0$ otherwise. From now one we suppose that we are in the first case.

LEMMA 2.2.2. *With the previous notation, $p/q = [b_1, \ldots, b_s]$.*

Both p/q and the sequence $(b_1, \ldots, b_s)$ are complete invariants of the pair (N, σ), up to isomorphisms (that is, isomorphisms of free groups that send one cone onto the other and preserve the order of the edges). We say that p/q is *the type* and $(b_1, \ldots, b_s)$ *the associated sequence* of the oriented cone (N, σ). If one changes the orientation of the cone (that is, σ into σ'), then the type of (N, σ') becomes p/q', where q' is the unique positive number such that $q' < p$ and $qq' \equiv 1 \pmod p$; the associated sequence becomes $(b_s, \ldots, b_1)$.

Consider now both the HJ-expansions

$$\frac{p}{q} = [b_1, \ldots, b_s] \quad \text{and} \quad \frac{p}{p-q} = [a_1, \ldots, a_r]. \tag{2.2.3}$$

There is a *duality* of the sequences $\underline{a}$ and $\underline{b}$. Its geometric interpretation is the following. Start from an oriented cone $\sigma \simeq \sigma_{p,q}$ in $N_{\mathbb{R}}$ of type p/q (well defined up to *unique* isomorphism). Consider its supplementary cone $\overline{\sigma}$, the polygonal line $P_{\overline{\sigma}}$ and the sequence $(\overline{v}_0, \ldots, \overline{v}_{r+1})$ of lattice points on it, starting from $\overline{v}_0 := -e_1$ and ending with $\overline{v}_{r+1} := e_2$ (see Figure 1).

LEMMA 2.2.4. *We have $\overline{\sigma} \simeq \sigma_{p,p-q}$, that is, the sequence associated with $(N, \overline{\sigma})$ is $(a_1, \ldots, a_r)$.*

There is an important point we wish to emphasize. In general, in torical presentations, cones and their dual cones are sitting in different lattices (dual to each other). Here, though the supplementary cone can canonically be identified with the dual cone (using the area-symplectic form equal to 1 on a positive basis of the lattice; see [**30**, p.145]), both σ and $\overline{\sigma}$ are represented in the *same* lattice N. This will allow us to connect by linear relations vectors from both cones in the same N (see Theorem 2.2.8). Such a relation interpreted in homology is important in the proof of the main result from § 10.

There is a canonical way to rewrite the entries of the continued fractions as follows, where $(2)^\ell$ means the constant sequence with ℓ terms all equal to 2, and $n_j \geqslant 3$ for all $1 \leqslant j \leqslant t$:

$$\frac{p}{p-q} = [(2)^{m_1-1}, n_1, (2)^{m_2-1}, n_2, \ldots, n_t, (2)^{m_{t+1}-1}]. \tag{2.2.5}$$

Geometrically, $t \in \mathbb{N}$ is the number of 'interior' vertices of the polygonal line $P_{\overline{\sigma}}$, and $(m_1, \ldots, m_{t+1})$ the sequence of integral lengths of the edges of $P_{\overline{\sigma}}$ (hence $m_i \geqslant 1$ for all $1 \leqslant i \leqslant t+1$).

From the arithmetical point of view of the HJ-expansions of p/q and $p/(p-q)$, the duality is reflected by Riemenschneider's point diagram [**31**], which basically says that

$$\frac{p}{q} = [m_1 + 1, (2)^{n_1-3}, m_2 + 2, (2)^{n_2-3}, \ldots, m_t + 2, (2)^{n_t-3}, m_{t+1} + 1]. \tag{2.2.6}$$

In particular,

$$r = 1 + \sum_{1 \leqslant i \leqslant s} (b_i - 2) = -1 + \sum_{1 \leqslant i \leqslant t+1} m_i \quad \text{and} \quad s = 1 + \sum_{1 \leqslant j \leqslant t} (n_j - 2). \tag{2.2.7}$$

However, there is an even deeper relation at the level of N (see [**30**, Proposition 5.3]).

Theorem 2.2.8. *Set* $w_l := v_{1+\sum_{1 \leqslant j \leqslant l-1}(n_j - 2)}$ *for all* $1 \leqslant l \leqslant t+1$. *Then*

$$\overline{v}_{i+1} - \overline{v}_i = w_l \quad \textit{if} \;\; m_1 + \ldots + m_{l-1} \leqslant i \leqslant m_1 + \ldots + m_l - 1. \tag{2.2.9}$$

For a detailed discussion of similar relations connecting a cone with its supplementary cone, and their relationship with continued fractions, see [**30**].

3. *Generalities on cyclic quotient singularities and lens spaces*

We recall the definitions of *cyclic quotient singularities* and *lens spaces*. Additionally, we introduce the notion of *order* associated with a lens space and we discuss its relationship with the group of automorphisms and dualities. See [**4**, pp. 99–105] for a classical presentation of cyclic quotient singularities, [**25**, **30**] for details about plumbings of links of surface singularities and [**8**, **30**] for the geometry of the splitting torus of lens spaces.

3.1. *The definitions*

Let p and q be coprime integers such that $p > q > 0$.

Definition 3.1.1. The *cyclic quotient* (or *Hirzebruch–Jung*) *singularity* $(\mathcal{X}_{p,q}, 0)$ is the germ of the quotient $\mathcal{X}_{p,q}$ of $\mathbb{C}^2$ by the action $\xi * (x, y) = (\xi x, \xi^q y)$ of the cyclic group $\{\xi \in \mathbb{C}\,,\ \xi^p = 1\} \simeq \mathbb{Z}/p\mathbb{Z}$. Its oriented link is the (oriented) *lens space* $L(p, q)$.

In particular, $L(p, q)$ is the quotient of $\mathbb{S}^3$ by the above action of $\mathbb{Z}/p\mathbb{Z}$ (this definition does not include $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$, which are sometimes also considered to be lens spaces). Bonahon [**8**] proved that *each lens space contains an embedded 2-dimensional torus — a so-called splitting torus —, unique up to an isotopy, which bounds on each side a solid torus*. The set $\mathcal{T}$ of solid tori bounded by a splitting torus, identified modulo isotopies of the ambient space, is a set of one or two elements. It has one element exactly when the solid tori can be interchanged by an isotopy. This happens exactly when $q \in \{1, p-1\}$; cf. [**8**, p.308].

Let us define an additional structure associated with an (unoriented) lens space. It has a similar nature as the notion of orientation ($\mathcal{T}$ is analogous to the set of connected components of the orientation bundle of a manifold), but it is independent of it.

Definition 3.1.2. An *order* of an (unoriented) lens space is a total order on the set $\mathcal{T}$.

Clearly, if $q \in \{1, p-1\}$, then this supports no additional information. In all other cases the order distinguishes *the first* and *the second* of the two (non-isotopic) solid tori bounded by *any* splitting torus.

The unicity of the splitting torus τ allows one to associate with any (unoriented) lens space L a free abelian group of rank 2, namely $N := H_1(\tau)$. In what follows $\nu_* : N \to H_1(L)$ will stay for the homological morphism induced by the inclusion $\nu : \tau \hookrightarrow L$.

REMARK 3.1.3. In fact, N is well defined up to the induced action of the isotopies of the lens space which move τ into itself. This is non-trivial only if $q \in \{1, p-1\}$, but even in those cases any such induced isomorphism $\varphi_N : N \to N$ satisfies $\nu_* \circ \varphi_N = \nu_*$.

3.2. *The order and its type*

Now we explain a way to extract the numbers $\{p/q, p/q'\}$ from an oriented lens space diffeomorphic to $L(p, q)$.

Let L be an *oriented* lens space. Choose an order of the two solid tori bounded by a splitting torus τ: denote by L_1 the first and by L_2 the second one. Orient τ as the boundary of L_1. Therefore, $N = H_1(\tau)$ gets an induced orientation (dual to the orientation of $H^1(\tau)$ such that the cup product of a positive basis is positive on the fundamental class of τ). There is up to isotopy a unique *meridian* of L_1 (that is, an unoriented simple closed curve on τ which is non-trivial homotopically on τ but is trivial in L_1). Orient it in an arbitrary way and denote by $e_1 \in N$ its homology class. Then orient the meridian of L_2 such that its homology class $e_2 \in N$ forms a positive basis (e_1, e_2) of N with respect to the orientation defined before. Denote by σ the oriented strictly convex cone in $N_{\mathbb{R}}$ generated by e_1 and e_2, taken in this order. Let p/q be the type of the oriented cone (N, σ); cf. (§ 2.2).

LEMMA 3.2.1. *The* 3*-manifold* L *is* (*orientation-preserving*) *diffeomorphic to* $L(p, q)$.

By choosing the opposite orientation of the meridian of L_1, one gets $(-e_1, -e_2)$ instead of (e_1, e_2), and hence $-\sigma$ instead of σ, which has a type that is also p/q. By changing the order of the solid tori, one gets as new type p/q'. This also reproves the classical fact that $L(p, q)$ is orientation-preserving diffeomorphic to $L(p, q_1)$ if and only if $q_1 \in \{q, q'\}$.

If $\#\mathcal{T} = 1$, then $p/q = p/q'$. If $\#\mathcal{T} = 2$ and we fix an order, then we get without ambiguity a unique element of $\{p/q, p/q'\}$. Hence, an order always provides a well defined element of the set $\{p/q, p/q'\}$, called the *type of the order*.

If $q' \neq q$, then from the type of the order one can recover the order itself. Indeed, the type contains the information regarding the oriented cone, which has ordered edges that correspond to an order of the two meridians. This is not the case for $q' = q \notin \{1, p-1\}$, since $\#\mathcal{T} = 2$, but $\#\{p/q, p/q'\} = 1$ (hence the order is a sharper invariant than its type).

Note that if we change the orientation of the above lens space $L(p, q)$, then in the above construction (e_1, e_2) can be replaced by $(-e_1, e_2)$ (that is, σ by $\overline{\sigma}$ of type $p/(p-q)$), and hence $\overline{L(p, q)} = L(p, p-q)$.

REMARK 3.2.2. The notation '$L(p, q)$' is not uniform in the literature: sometimes, what we call $L(p, q)$ is denoted by $L(p, p-q)$; the ambiguity originates in the orientation choice.

3.3. *Self-diffeomorphisms*

The previous discussion also provides the group of orientation-preserving self-diffeomorphisms $\mathrm{Diff}^+(L)$ of an oriented lens space L; cf. [8]. For any $\varphi \in \mathrm{Diff}^+(L)$ write φ_* for the induced

FIGURE 2. *The graph* $G(\underline{b})$.

morphism at the level of $H_1(L) \simeq \mathbb{Z}/p\mathbb{Z}$. Then the isotopy class of φ is uniquely determined by φ_*, and φ_* can only be multiplication by ± 1 or $\pm q$, and $\pm q$ occurs only if $q' = q$ (corresponding to how φ changes the orientation of the meridians and/or the solid tori).

If $q' \neq q$, then $\mathrm{Diff}^+(L)$ preserves automatically the order (that is, any of the two possible orders is left invariant). If $q = q' \notin \{1, p-1\}$, then φ reverses it exactly when φ_* is the multiplication by $\pm q$. Hence, once an order o is fixed, the subgroup $\mathrm{Diff}^{+,o}(L)$ of $\mathrm{Diff}^+(L)$ that preserves o is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and its elements induce $\varphi_* = \pm 1$ on $H_1(L) \simeq \mathbb{Z}/p\mathbb{Z}$, with the exception of $p = 2$, when $\mathrm{Diff}^{+,o}(L) \simeq \{\mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}}\}$.

3.4. *The order and the sequence* $\{v_1, \ldots, v_s\}$

If both an orientation and an order are fixed on a lens space (say by the choice of the numbering L_1 and L_2 of the solid tori bounded by a splitting torus), then the discussion from § 3.2 shows that the sequence $(v_1, \ldots, v_s)$ is well defined up to a sign and the automorphisms φ_N of N from Remark 3.1.3.

For such a situation later we use the following (ordered) set of elements of $H_1(L)$, well defined up to a sign (cf. Remark 3.1.3), associated with an oriented and ordered L:

$$(\alpha_1^L, \ldots, \alpha_s^L) := \pm(\nu_*(v_1), \ldots, \nu_*(v_s)) \in H_1(L)^s. \tag{3.4.1}$$

If in the above construction one changes the orientation of L and one keeps *the same order*, then one gets

$$(\alpha_1^{\overline{L}}, \ldots, \alpha_r^{\overline{L}}) := \pm(\nu_*(\overline{v}_1), \ldots, \nu_*(\overline{v}_r)) \in H_1(L)^r, \tag{3.4.2}$$

where the v_i and $\overline{v}_i$ are related as in § 2.2.

3.5. *The order and plumbings*

Consider again the surface $\mathcal{X}_{p,q}$. The sequences $\underline{a}$ and $\underline{b}$ from (2.2.3) guide two different geometrical packages: $\underline{a}$ is related more to the equations and deformations of $\mathcal{X}_{p,q}$, for example, see § 5 [**12**, **16**] use even the notation $\mathcal{X}_{p,q} = X(a_1, \ldots, a_r)$), while $\underline{b}$ is related to the resolution.

The dual graph $G(\underline{b})$ of the minimal resolution is a string; see Figure 2. Let $\Pi(\underline{b})$ be the oriented compact 4-manifold with boundary obtained by plumbing according to $G(\underline{b})$ (it is diffeomorphic to the minimal resolution of a Milnor representative of $(\mathcal{X}_{p,q}, 0)$). It contains oriented 2-spheres $\{S_i\}_{1 \leqslant i \leqslant s}$ which intersect according to the graph $G(\underline{b})$ and which are realized in the minimal resolution by the irreducible exceptional curves. By construction $\partial\Pi(\underline{b}) = L(p,q) = \overline{\partial\Pi(\underline{a})}$.

Note also that the permutation $(x, y) \mapsto (y, x)$ of the coordinates of $\mathbb{C}^2$ realizes an isomorphism $\mathcal{X}_{p,q} \to \mathcal{X}_{p,q'}$. Hence, *a priori* there is no preferred order of the coordinate axes or of the 2-spheres $\{S_i\}_{1 \leqslant i \leqslant s}$ if $s > 1$: the marking $\{S_i\}_{1 \leqslant i \leqslant s}$ can be replaced freely by $\{S_{s-i}\}_{1 \leqslant i \leqslant s}$p; but the ambiguity disappears (for both plumbing graphs $G(\underline{a})$ and $G(\underline{b})$) if $\#\mathcal{T} = 2$ and an order is fixed. We explain this in the following paragraphs.

With any marking $\{S_i\}_{1 \leqslant i \leqslant s}$ of the 2-spheres, associate a collection $(R_i)_{1 \leqslant i \leqslant s}$ of pairwise disjoint oriented discs properly embedded inside $\Pi(\underline{b})$, with R_i being a normal slice of S_i inside the associated part of the plumbing decomposition of $\Pi(\underline{b})$, and with $R_i \cdot_{\Pi(\underline{b})} S_i = +1$.

Assume that $q \notin \{1, p-1\}$, or equivalently, $s \neq 1 \neq r$, that is, the lengths of both $\underline{a}$ and $\underline{b}$ are greater than one. Let us concentrate now on $G(\underline{b})$ (there is a symmetric discussion for

$G(\underline{a})$ too). Any edge of the graph, via the plumbing construction, determines a splitting torus. Then ∂R_1 and ∂R_s sit in two different (non-isotopic) solid tori.

Each edge of $G(\underline{b})$ corresponds to a splitting torus of L. Choose an ordering (L_1, L_2) of the solid tori bounded by it such that the type of L endowed with the associated order is p/q. Then we mark the 2-spheres in such a way that $\partial R_1 \subset L_1$ and $\partial R_s \subset L_2$. This defines an order of 2-spheres. By this convention, not only the type p/q of the order and the ordered sequence $(b_1, \ldots, b_s)$ of weights of the graph correspond by Lemma 2.2.2, but even if this sequence is symmetric, we indicate in the plumbed manifold $\Pi(\underline{b})$ which 2-sphere has index 1. In fact, this specification is equivalent with a choice of an order.

REMARK 3.5.1. Regarding Figure 2, note the following. If $s > 1$, and we replace the 'symbols' b_i by some integers at least 2, then we get an unordered graph. On the other hand, with the present decoration we indicate which end-edge is S_1, respectively, S_s, and hence Figure 2 in fact represents an ordered graph providing an order of its plumbed lens space.

All the constructions of the present paper are guided by plumbing graphs. Using this, we introduce uniformly an order in all the lens spaces involved.

Assume that $s > 1$ and $r > 1$. Fix the graph $G(\underline{b})$ and its marking as in Figure 2. It defines an order (via the rule $R_1 \subset L_1$) of the plumbed manifold $\partial\Pi(\underline{b})$.

Construct compatible markings on $G(\underline{a})$ and $G(\underline{b})$ together with a canonical identification of the plumbing manifolds $\partial\Pi(\underline{b})$ and $\partial\Pi(\underline{a})$ as follows. Fix $G(\underline{b})$ with its marking, which provides an order of $\Pi(\underline{b})$. The orientation change of $\Pi(\underline{b})$ corresponds to the change of each weight $-b_i$ into b_i (but keeping the 1, respectively, s marking of the ends). Note that by a plumbing calculus of the plumbing graphs, we may keep track of the order of the lens space (the order of the end-vertices of the graph). Hence, when we do a plumbing calculus in order to replace the graph marked with the $\underline{b}$ into one which has weights $\underline{a}$, there is only one way to order/mark the sequence $\underline{a}$ in such a way that the order of the lens space will reflect properly the order of the sequence $\underline{a}$.

Finally, if a plumbing graph contains a subgraph having the information regarding the above markings of $G(\underline{a})$ or $G(\underline{b})$, we mark it in a compatible way.

DEFINITION 3.5.2. The order fixed in this way by the marked graphs is called the *preferred order* of the lens spaces; L endowed with a preferred order will be denoted by L^*. Similarly, $L(p,q)^*$ denotes the *ordered* lens space of type p/q.

3.6. *The order and coordinates*

An order of the plumbing/resolution graph can be related with a choice of an order of the coordinates of $\mathbb{C}^2$ in Definition 3.1.1 as follows.

Assume first that $\underline{b}$ is not symmetric. If we fix p and q and the coordinates (x, y) as in Definition 3.1.1, and we mark/order the 2-spheres of the resolution in such a way that the expansion $p/q = [b_1, \ldots, b_s]$ holds, then the strict transform of the image of the x-axis or of $\{x = 0\}$ can be isotoped to R_1 or R_s, respectively. We take this as a general compatibility property even if $\underline{b}$ is symmetric: the (order of the) coordinates (x, y) is *compatible* with the ordering of the 2-spheres if the above fact holds.

Finally, we verify another compatibility property. Note that each ∂R_i can be isotoped in $\partial\Pi(\underline{b})$ in a tubular neighbourhood of τ (which has N as its first homology).

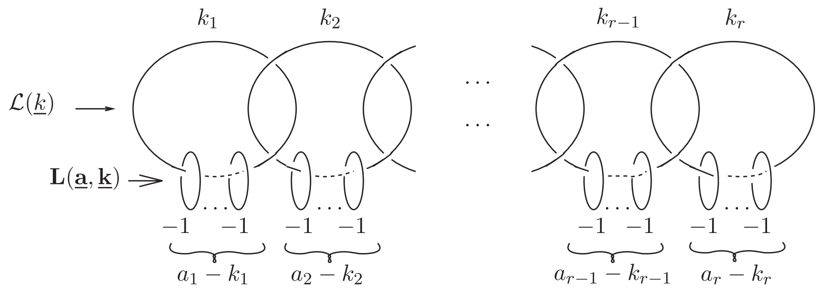


FIGURE 3. *The framed link* $L(\underline{a}, \underline{k}) \subset N(\underline{k})$.

PROPOSITION 3.6.1 [**30**, pp. 176–177]. *Let* L *be an ordered and oriented lens space. If* $q \notin \{1, p-1\}$, *then* $([\partial R_1], \dots, [\partial R_s]) = \pm(v_1, \dots, v_s)$ *in* N. *In fact, in all cases,* $(\nu_*[\partial R_1], \dots, \nu_*[\partial R_s]) = \pm(\nu_*(v_1), \dots, \nu_*(v_s))$ *in* $H_1(L)$.

4. *The Stein fillings of lens spaces, following Lisca*

4.1. *The set* K_r

The next parameter set supports all the three main constructions presented in the body of the paper.

DEFINITION 4.1.1 (Christophersen [**12**]). For $r \geqslant 1$ denote by

$$K_r := \{\underline{k} = (k_1, \dots, k_r) \in \mathrm{adm}(\mathbb{N}^r) \mid [k_1, \dots, k_r] = 0\} \tag{4.1.2}$$

the set of *admissible sequences which represent* 0. For $\underline{k} = (k_1, \dots, k_r) \in K_r$ set $\underline{k}' := (k_r, \dots, k_1) \in K_r$.

We wish to emphasize that the condition of *admissibility* (cf. Definition 2.1.5) is really restrictive. For example, $\underline{k} = (2, 1, 1, 1, 1, 2) \notin K_6$ although $[\underline{k}] = 0$. By admissibility, if $r > 1$, then each $k_i > 0$ and K_1 has only one element, namely (0). For any $r > 1$ the number of elements of K_r is the *Catalan number* $(1/r)\binom{2(r-1)}{r-1}$.

For two coprime integers p and q with $p > q > 0$, and the HJ-expansion $p/(p-q) = [a_1, \dots, a_r]$, set:

$$K_r(\frac{p}{p-q}) = K_r(\underline{a}) := \{\underline{k} \in K_r \mid \underline{k} \leqslant \underline{a}\} \subset K_r. \tag{4.1.3}$$

(Here, $\underline{k} \leqslant \underline{a}$ means that $k_i \leqslant a_i$ for all i.)

4.2. *Lisca's construction*

The field of complex lines tangent to $\mathbb{S}^3$ is left invariant by the cyclic action used in Definition 3.1.1, and hence it descends to the so-called *standard contact structure* ξ_{st} on $L(p, q)$.

Fix an element $\underline{k} \in K_r(\underline{a})$. Let $\mathcal{L}(\underline{k})$ be the framed link of Figure 3 with r components and decorations $k_1, \dots, k_r$ (that is, the thick components are neglected for a moment). Let $N(\underline{k})$ be the closed oriented 3-manifold given by surgery on $\mathbb{S}^3$ along $\mathcal{L}(\underline{k})$. Using the slam-dunk operation on rationally framed links in $\mathbb{S}^3$ (see [**14**, p.163]), one obtains an orientation-preserving diffeomorphism as follows:

$$\eta : N(\underline{k}) \longrightarrow \mathbb{S}^1 \times \mathbb{S}^2. \tag{4.2.1}$$

DEFINITION 4.2.2 [**21**, p.766]. Consider the diffeomorphism η from (4.2.1) and denote by $L(\underline{a},\underline{k}) \subset N(\underline{k})$ the thick framed link drawn in Figure 3. Define $W_{p,q}(\underline{k})$ to be the smooth oriented 4-manifold with boundary obtained by attaching 2-handles to $\mathbb{S}^1 \times \mathbb{D}^3$ along the framed link $\eta(L(\underline{a},\underline{k})) \subset \mathbb{S}^1 \times \mathbb{S}^2$.

From the Main Theorem 1.1 of [**21**], one can extract the following.

THEOREM 4.2.3.
(a) *All the manifolds* $W_{p,q}(\underline{k})$ *admit Stein structures that fill* $(L(p,q),\xi_{st})$, *and any Stein filling of* $(L(p,q),\xi_{st})$ *is diffeomorphic to one of the manifolds* $W_{p,q}(\underline{k})$.
(b) *The manifold* $W_{p,q_1}(\underline{k}_1)$ *is orientation-preserving diffeomorphic to* $W_{p,q_2}(\underline{k}_2)$ *if and only if* $(q_2,\underline{k}_2) = (q_1,\underline{k}_1)$ *or* $(q_2,\underline{k}_2) = (q_1',\underline{k}_1')$.

4.3. *Lisca's criterion to recognize* $W_{p,q}(\underline{k})$

Once Theorem 4.2.3(a) is proved, Theorem 4.2.3(b) follows from a straightforward homological computation, the essence of which is highlighted by the next criterion. Let W be a Stein filling of $(L(p,q),\xi_{st})$. Let V be the closed 4-manifold obtained by gluing W and $\Pi(\underline{a})$ via an orientation-preserving diffeomorphism $\phi : \partial W \to \partial\overline{\Pi(\underline{a})}$ of their boundaries. Let $\mu : \Pi(\underline{a}) \hookrightarrow V$ be the inclusion morphism.

PROPOSITION 4.3.1 [**21**, § 7]. *Denote by* $\{s_i\}_{1\leqslant i\leqslant r}$ *the classes of 2-spheres* $\{S_i\}_{1\leqslant i\leqslant r}$ *in* $H_2(\Pi(\underline{a}))$ (*listed in the same order as* $\{a_i\}_{1\leqslant i\leqslant r}$), *and also their images via the monomorphism* $\mu_* : H_2(\Pi(\underline{a})) \to H_2(V)$. *Then there is a* $\underline{k} \in K_r(\underline{a})$ *such that*

$$\#\{e \in H_2(V) \mid e^2 = -1,\ s_i \cdot e \neq 0,\ s_j \cdot e = 0 \text{ for all } j \neq i\} = 2(a_i - k_i) \tag{4.3.2}$$

for all $i \in \{1,\dots,r\}$. *In this way one gets the pair* $(\underline{a},\underline{k})$, *and* W *is orientation-preserving diffeomorphic to* $W_{p,q}(\underline{k})$.

Note that, as $\{S_i\}_{1\leqslant i\leqslant r}$ and $\{S_{r-i}\}_{1\leqslant i\leqslant r}$ cannot be distinguished, the algorithm does not differentiate $(\underline{a},\underline{k})$ from $(\underline{a}',\underline{k}')$, and hence $W_{p,q}(\underline{k})$ from $W_{p,q'}(\underline{k}')$.

One verifies that the above criterion is independent of the choice of the diffeomorphism ϕ, which has ambiguities that correspond to $\mathrm{Diff}^+(L(p,q))$. In fact, Lisca shows that even the diffeomorphism type of the resulting manifold V is independent of the possible choices of ϕ. For this he proves that $W_{p,q}(\underline{k})$ admits an orientation-preserving self-diffeomorphism that induces multiplication by -1 on $H_1(L(p,q))$ (cf. [**21**, (7.2)]), and clearly $\Pi(\underline{a})$ has a self-diffeomorphism that induces multiplication by q on $H_1(L(p,q))$, provided that $q' = q$.

4.4. *Compatibility with the order*

One can eliminate the ambiguity left by Theorem 4.2.3(b) using the order of the boundary. Note that if $r = 1$, or even if $r > 1$ but both sequences $\underline{a}$ and $\underline{k}$ are symmetric, then there is no ambiguity, since $(p,q,\underline{k}) = (p,q',\underline{k}')$.

Assume that we are in the remaining situations. Recall that all the time $\underline{a}$ and q are related by the expansion $[a_1,\dots,a_r] = p/(p-q)$. The point is that the framed link from Figure 3 is not symmetric. If we mark the link components of $\mathcal{L}(\underline{k})$ as in Figure 3, then by the rule described in § 3.5 (namely, by imposing $\partial R_1 \subset L_1$), we appoint the preferred order of the boundary of $W_{p,q}(\underline{k})$; cf. Definition 3.5.2. The filling obtained in this way (the space $W_{p,q}(\underline{k})$ with the preferred order on its boundary) is denoted by $W_{p,q}(\underline{k})^*$.

Note that no orientation-preserving diffeomorphism $W_{p,q}(\underline{k})^* \to W_{p,q'}(\underline{k'})^*$ (from Theorem 4.2.3(b)) preserves the preferred orders of the boundaries. Hence we have the following theorem.

THEOREM 4.4.1. *All the spaces* $W_{p,q}(\underline{k})^*$ *are different, and hence their boundaries* $L(p,q)^*$ *and* $\underline{k} \in K_r(\underline{a})$ *uniquely determine all the Stein fillings up to orientation-preserving diffeomorphisms that preserve the order of the boundary.*

Proposition 4.3.1 will have the following new form. Let W^* be a Stein filling of $(L(p,q), \xi_{st})$ with an order on its boundary. Consider $\Pi(\underline{a})^*$ with its preferred order (providing a well determined order of the s_i). Construct V as in Proposition 4.3.1, and consider the two pairs $(q, \underline{k})$ and $(q', \underline{k}')$ provided (but undecided) by Proposition 4.3.1.

PROPOSITION 4.4.2. *If* ϕ *preserves or reverses the orders of the boundary then* W^* *is orientation and order-preserving diffeomorphic to* $W_{p,q}(\underline{k})^*$ *or to* $W_{p,q'}(\underline{k}')^*$*, respectively.*

REMARK 4.4.3. Assume that $q = q'$. Then the permutation of the coordinates $(x, y) \mapsto (y, x)$ of $\mathbb{C}^2$ induces an automorphism of $\mathcal{X}_{p,q}$, and also of the miniversal deformation space. The permutation on its reduced components corresponds to $\underline{k} \mapsto \underline{k}'$ inducing an orientation-preserving diffeomorphism $W_{p,q}(\underline{k}) \to W_{p,q}(\underline{k}')$. As it follows from the above discussion, this diffeomorphism *does not* preserve the order, provided that $\underline{k} \neq \underline{k}'$.

5. *The smoothings of cyclic quotient singularities, following Christophersen and Stevens*

In this section we recall some results of Christophersen and Stevens on the structure of the reduced miniversal base space of cyclic quotients. For more details, see [**6**, **12**, **36**].

5.1. *Generalities on versal deformations*

DEFINITION 5.1.1. Let (X, x) be a germ of a complex analytic space. A *deformation* of (X, x) is a germ of flat morphism $\pi : (Y, y) \to (S, s)$ together with an isomorphism between (X, x) and the special fibre $\pi^{-1}(s)$. A deformation of (X, x) is *versal* if any other deformation is obtainable from it by a base-change. A versal deformation is *miniversal* if the Zariski tangent space of its base (S, s) has the smallest possible dimension. A *smoothing component* is an irreducible component of the miniversal base space over which the generic fibres are smooth.

If (X, x) is a germ of a reduced complex analytic space *with an isolated singularity*, then the following well known facts hold:

(i) (Grauert [**15**], Schlessinger [**33**]) The miniversal deformation π exists and is unique up to (non-unique) isomorphism.

(ii) (Artin [**2**]) If (X, x) is a rational surface singularity, then all the components of the reduced miniversal base space are smoothing ones.

(iii) (Looijenga [**22**]) There exist (Milnor) representatives Y_{red} and S_{red} of the reduced total and base spaces of π such that the restriction $\pi : \partial Y_{\text{red}} \cap \pi^{-1}(S_{\text{red}}) \to S_{\text{red}}$ is a trivial C^∞-fibration.

Hence, for each smoothing component, the oriented diffeomorphism type of the oriented manifold with boundary $(\pi^{-1}(s) \cap Y_{\text{red}}, \pi^{-1}(s) \cap \partial Y_{\text{red}})$ does not depend on the choice of the generic element s: it is called *the Milnor fibre* of that component. Moreover, its boundary is canonically identified with the link *up to isotopy*. In particular, the Milnor fibre over a smoothing component is a Stein filling of the link endowed with its standard contact structure (provided that the representatives are carefully chosen; see [**10**]).

5.2. *The equations of $\mathcal{X}_{p,q}$*

The singularity $\mathcal{X}_{p,q}$ may also be seen as the germ at the 0-dimensional orbit of the toric variety $\mathcal{Z}_{\sigma_{p,q}} = \text{Spec}\,\mathbb{C}[\check{\sigma}_{p,q} \cap M]$, where $\sigma_{p,q} \subset N_{\mathbb{R}}$ is an oriented cone in N of type p/q, and $M := \text{Hom}(N, \mathbb{Z})$. We identify $\check{\sigma}_{p,q}$ and $\overline{\sigma}_{p,q} \simeq \sigma_{p,p-q}$; cf. Figure 1. The lattice points $(\overline{v}_0, \ldots, \overline{v}_{r+1})$ are the minimal generating set of the semi-group $\overline{\sigma}_{p,q} \cap N \simeq \check{\sigma}_{p,q} \cap M$. Therefore, the monomials

$$z_i := \chi^{\overline{v}_i} \quad \text{for all } i \in \{0, \ldots, r+1\}, \tag{5.2.1}$$

generate the toric algebra $\mathbb{C}[\check{\sigma}_{p,q} \cap M]$. Hence, the toric surface $\mathcal{Z}_{\sigma_{p,q}}$ may be embedded inside $\mathbb{C}^{r+2}$ using the regular functions $z_0, \ldots, z_{r+1}$. A very elegant way to write the equations of $\mathcal{X}_{p,q}$ is given by Riemenschneider [**32**] via a *quasi-determinant* as follows:

$$\begin{vmatrix} z_0 & & z_1 & \ldots & z_{r-1} & & z_r \\ z_1 & & z_2 & \ldots & z_r & & z_{r+1} \\ & z_1^{a_1-2} & & \ldots & & z_r^{a_r-2} & \end{vmatrix} \tag{5.2.2}$$

The *generalized minors* of the quasi-determinant

$$\begin{vmatrix} f_0 & & f_1 & \ldots & f_{r-1} & & f_r \\ g_0 & & g_1 & \ldots & g_{r-1} & & g_r \\ & h_{0,1} & & \ldots & & h_{r-1,r} & \end{vmatrix}$$

are given by

$$E^{(i,j)} := f_{i-1}g_j - g_{i-1}f_j \cdot \prod_{\ell=i}^{j} h_{\ell-1,\ell} \quad \text{for } 1 \leqslant i \leqslant j \leqslant r. \tag{5.2.3}$$

The equations of $\mathcal{X}_{p,q}$ are given by the vanishing of the generalized minors of (5.2.2); we refer to them as $\mathcal{E}$. They include the equations

$$z_{i-1}z_{i+1} - z_i^{a_i} = 0 \quad \text{for all } i \in \{1, \ldots, r\}. \tag{5.2.4}$$

REMARK 5.2.5. Once the preferred order of the coordinates (x, y) is fixed (cf. Definition 3.5.2 and § 3.6), they also induce an order/marking of the coordinates z_i via the identities (5.2.1).

5.3. *The equations of the reduced miniversal base space*

Denote by $S_{\text{red}}(p, q)$ the reduced base space of the miniversal deformation of the cyclic quotient singularity $\mathcal{X}_{p,q}$. It was determined via several steps.

In [**31**] Riemenschneider determined the infinitesimal deformations of $\mathcal{X}_{p,q}$. Fifteen years later Arndt in his thesis [**1**] gave an algorithm to find equations of the base space. Although the structure of $S_{\text{red}}(p, q)$ is hard to find from its equations, Arndt conjectured (on the basis of computations for particular cases) that the number of irreducible components should not exceed the Catalan number $1/r\binom{2(r-1)}{r-1}$. This conjecture was proved by Stevens [**35**], based in an essential way on the work [**18**] of Kollár and Shepherd-Barron on P-resolutions. Moreover, at the same time, Christophersen in [**12**] and Stevens in [**35**] provided (conceptual) equations. It was not at all obvious that the two sets of equations are the same; this was later explicitly proved in the thesis of Brohme [**9**]. In fact, Christophersen and Stevens defined for each $\underline{k} \in K_r(\underline{a})$ an explicit system $\mathcal{E}_{\underline{k}}$ (equivalent to $\mathcal{E}$) of equations which define $\mathcal{X}_{p,q}$, and an explicit deformation $\tilde{\mathcal{E}}_{\underline{k}}$ of these equations with smooth parameter space. (For the completeness of the discussion, we review $\mathcal{E}_{\underline{k}}$ and $\tilde{\mathcal{E}}_{\underline{k}}$ in § 5.4.) The result of Stevens [**35**] mentioned above shows that one gets in this way all the irreducible components of $S_{\mathrm{red}}(p,q)$. Hence we have the following theorem.

THEOREM 5.3.1. *The reduced base space $S_{\mathrm{red}}(p,q)$ of the miniversal deformation of $\mathcal{X}_{p,q}$ has exactly $\#K_r(\underline{a})$ irreducible components.*

As $\mathcal{X}_{p,q}$ is a rational singularity, fact (ii) of § 5.1 shows that all the irreducible components of $S_{\mathrm{red}}(p,q)$ are smoothing components. *We denote by $S^{\mathrm{CS}}_{\underline{k}}$ the irreducible component that corresponds to $\underline{k}\in K_r(\underline{a})$ through the equations of Christophersen and Stevens.*

Through these equations one has in fact an *explicit* bijection between the set $K_r(\underline{a})$ and the irreducible components of $S_{\mathrm{red}}(p,q)$. This (together with (4.1) and (4.2)) allows to understand the meaning of Lisca's conjecture from § 1.

5.4. *The system $\mathcal{E}_{\underline{k}}$ and its deformations*

We follow Stevens' version from [**35**]. One starts with the identification of K_r with the triangulations of the $(r+1)$-gons.

Consider a convex polygon $\mathcal{P}_{r+1}$ in the plane with $r+1$ vertices marked successively by $A_1,\dots,A_{r+1}$. Denote by $T(\mathcal{P}_{r+1})$ the set of triangulations of $\mathcal{P}_{r+1}$ with vertices $A_1,\dots,A_{r+1}$ and edges that are diagonals of $\mathcal{P}_{r+1}$. With each triangulation $\theta\in T(\mathcal{P}_{r+1})$ associate the sequence $(k_1,\dots,k_r)$ such that k_i is the number of triangles containing the vertex A_i. Then $\theta\mapsto\underline{k}$ realizes a bijection between $T(\mathcal{P}_{r+1})$ and K_r; cf. [**35**].

We fix a triangulation $\theta\in T(\mathcal{P}_{r+1})$ corresponding to $\underline{k}$. We define *weights* of the vertices A_i $(1\leqslant i\leqslant r)$ and of edges A_iA_j of the triangles (for some $1\leqslant i<j\leqslant r$). They are rational monomials in the variables z_i $(1\leqslant i\leqslant r)$ and in some 'new' variables $\mathfrak{z}_i$ $(1\leqslant i\leqslant r)$. By definition, the weight of A_i is $\mathfrak{z}_i$. In order to define the weight of A_iA_j, first one collapses in θ the vertices $A_1,\dots,A_{i-1},A_{j+1},\dots,A_{r+1}$ into one vertex (and the corresponding edges and triangles too) to get a $(j-i+2)$-gon $\theta^{(i,j)}$. Denote its $\underline{k}$ invariants (associated to $(j-i+2)$-gon $\theta^{(i,j)}$ similarly as in the above Stevens' correspondence) by $(k_i^{(i,j)},\dots,k_j^{(i,j)})$. Let $\alpha_\ell^{(i,j)}$ $(i-1\leqslant\ell\leqslant j+1)$ be defined by the rule as follows:

$$\begin{aligned}\alpha_\ell^{(i,j)}&=0\quad\text{for } \ell=i-1 \text{ and } \ell=j+1,\\ \alpha_\ell^{(i,j)}&=1\quad\text{for } \ell=i \text{ and } \ell=j,\\ \alpha_{\ell-1}^{(i,j)}+\alpha_{\ell+1}^{(i,j)}=k_\ell^{(i,j)}\alpha_\ell^{(i,j)}&\quad\text{for } i\leqslant\ell\leqslant j\end{aligned}$$

(for a continued fraction interpretation of the integers $\alpha_\ell^{(i,j)}$, see [**35**, (1.1)]). Then the weight of A_iA_j (which equals the product $w_{i,j}w_{j,i}$ of [**35**]) is given by (cf. [**35**, 6.2.1])

$$w(A_iA_j):=\frac{1}{z_i^{\alpha_{i+1}^{(i,j)}-1}z_j^{\alpha_{j-1}^{(i,j)}-1}}\cdot\prod_{\ell=i+1}^{j-1}\left(\mathfrak{z}_\ell\cdot z_\ell^{2-k_\ell^{(i,j)}}\right)^{\alpha_\ell^{(i,j)}}.$$

Then the system $\mathcal{E}_{\underline{k}}$ consists of the equations $E_{\underline{k}}^{(i,j)}$ $(1\leqslant i\leqslant j\leqslant r)$ having the form

$$z_{i-1}z_{j+1}=z_iz_j\quad\cdot\prod(\text{all weights of vertices and edges on the shortest path in }\theta\text{ from }A_i\text{ to }A_j),\tag{5.4.1}$$

where one substitutes $\mathfrak{z}_\ell=z_\ell^{a_\ell-2}$. Note that for $i=j$ one gets the equation (5.2.4).

One can check that the system $\mathcal{E}_{\underline{k}}$ is equivalent with the system $\mathcal{E}$. Indeed, if $E^{(i,j)}$ is the generalized minor of (5.2.2) (see formulae (5.2.3)), then $E_{\underline{k}}^{(i,j)}$ is obtained from $E^{(i,j)}$ via some substitutions of equations of type $E^{(i',j')}$, with $i<i'\leqslant j'<j$.

For example, if $r = 3$ and $\underline{k} = (1, 2, 1)$, then $E^{(i,j)}_{\underline{k}} = E^{(i,j)}$ for all i and j. However, if $\underline{k} = (2, 1, 2)$, then $E^{(1,3)}_{\underline{k}} : \{z_0 z_4 = \mathfrak{z}_1\mathfrak{z}_2^2\mathfrak{z}_3 z_2^2\}$ is obtained from $E^{(1,3)} : \{z_0 z_4 = \mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_3 z_1 z_3\}$ with substitution $E^{(2,2)} : \{z_1 z_3 = \mathfrak{z}_2 z_2^2\}$; the others coincide (here each $E^{(i,j)}$ is the generalized minor of (5.2.2) before the substitution, that is, the last row of (5.2.2) consists of $\mathfrak{z}_1, \ldots, \mathfrak{z}_r$).

For more details or different presentations, see [**6**, p. 8–11, **12**, pp. 83–84, **35**, pp. 316–317].

Clearly, the space $\mathcal{X}_{p,q}$ is independent of $\underline{k}$, but for each $\underline{k}$ one deforms different sets of equations of $\mathcal{X}_{p,q}$, namely $\mathcal{E}_{\underline{k}}$. For simplicity, for each $\underline{k}$ we provide only a subspace of the deformation associated with $\underline{k}$ (in the language of [**35**] it corresponds to the vanishing of the deformation parameters s_ϵ), but it already contains the 1-parameter smoothing needed in the proof of the main statement. The deformations are obtained from the equations (5.4.1) via the substitutions

$$\mathfrak{z}_\ell = z_\ell^{a_\ell - 2} + t_{\ell,1} \cdot z_\ell^{a_\ell - 1} + \ldots + t_{\ell, a_\ell - k_\ell} \cdot z^{k_\ell - 2}, \tag{5.4.2}$$

where the variables $t_{\ell,m}$ are deformation parameters.

As an example, consider the above case when $r = 3$ and $\underline{k} = (2, 1, 2)$. Then $\mathfrak{z}_2 = z_2^{a_2-2} + \ldots + t/z_2$. Hence, substituting this in $E^{(1,3)}$, the pole z_2 survives, that is, $E^{(1,3)}$ cannot be deformed by this substitution. On the other hand, substituting in $E^{(2,2)}$, or in $E^{(1,3)}_{\underline{k}}$, the denominator disappears. This partly shows the advantage and role of $\mathcal{E}_{\underline{k}}$. (The point is that exactly this very last term in (5.4.2) will be crucial in what follows.)

We wish to emphasize that in our proof in § 8 from the (technical) formulas/definitions of this subsection we need merely the fact that the deformation component associated with $\underline{k}$ contains a 1-parameter deformation which has a set of equations that contains

$$z_{i-1} z_{i+1} = z_i^{a_i} + t \cdot z_i^{k_i} \quad \text{for all } i \in \{1, \ldots, r\}. \tag{5.4.3}$$

6. *The smoothings of sandwiched surface singularities, following de Jong and van Straten*

Cyclic quotient singularities are particular cases of *sandwiched surface singularities*. de Jong and van Straten related in [**16**] the deformation theory of sandwiched surface singularities to the deformation theory of so-called *decorated plane curve singularities*. They show that *1-parameter deformations of decorated curves provide 1-parameter deformations for sandwiched singularities, and all of these later ones can be obtained in this way*. Moreover, the Milnor fibres of those that are smoothings can be combinatorially described by the so-called *picture deformations*.

In this section we explain the general framework, while in § 7 we specialize it to cyclic quotient singularities.

6.1. *Sandwiched singularities*

The normal surface singularity $(X, 0)$ is called *sandwiched* if it is a germ of an algebraic surface that admits a birational map $X \to \mathbb{C}^2$. They were introduced in [**34**] by Spivakovsky; see also [**16**, **19**, **24**] for different view points.

Sandwiched singularities are rational. They are characterized (like the rational singularities) by their dual resolution graphs. Hence one may speak about *sandwiched graphs*.

PROPOSITION 6.1.1. *Sandwiched graphs are characterized as follows: by adding new vertices with weights (self-intersections) -1 (on the 'right places') one may obtain a 'smooth graph', that is, the dual tree of a configuration of smooth rational curves which blows down to a smooth point.*

6.2. *Decorated curves and their deformations*

Any sandwiched singularity may be obtained from a weighted curve (C, l). Here $(C, 0) \subset (\mathbb{C}^2, 0)$ denotes a reduced germ of plane curve with branches $\{C_i\}_{1 \leqslant i \leqslant r}$ and a function $\{1, \ldots, r\} \ni i \mapsto l_i \in \mathbb{N}^*$.

Consider the minimal resolution of C. The *multiplicity sequence* associated with C_i is the sequence of multiplicities on the successive strict transforms of C_i, starting from C_i itself and not counting the last strict transform. The *total multiplicity* $m(i)$ of C_i with respect to C is the sum of multiplicities of C_i already defined.

Definition 6.2.1 [**16**, (1.3)]. A *decorated germ* of plane curve is a weighted germ (C, l) such that $l_i \geqslant m(i)$ for all $i \in \{1, \ldots, r\}$.

The point is that starting from a decorated germ, one can blow up iteratively points infinitely near 0 on the strict transform of C, such that the number of such points sitting on the strict transform of C_i is exactly l_i. If l_i is sufficient large (in general, larger than $m(i)$), then the union of the exceptional components that do not meet the strict transform of C forms a connected configuration of curves. After its contraction one gets necessarily a sandwiched singularity $X(C, l)$, determined uniquely by (C, l) (for details see [**16**]).

The total multiplicity of C_i with respect to C may be encoded also as the unique subscheme of length $m(i)$ supported on the preimage of 0 on the normalization of C_i. The same thing is valid for l_i. This allows to define the *total multiplicity scheme* $m(C)$ of any reduced curve contained in a smooth complex surface, as the union of the total multiplicity schemes of all its germs.

Definition 6.2.2.

(i) Given a smooth complex analytic surface Σ, a pair (C, l) consisting of a reduced curve $C \hookrightarrow \Sigma$ and a subscheme l of the normalization $\tilde{C}$ of C is called *a decorated curve* if $m(C)$ is a subscheme of l (see [**16**, (4.1)]).

(ii) A 1-*parameter deformation* of a decorated curve (C, l) over a germ of smooth curve $(S, 0)$ consists of the following:

(1) a δ-constant deformation $C_S \to S$ of C;
(2) a flat deformation $l_S \subset \tilde{C}_S = \tilde{C} \times S$ of the scheme l, such that
(3) $m_S \subset l_S$, where the *relative total multiplicity scheme* m_S of $\tilde{C}_S \to C_S$ is defined as the closure $\overline{\bigcup_{s \in S \setminus 0} m(C_s)}$ (see [**16**, p.476]).

(iii) A 1-parameter deformation (C_S, l_S) is called a *picture deformation* if for generic $s \neq 0$ the divisor l_s is reduced.

In [**16**, (4.4)] the authors prove that all the 1-parameter deformations of $X(C, l)$ are obtained by 1-parameter deformations of the decorated germ (C, l). Moreover, *picture deformations* provide smoothings of $X(C, l)$.

6.3. *Picture deformations and their Milnor fibres*

Consider a decorated germ (C, l) with *all the components* $\{C_i\}_{1 \leqslant i \leqslant r}$ *smooth*, and one of its picture deformations (C_S, l_S). Fix a closed Milnor ball B for the germ $(C, 0)$. For $s \neq 0$ sufficiently small, C_s will have a representative in B, denoted by D, which meets ∂B transversally. It is a union of embedded *discs* $\{D_i\}_{1 \leqslant i \leqslant r}$ canonically oriented by their complex structures (and which has a set of indices that correspond canonically to those of $\{C_i\}_{1 \leqslant i \leqslant r}$). The singularities of D consist of ordinary m-tuple points, for various m.

Denote by $\{P_j\}_{1\leqslant j\leqslant n}$ the images in B of the points in the support of l_s. It is a finite set of points that contains the singular set of D (because $m_s \subset l_s$), but it contains some other 'free' points as well. There is *a priori* no preferred choice of their ordering. Hence, the matrix introduced next is well defined only up to permutation of columns.

Definition 6.3.1 [**16**, p.483]. The *incidence matrix* of a picture deformation (C_S, l_S) is the matrix $\mathcal{I}(C_S, l_S) \in \mathrm{Mat}_{r,n}(\mathbb{Z})$ which has an entry at the intersection of the ith row and the jth column equal to 1 if $P_j \in D_i$ and to 0 if $P_j \notin D_i$.

Definition 6.2.2(2) implies that

$$\text{the sum of entries on the } i\text{th row of } \mathcal{I}(C_S, l_S) \text{ is } l_i. \tag{6.3.2}$$

The Milnor fibre of such a smoothing is recovered as follows. Let

$$(\tilde{B}, \tilde{D}) \xrightarrow{\beta} (B, D) \tag{6.3.3}$$

be the simultaneous blow-up of the points P_j of D. Here $\tilde{D} := \bigcup_{1\leqslant i\leqslant r} \tilde{D}_i$, where $\tilde{D}_i$ is the strict transform by the modification β of the disc D_i. Let T_i be a sufficiently small open tubular neighbourhood of $\tilde{D}_i$ in $\tilde{B}$.

Proposition 6.3.4 [**16**, (5.1)]. *Suppose that all the irreducible components of C are smooth. The Milnor fibre of the smoothing of $X(C, l)$ corresponding to the picture deformation (C_S, l_S) is orientation-preserving diffeomorphic to the compact oriented manifold with boundary $W := \tilde{B} \setminus (\bigcup_{1\leqslant i\leqslant r} T_i)$ (which has corners that are smoothed).*

7. *The smoothings of cyclic quotient singularities following de Jong and van Straten*

7.1. *How to find (C, l)?*

The construction of the decorated germ (C, l) used for cyclic quotients is in a natural way valid for the more general class of *minimal singularities*. Hence, it is natural to present it in this context. Minimal singularities were introduced by Kollár [**17**] in arbitrary dimension. In the case of normal surfaces, they are exactly those rational germs which have reduced fundamental cycle (for example, see [**19**]). They are special sandwiched singularities, characterized by their graphs as follows.

Consider the minimal resolution of a rational singularity (X, x). Let Γ be its dual graph, let J be the set of its vertices, let ν_j be the valency and let $-e_j < 0$ be the weight (self-intersection) of the vertex $j \in J$. Then (X, x) is minimal if and only if its minimal resolution satisfies $\nu_j \leqslant e_j$ for all $j \in J$.

For any minimal singularity (X, x), there is an easy algorithm that provides a decorated curve (C, l) defining (X, x), starting from Γ. Its steps are the following.

(I) Construct a new graph Γ' by connecting the vertex j of Γ to $e_j - \nu_j$ new vertices, with the exception of one vertex j_0 satisfying $e_{j_0} - \nu_{j_0} > 0$, which must be connected to $e_{j_0} - \nu_{j_0} - 1$ new vertices. Each new vertex is endowed with the weight -1.

(II) Construct another graph Γ'' by endowing each new vertex with one new arrowhead.

Then Γ' is a smooth graph (hence by Proposition 6.1.1 minimal singularities are sandwiched indeed), and Γ'' is the (not necessarily minimal) dual graph of an embedded resolution of a germ of a plane curve C with all components C_i smooth. They are obtained by blowing-down 'curvettas' corresponding to the arrowheads of Γ''.

The graph Γ'' also has the following properties.

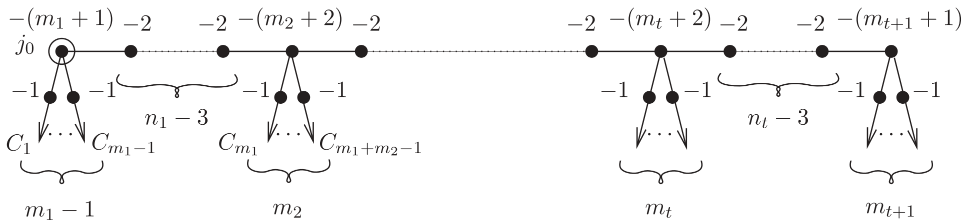


FIGURE 4. *The dual graph associated with (C, l), with $\mathcal{X}_{p,q} = X(C, l)$.*

LEMMA 7.1.1.

(1) *The intersection number of the irreducible curves C_i and C_j associated with two distinct arrowheads is equal to the number of vertices on the intersection of the geodesics from j_0 to the two arrowheads.*

(2) *The weight l_i associated with the irreducible curve corresponding to an arrowhead is equal to the distance from the vertex j_0 to that arrowhead.*

We apply the previous procedure to $\mathcal{X}_{p,q}$ by choosing as j_0 that vertex of $\Gamma = G(\underline{b})$ which is marked by 1, that is, which has weight $-b_1$ (see Figure 2). We use the notation (2.2.6) for the sequence $\underline{b}$. Hence, $\mathcal{X}_{p,q}$ may be presented as a sandwiched singularity $X(C, l)$, where the components of C are indicated in the graph Γ'' shown in Figure 4. By (2.2.7), the number of curves is exactly r (which explains why we have chosen this notation for the number of components of C in § 6). By Lemma 7.1.1 one has

$$C_i \cdot C_j = l_i - 1 \quad \text{for all } 1 \leqslant i < j \leqslant r; \tag{7.1.2}$$

$$l_i = 2 + \sum_{1 \leqslant j \leqslant h-1} (n_j - 2) \quad \text{whenever} \quad \sum_{1 \leqslant j \leqslant h-1} m_j \leqslant i < \sum_{1 \leqslant j \leqslant h} m_j. \tag{7.1.3}$$

By (2.2.5), these last relations (7.1.3) transform into

$$l_i = 2 + \sum_{1 \leqslant j \leqslant i} (a_i - 2) \quad \text{for all } i \in \{1, \ldots, r\}. \tag{7.1.4}$$

7.2. *From triangulations to the incidence matrix*

We describe the *incidence matrix* using an interpretation by Stevens of the elements of K_r via triangulations of a polygon [**35**], and the following notation: If $M \in \mathrm{Mat}_{r,n}(\mathbb{Z})$ then $\int M \in \mathrm{Mat}_{r,n}(\mathbb{Z})$ will denote that matrix which has an ith row that is the sum of the first i rows of M. By our construction, we get the *same* incidence matrix as in [**16**], nevertheless, we arrive to it slightly differently (perhaps, more conceptually). More precisely, [**16**] starts with a 'difference matrix' (with non-negative entries), considers its $\int$, and the *modulo* 2 remainders of the entries of this second matrix constitute the incidence matrix. In our case, we conceptually assign to each entry of the 'difference matrix' a sign such that its $\int$ will be exactly the incidence matrix; cf. Remark 7.2.3.

Assume that $r > 1$ (since $\# K_1 = 1$, in the identifications we wish to get we lose nothing).

Consider a convex polygon $\mathcal{P}_{r+1}$, its set of triangulations $T(\mathcal{P}_{r+1})$, and Stevens' bijection $T(\mathcal{P}_{r+1}) \to K_r$ ($\theta \mapsto \underline{k}$) as in § 5.4. Fix $\theta \in T(\mathcal{P}_{r+1})$. To each triangle Δ of θ and vertex A of $\mathcal{P}_{r+1}$ we define a 'sign' $\alpha = \alpha(A, \Delta) \in \{0, -1, +1\}$ as follows. If A is not a vertex of Δ, we take $\alpha = 0$. Then we (totally) order the vertices of Δ by restricting to them the order $A_1, \ldots, A_{r+1}$ of the vertices of $\mathcal{P}_{r+1}$. We set $\alpha = +1$ for the first and third vertex, while $\alpha = -1$ for the

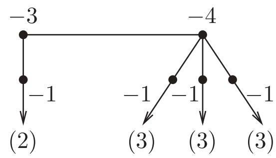


FIGURE 5. *Dual graph associated with* (C, l), *with* $\mathcal{X}_{11,4} = X(C, l)$.

second one. Next, we order the triangles $\{\Delta_j\}_{j=1}^{r-1}$ of θ in an arbitrary way, and we define the 'sign-incidence matrix' between the vertices $A_1, \dots, A_r$ (corresponding to the rows) and the triangles $\Delta_1, \dots, \Delta_{r-1}$ (corresponding to the columns) by $d_{i,j} := \alpha(A_i, \Delta_j)$. If θ corresponds to $\underline{k}$, then denote this matrix by $D(\underline{k}) \in \mathrm{Mat}_{r,r-1}(\mathbb{Z})$ (well defined up to a permutation of its columns). Moreover, for each $\ell \in \mathbb{N}$, denote by $M_{r,\ell}(i) \in \mathrm{Mat}_{r,\ell}(\mathbb{Z})$ the matrix which has entries all of which are equal to $+1$ on the ith row and all of which are equal to 0 elsewhere.

Then, for each $\underline{k} \in K_r(\underline{a})$, consider the 'block-matrix'

$$D(\underline{a}; \underline{k}) := (D(\underline{k}) \mid M_{r,a_1-k_1}(1) \mid \dots \mid M_{r,a_r-k_r}(r)) \in \mathrm{Mat}_{r,r-1+\sum(a_i-k_i)}(\mathbb{Z}). \tag{7.2.1}$$

The following theorem, valid for any fixed (C, l) as in § 7.1, follows from [**16**, Section 6.4].

THEOREM 7.2.2 [**16**, (6.18)]. *For every* $\underline{k} \in K_r(\underline{a})$, *the matrix* $\int D(\underline{a}; \underline{k})$ *is* (*up to a permutation of columns*) *the incidence matrix of some picture deformation of* (C, l). *In particular, the number* n *of points* $\{P_i\}_i$ *is* $n = r - 1 + \sum_{i=1}^{r}(a_i - k_i)$. *Moreover, varying* $\underline{k} \in K_r(\underline{a})$, *one gets all incidence matrices of picture deformations of* (C, l).

REMARK 7.2.3. In [**16**] the authors call the matrices $\int D(\underline{a}; \underline{k})$ *CQS-matrices*, and denote them by M. Up to the signs of the entries, $D(\underline{a}; \underline{k})$ are their *difference matrices* ΔM. More precisely, the entries of ΔM are the absolute values of the entries of $D(\underline{a}; \underline{k})$.

REMARK 7.2.4. By Definition 6.3.1 and (6.3.2), an incidence matrix has all its entries equal to 0 or $+1$ and the sum of all entries of the ith row is l_i. Let us verify that this is indeed the case for $\int D(\underline{a}; \underline{k})$. The first property is a consequence of the fact that on each column of $D(\underline{k})$, the non-zero entries are either $(+1, -1)$ or $(+1, -1, +1)$, always in this order, depending on the fact that A_{r+1} is a vertex of the corresponding triangle or not. The second property is a consequence of (7.1.4) and the following elementary fact regarding the above sign-assigning procedure: A_1 has no sign equal to -1, and A_i has exactly one sign equal to -1 for any $i > 1$.

Since, via picture deformations, we hit all the components of the reduced miniversal base space of $\mathcal{X}_{p,q}$, via the correspondence Theorem 7.2.2 de Jong and van Straten parametrize the smoothing components by the elements of $K_r(\underline{a})$. Denote by $S_{\underline{k}}^{JS}$ *the component parametrized by* $\underline{k} \in K_r(\underline{a})$. Denote by $W(\underline{a}, \underline{k})$ *the Milnor fibre* of the corresponding smoothing, that is, the manifold constructed in § 6.3, specialized to the present situation.

EXAMPLE 7.2.5. Here we list all the objects presented in this section, applied for the singularity $\mathcal{X}_{11,4}$, that is, for $\underline{a} = (2, 3, 2, 2)$ and $\underline{b} = (3, 4)$. The dual graph of the minimal resolution is $G(3, 4)$ (see Figure 2), while the resolution graph of the decorated germ (C, l) is drawn in Figure 5 (where between parenthesis we inserted the integers $\{l_i\}_{i=1}^{4}$):

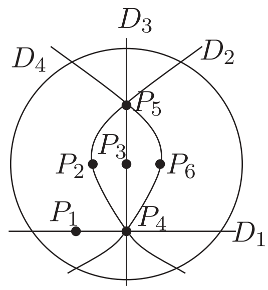


FIGURE 6. *A general member of a picture deformation with* $\underline{k} = (1, 2, 2, 1)$.

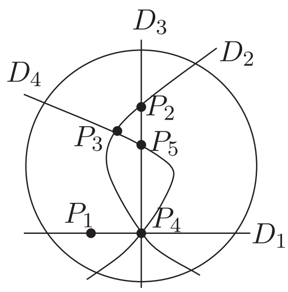


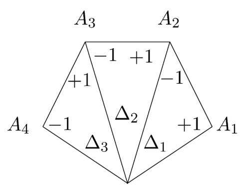
FIGURE 7. *A general member of a picture deformation with* $\underline{k} = (1, 3, 1, 2)$.

The associated set of sequences representing zero is given by

$$K_5(2, 3, 2, 2) = \{(1, 2, 2, 1), (1, 3, 1, 2)\}.$$

One has the following associated triangulations of a pentagon (with the corresponding signs), matrices $D(\underline{a}; \underline{k})$ and $\int D(\underline{a}; \underline{k})$ (with blocks $M_{r,a_i - k_i}$ separated, those with $a_i = k_i$ being empty), and a generic member of a picture deformation with $\int D(\underline{a}; \underline{k})$ as the incidence matrix (see Figures 6 and 7) as follows:

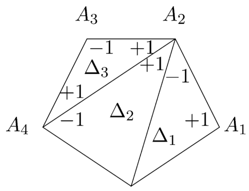
(i) $\underline{k} = (1, 2, 2, 1)$.



$$D(2, 3, 2, 2; 1, 2, 2, 1) = \left(\begin{array}{ccc|c|c||c} +1 & 0 & 0 & +1 & 0 & 0 \\ -1 & +1 & 0 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \end{array}\right)$$

$$\int D(2, 3, 2, 2; 1, 2, 2, 1) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(ii) $\underline{k} = (1, 3, 1, 2)$.



$$D(2,3,2,2;1,3,1,2) = \left(\begin{array}{ccc|c||c||} +1 & 0 & 0 & +1 & 0 \\ -1 & +1 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & -1 & +1 & 0 & 0 \end{array}\right)$$

$$\int D(2,3,2,2;1,3,1,2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

8. *'From Christophersen and Stevens to Lisca'*

8.1. *Some inequalities*

Before we start our discussion regarding deformations of cyclic quotients, we state a technical lemma, which will be used several times.

LEMMA 8.1.1. *Assume that* $\underline{x} \in \mathbb{N}^n$ *is an admissible sequence* (*cf. Definition 2.1.5*). *Then we have the following.*

(1) *For any* $x'_i \geqslant x_i$ $(1 \leqslant i \leqslant n)$, $\underline{x}'$ *is admissible too.*

(2) *Assume that* $\{\nu_i\}_{i=0}^{n+1}$ *satisfy the inequalities* $\nu_{i+1} \geqslant x_i\,\nu_i - \nu_{i-1}$ *for all* $1 \leqslant i \leqslant n$. *Then for all* i *one also has*
$$\nu_{i+1} \geqslant Z_i(x_i, \ldots, x_1)\,\nu_1 - Z_{i-1}(x_i, \ldots, x_2)\,\nu_0. \tag{8.1.2}$$

(3) *Assume that* $x_i \geqslant 2$ *for all* i *and set* $\mu_i := Z_{i-1}(x_1, \ldots, x_{i-1}) - Z_{i-2}(x_2, \ldots, x_{i-1})$ *for* $1 \leqslant i \leqslant n$. *Then*
$$\begin{aligned} &1 = Z_0 < Z_1(x_1) < Z_2(x_1, x_2) < \ldots < Z_n(x_1, \ldots, x_n)\,, \text{ and} \\ &1 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n\,. \end{aligned}$$

Proof. (1) We have that $M(\underline{x}')$ is the sum of $M(\underline{x})$ and a diagonal matrix with non-negative entries. Therefore, if $M(\underline{x})$ is positive semi-definite of rank at least $(n-1)$, then $M(\underline{x}')$ will have the same property.

(2) We prove by decreasing induction on j that, for fixed i and for all $1 \leqslant j \leqslant i$, one has
$$\nu_{i+1} \geqslant Z_{i-j+1}(x_i, x_{i-1}, \ldots, x_j)\,\nu_j - Z_{i-j}(x_i, x_{i-1}, \ldots x_{j+1})\,\nu_{j-1}. \tag{8.1.3}$$
For $j = i$ this is clear. The induction runs as follows. Since $M(\underline{x})$ is positive semi-definite $Z_{i-j+1}(x_i, \ldots, x_j) \geqslant 0$, it follows that the right-hand side of (8.1.3) is greater than
$$\begin{aligned} &Z_{i-j+1}(x_i, \ldots, x_j)\,(x_{j-1}\nu_{j-1} - \nu_{j-2}) - Z_{i-j}(x_i, \ldots x_{j+1})\,\nu_{j-1} \\ &\quad = [x_{j-1}Z_{i-j+1}(x_i, \ldots, x_j) - Z_{i-j}(x_i, \ldots x_{j+1})]\,\nu_{j-1} - Z_{i-j+1}(x_i, \ldots, x_j)\,\nu_{j-2}. \end{aligned}$$

Since in the parenthesis we have exactly $Z_{i-j+2}(x_i, \ldots, x_{j-1})$ (cf. equation (2.1.3)), the proof of (8.1.3) is finished for all j. For $j = 1$ we get the wished inequality.

(3) The first part follows by induction: if $Z_{i-1}(x_1,\dots,x_{i-1}) > Z_{i-2}(x_1,\dots,x_{i-2})$, then

$$Z_i(x_1,\dots,x_i) = x_i Z_{i-1}(x_1,\dots,x_{i-1}) - Z_{i-2}(x_1,\dots,x_{i-2}) > (x_i-1)Z_{i-1}(x_1,\dots,x_{i-1}).$$

This reinterpreted also shows that $\mu_i \geqslant 0$. Then the identity $\mu_{i+1} - \mu_i = (x_i - 2)\mu_i + (\mu_i - \mu_{i-1})$ and induction completes the proof. □

Since $\underline{k}$ is admissible and $\underline{k} \leqslant \underline{a}$, the above lemma can be applied for both $\underline{k}$ and $\underline{a}$.

8.2. $\mathcal{X}^t_{\underline{k}}$ *as the Milnor fibre*

In what follows, will follow the notation of § 5. First we concentrate on $\mathcal{X}_{p,q}$. Using equations (5.2.4) and induction, one shows that the restriction of each z_i to $\mathcal{X}_{p,q}$ is a rational function in (z_0, z_1) of the form

$$z_i = z_1^{Z_{i-1}(a_1,\dots,a_{i-1})} \cdot z_0^{-Z_{i-2}(a_2,\dots,a_{i-1})} \quad \text{for } i \in \{1,\dots,r+1\}. \tag{8.2.1}$$

The equations $\mathcal{E}_{\underline{k}}$ are weighted homogeneous, however, the weights $w_i := w(z_i)$ are not unique. With the choice $w_0 = w_1 = 1$ one has the following lemma.

Lemma 8.2.2. *We have the following:*
(a) $w_i = Z_{i-1}(a_1,\dots,a_{i-1}) - Z_{i-2}(a_2,\dots,a_{i-1})$ *for all* $i \geqslant 1$*;*
(b) $1 = w_0 = w_1 \leqslant w_2 \leqslant \dots \leqslant w_{r+1} = q$.

Proof. We see that (a) follows from (8.2.1), $w_{r+1} = q$ from (2.1.2), and the rest of (b) from Lemma 8.1.1. □

We consider a special 1-parameter deformation $\mathcal{E}^t_{\underline{k}}$ of the equations $\mathcal{E}_{\underline{k}}$. This deformation is uniquely determined by the deformed equations of (5.2.4) (cf. [**12**; **35**, (2.2)]). These are (see (5.4.3)) as follows:

$$z_{i-1}z_{i+1} = z_i^{a_i} + t\cdot z_i^{k_i} \quad \text{for all } i \in \{1,\dots,r\}, \tag{8.2.3}$$

where $t \in \mathbb{C}$. Note that, although (5.2.4) did not depend on $\underline{k}$, this is not the case for their deformations. Let $\mathcal{X}^t_{\underline{k}}$ be the affine space determined by the equations $\mathcal{E}^t_{\underline{k}}$ in $\mathbb{C}^{r+2}$.

Lemma 8.2.4. *The deformation* $t \mapsto \mathcal{X}^t_{\underline{k}}$ *has negative weight and is a smoothing belonging to the component* $S^{\mathrm{CS}}_{\underline{k}}$*. In particular,* $\mathcal{X}^t_{\underline{k}}$ *is a smooth affine variety for* $t \neq 0$.

Proof. The first statement just means that the weight of the added monomial $z_i^{k_i}$ is not larger than the weight of $z_{i-1}z_{i+1} - z_i^{a_i}$, that is, $w_i > 0$ and $k_i \leqslant a_i$. For the second part one checks the general form of the equations of $S^{\mathrm{CS}}_{\underline{k}}$ from [**35**, (2.2)] or [**12**], and the fact that the present deformation does not belong to the discriminant of $S^{\mathrm{CS}}_{\underline{k}}$ described in [**12**].

In fact, the smoothness also follows from our direct computation, as a byproduct of Theorem 8.4.6. Indeed, $\mathcal{X}^t_{\underline{k}}$ as a fibre of the miniversal deformation is normal. In § 8.3 we construct a resolution of it, which has no exceptional curve by Theorem 8.4.6. Hence $\mathcal{X}^t_{\underline{k}}$ is smooth. □

In particular, the above smoothing has a series of pleasant properties (for example, it induces a projective deformation that is locally trivial 'near ∞'). Moreover, by [**37**, (2.2)] we have that

$$\mathcal{X}^t_{\underline{k}} \text{ is diffeomorphic to the Milnor fibre of } S^{\mathrm{CS}}_{\underline{k}}. \tag{8.2.5}$$

In what follows we denote by $\widehat{\mathcal{X}_{\underline{k}}^t}$ the closure of $\mathcal{X}_{\underline{k}}^t$ in $\mathbb{P}^{r+2}$, and let $C_{\underline{k}}^\infty = \widehat{\mathcal{X}_{\underline{k}}^t} \setminus \mathcal{X}_{\underline{k}}^t$ be its curve at infinity (as may be seen by a computation, or by the equisingularity at infinity mentioned above, $C_{\underline{k}}^\infty$ is topologically independent of $\underline{k}$ and t).

8.3. $\mathcal{X}_{\underline{k}}^t$ *as a rational surface*

Similarly as for $\mathcal{X}_{p,q}$ one shows by induction that on $\mathcal{X}_{\underline{k}}^t$ all the restrictions of the coordinates z_i can be expressed as rational functions in (z_0, z_1)

LEMMA 8.3.1. *For each $i \in \{1, \dots, r+1\}$, on $\mathcal{X}_{\underline{k}}^t$ one has*

$$z_i = z_0^{-Z_{i-2}(a_2,\dots,a_{i-1})} P_i \tag{8.3.2}$$

for some $P_i \in \mathbb{Z}[t, z_0, z_1]$. The polynomials P_i satisfy the inductive relations as follows:

$$P_{i-1} \cdot P_{i+1} = P_i^{a_i} + t P_i^{k_i} \cdot z_0^{(a_i-k_i)\cdot Z_{i-2}(a_2,\dots,a_{i-1})} \tag{8.3.3}$$

with $P_1 = z_1$ and with the convention $P_0 = 1$. Moreover $z_0 \nmid P_i$.

Proof. Define P_i by (8.3.2). By a substitution it is clear that (8.3.3) follows from (8.2.3) and (8.3.2). By (8.3.3) and induction, P_i is an (*a priori* rational) function in the variables (t, z_0, z_1). Hence, we only have to prove that P_i is a polynomial and $z_0 \nmid P_i$. Let R be an irreducible polynomial in (t, z_0, z_1) and let $\nu_R : \mathbb{C}(t, z_0, z_1)^* \to \mathbb{Z}$ be the valuation associated with it. We have to show that $\nu_R(z_i) \geqslant 0$ for $R \neq z_0$ and $\nu_{z_0}(z_i) = -Z_{i-2}(a_2, \dots, a_{i-1})$.

Set $\nu_i := \nu_R(z_i)$, and consider first $R = z_0$. Then analysing (8.2.3), we get that z_0 is a pole of z_i for $i \geqslant 2$, and hence $\nu_{z_0}(z_i^{a_i} + t z_i^{k_i}) \geqslant \nu_{z_0}(z_i^{a_i})$. This shows that $\nu_{i+1} \geqslant a_i \nu_i - \nu_{i-1}$, and hence (8.1.2) can be applied. Since $\nu_0 = 1$ and $\nu_1 = 0$, we get $\nu_i \geqslant -Z_{i-2}(a_2, \dots, a_{i-1})$.

If $R \neq z_0$, then $\nu_0 = 0$ and $\nu_1 \geqslant 0$. Assume that $\nu_j \geqslant 0$ for $0 \leqslant j \leqslant i$. Then by (8.2.3) we have $\nu_{j+1} \geqslant k_j \nu_j - \nu_{j-1}$ for all $1 \leqslant j \leqslant i$, and hence (8.1.2) can again be applied (which has 'Z-coefficients' that are non-negative by the admissibility of $\underline{k}$). In particular $\nu_{i+1} \geqslant 0$ too. Hence P_i is a polynomial. Finally, (8.3.3) shows that $P_{i+1} P_{i-1} \equiv c P_i^{a_i} \pmod{z_0}$ for some non-zero constant c. Since z_0 does not divide P_0 and P_1, by induction it does not divide P_i either. □

In fact, by Lemma 8.1.1(3) and (2.1.2), the different exponents of z_0 in (8.3.2) satisfy:

$$1 = Z_0 < Z_1(a_2) < \dots < Z_{i-2}(a_2, \dots, a_{i-1}) < \dots < Z_{r-1}(a_2, \dots, a_r) = p - q. \tag{8.3.4}$$

Define now the application $\pi : \mathbb{C}^2 \setminus \{z_0 = 0\} \longrightarrow \mathcal{X}_{\underline{k}}^t$ by $(z_0, z_1) \mapsto (z_0, z_1, \dots, z_{r+1})$, or

$$(z_0, z_1) \longrightarrow (z_0, z_1, z_0^{-1} P_2, \dots, z_0^{-Z_{i-2}(a_2,\dots,a_{i-1})} P_i, \dots, z_0^{-(p-q)} P_{r+1}) \in \mathbb{C}^{r+2}, \tag{8.3.5}$$

and the induced birational map $\widehat{\pi} : \mathbb{P}^2 \dashrightarrow \widehat{\mathcal{X}_{\underline{k}}^t}$, which sends $[z_{-1} : z_0 : z_1]$ into

$$\Big[1 : \frac{z_0}{z_{-1}} : \frac{z_1}{z_{-1}} : \frac{z_1^{a_1} + t z_1^{k_1}}{z_{-1}^{a_1-1} z_0} : \dots : \frac{P_i}{z_{-1}^{w_i} z_0^{Z_{i-2}(a_2,\dots,a_{i-1})}} : \dots : \frac{P_{r+1}}{z_{-1}^{q} z_0^{p-q}}\Big]. \tag{8.3.6}$$

Let $\rho'_{\underline{k}} : B'\mathbb{P}^2 \to \mathbb{P}^2$ be the minimal sequence of blow-ups such that $\widehat{\pi} \circ \rho'_{\underline{k}}$ extends to a regular map $B'\mathbb{P}^2 \to \widehat{\mathcal{X}_{\underline{k}}^t}$. Let $L_\infty \subset \mathbb{P}^2$ be the line at infinity (defined by $z_{-1} = 0$) and let L_0 be the closure in $\mathbb{P}^2$ of $\{z_0 = 0\}$. We use the same notation for their strict transforms via blow-ups of $\mathbb{P}^2$.

LEMMA 8.3.7. *The morphism $\widehat{\pi} \circ \rho'_{\underline{k}}$ sends L_0 and the total transform of L_∞ in $C_{\underline{k}}^\infty$.*

Proof. Use (8.3.6) or the fact that the projection $\mathrm{pr} : \mathcal{X}^t_{\underline{k}} \to \mathbb{C}^2$ is regular and the corresponding restrictions of $\mathrm{pr} \circ (\widehat{\pi} \circ \rho'_{\underline{k}})$ and $\rho'_{\underline{k}}$ are equal. □

Hence, from the point of view of $\mathcal{X}^t_{\underline{k}}$, resolving the indeterminacy points of $\widehat{\pi}$ above L_∞ is irrelevant. Let $\rho_{\underline{k}} : B\mathbb{P}^2 \to \mathbb{P}^2$ be the *minimal* sequence of blow-ups which resolve the indeterminacies of $\widehat{\pi}$ *sitting in* $\mathbb{C}^2$ (hence $\rho'_{\underline{k}}$ and $\rho_{\underline{k}}$ over $\mathbb{C}^2$ coincide). Denote by E_π its exceptional curve and by C_π the union of those irreducible components of E_π that are sent to $C^\infty_{\underline{k}}$. Summing up the above discussions, one obtains the following corollary.

Corollary 8.3.8. *The restriction of* $\widehat{\pi} \circ \rho_{\underline{k}}$ *induces an isomorphism* $B\mathbb{P}^2 \setminus (L_\infty \cup L_0 \cup C_\pi) \to \mathcal{X}^t_{\underline{k}}$. *In particular, the Milnor fibre can be realized as the complement of the projective curve* $L_\infty \cup L_0 \cup C_\pi$ *in* $B\mathbb{P}^2$.

Proof. Use the fact that $\mathcal{X}^t_{\underline{k}}$ is smooth; cf. Lemma 8.2.4. □

For the convenience of the reader, we represent in the following diagram all the maps introduced in the previous discussion:

$$\begin{tikzcd}
B\mathbb{P}^2 \arrow[dr, "\rho_{\underline{k}}"'] & B'\mathbb{P}^2 \arrow[l] \arrow[r] \arrow[d, "\rho'_{\underline{k}}"] & \widehat{\mathcal{X}^t_{\underline{k}}} = \mathcal{X}^t_{\underline{k}} \cup C^\infty_{\underline{k}} & \mathcal{X}^t_{\underline{k}} \arrow[l, hook'] \arrow[d, "pr"] \\
 & \mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty \arrow[ur, dashed, "\widehat{\pi}"'] & & \mathbb{C}^2 \arrow[ll, hook']
\end{tikzcd}$$

8.4. *The curve-configurations E_π and C_π*

The equations (8.3.4) and (8.3.5) show that the indeterminacy points of $\widehat{\pi}$ sitting in $\mathbb{C}^2$ are given by $\{z_0 = P_{r+1} = 0\}$. By equations (8.3.3) and induction, this set equals $\{z_0 = P_2 = 0\} = \{z_0 = z_1^{a_1} + tz_1^{k_1} = 0\}$ sitting in L_0. The indeterminacy at the points $(0, \xi_j)$, where $\{\xi_j\}_j$ are the roots of $z_1^{a_1-k_1} + t = 0$, can be eliminated by a single blow-up (for example, see below). The indeterminacy at $(0,0)$ (which appears exactly when $k_1 > 0$, that is, when $r > 1$) requires, in general, more blow-ups. The structure of $\widehat{\pi}$ at these points will be revealed in the following paragraphs.

The modification $\rho_{\underline{k}} : B\mathbb{P}^2 \to \mathbb{P}^2$ will be constructed in two steps. First, we define a toric modification of $\mathbb{P}^2$ with exceptional curves $\bigcup_{j=2}^r V_j$, all above $[1:0:0]$, such that $L_0 \cup (\bigcup_{j=2}^r V_j)$ form a string. After this modification, $\sum_{1 \leqslant i \leqslant r}(a_i - k_i)$ indeterminacy points survive; they will be eliminated in the second step by blowing up each point once.

Recall that $z_0 = \chi^{\overline{v}_0}$ and $z_1 = \chi^{\overline{v}_1}$. Denote by $(u_1, u_{r+1}) \in N$ the dual basis of $(\overline{v}_0, \overline{v}_1)$ and by $\tilde{\sigma}$ the cone generated by it. Hence, the affine plane $\mathbb{C}^2$ of coordinates (z_0, z_1) is identified with the toric surface $\mathcal{Z}_{\tilde{\sigma},N}$. Take also $u_0 := -(u_{r+1} + u_1)$ and the complete regular fan $\mathcal{F}_0$ which has 1-dimensional cones that are generated by u_0, u_1 and u_{r+1}. Then $\mathcal{Z}_{\mathcal{F}_0,N} = \mathbb{P}^2$.

Next, consider the complete regular fan $\mathcal{F}_{\underline{k}}$ subdividing $\mathcal{F}_0$, which has 1-dimensional cones that are generated by the primitive elements $u_0, u_1, \ldots, u_{r+1}$ of N such that (see Figure 8)

$$\begin{aligned}
&u_0 + u_2 = (k_1 - 1)u_1, \\
&u_{j-1} + u_{j+1} = k_j u_j \quad \text{for all } j \in \{2, \ldots, r\}, \\
&u_{r+1} + u_1 = -u_0.
\end{aligned} \tag{8.4.1}$$

Its existence is ensured by the fact that $\underline{k}$ is an admissible sequence that represents 0.

Now, $\mathcal{F}_{\underline{k}}$ being a subdivision of $\mathcal{F}_0$, it induces a proper birational toric morphism

$$\mathcal{Z}_{\mathcal{F}_{\underline{k}},N} \xrightarrow{\psi_{\underline{k}}} \mathcal{Z}_{\mathcal{F}_0,N} = \mathbb{P}^2. \tag{8.4.2}$$

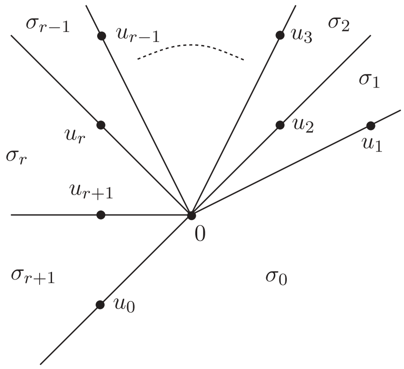


FIGURE 8. *The complete regular fan $\mathcal{F}_{\underline{k}}$.*

For any $j \in \{0, \ldots, r\}$, denote by $O_j \simeq \mathbb{C}^*$ the orbit in $\mathcal{Z}_{\mathcal{F}_{\underline{k}},N}$ corresponding to the 1-dimensional cone generated by u_j, and by V_j its closure. Clearly $V_0 = L_\infty$ and $V_1 = L_0$.

In fact, $\psi_{\underline{k}}$ can also be characterized independently of toric geometry: it is the unique modification with exceptional divisors $\{V_j\}_{j=2}^r$, all above the point $(0, 0)$, such that $L_0 \cup V_2 \cup \ldots \cup V_r$ form a string with self-intersections $1 - k_1, -k_2, \ldots, -k_r$; cf. [**26**, (1.6)].

For all $j \in \{0, \ldots, r\}$ let σ_j be the cone generated by (u_j, u_{j+1}) and let (ξ_j, η_j) be the dual basis of (u_j, u_{j+1}). Set the corresponding monomials $x_j := \chi^{\xi_j}$ and $y_j := \chi^{\eta_j}$. Therefore, in $\mathcal{Z}_{\sigma_j,N}$ one has $\{x_j = 0\} = V_j$ and $\{y_j = 0\} = V_{j+1}$. Moreover, the restriction

$$\psi_j : \mathcal{Z}_{\sigma_j,N} \longrightarrow \mathcal{Z}_{\tilde{\sigma},N} = \mathbb{C}^2, \tag{8.4.3}$$

of $\psi_{\underline{k}}$ to $\mathcal{Z}_{\sigma_j,N} \subset \mathcal{Z}_{\mathcal{F}_{\underline{k}},N}$ is described by the following monomial changes of variables:

$$\left.\begin{aligned} z_0 &= x_j^{Z_{j-1}(k_1,\ldots,k_{j-1})} y_j^{Z_j(k_1,\ldots,k_j)} \\ z_1 &= x_j^{Z_{j-2}(k_2,\ldots,k_{j-1})} y_j^{Z_{j-1}(k_2,\ldots,k_j)} \end{aligned}\right. \quad \text{for all } j \in \{1, \ldots, r\}. \tag{8.4.4}$$

This is a consequence of the fact that, for all $j \in \{1, \ldots, r\}$, one has

$$u_j = Z_{j-1}(k_1, \ldots, k_{j-1})u_1 + Z_{j-2}(k_2, \ldots, k_{j-1})u_{r+1}. \tag{8.4.5}$$

The relation (8.4.5) follows by (increasing) induction, (8.4.1) and determinantal relations as in (2.1.3).

THEOREM 8.4.6. *Consider the birational map $\Psi_{\underline{k}} := \psi_{\underline{k}} \circ \widehat{\pi} : \mathcal{Z}_{\mathcal{F}_{\underline{k}},N} \dashrightarrow \widehat{\mathcal{X}_{\underline{k}}^t}$. The indeterminacy points of $\Psi_{\underline{k}}$ are contained in $\bigcup_{j=1}^r O_j$. Moreover, each orbit O_j contains precisely $a_j - k_j$ indeterminacy points, which are eliminated by blowing up each point once. Let $\rho_{\underline{k}}$ be the composition of $\Psi_{\underline{k}}$ with these blow-ups and let C be any of the new exceptional curves obtained by one of these $\sum_{j=1}^r (a_j - k_j)$ blow-ups. Then $\rho_{\underline{k}}(C)$ is a curve with $\rho_{\underline{k}}(C) \not\subset C_{\underline{k}}^\infty$.*

Moreover, $\Psi_{\underline{k}}$ and $\rho_{\underline{k}}$ map $\bigcup_{j=0}^r V_j$ and their strict transforms, respectively, to $C_{\underline{k}}^\infty$.

Proof. Note that, for any $j \in \{1, \ldots, r\}$, in the chart $\mathcal{Z}_{\sigma_j,N}$ with affine coordinates (x_j, y_j) one has $\{x_j = 0\} = O_j \cup \{V_j \cap V_{j+1}\}$, and hence these affine coordinate axes cover all the exceptional locus and indeterminacy points. Hence, it is enough to analyse in each chart (x_j, y_j) the behaviour of $\Psi_{\underline{k}}$ along $\{x_j = 0\}$. For each $i \in \{1, \ldots, r+1\}$ and $j \in \{1, \ldots, r\}$, define

$$m_i^{(j)} := \begin{cases} Z_{j-i-1}(k_{i+1}, \ldots, k_{j-1}) & \text{if } i \leqslant j, \\ -Z_{i-j-1}(a_{j+1}, \ldots, a_{i-1}) & \text{if } i > j. \end{cases} \tag{8.4.7}$$

The next technical lemma will not only guarantee that $m_i^{(j)}$ is the valuative order of $z_i \circ \psi_{\underline{k}}$ along V_j, but also it gives a rather complete structure of the pull-back $z_i \circ \psi_j$ as well, where the maps ψ_j are defined by (8.4.3).

LEMMA 8.4.8. *For any fixed j, one has*

$$z_i \circ \psi_j = x_j^{m_i^{(j)}} \, y_j^{m_i^{(j+1)}} \, Q_i^{(j)} \tag{8.4.9}$$

for some $Q_i^{(j)} \in \mathbb{Z}[t, x_j, y_j]$, which has the following properties too.

(a) *We have*

$$Q_i^{(j)}\Big|_{x_j=0} = \begin{cases} c_1 & \text{for } i \leqslant j, \\ c_1'(c_2 y_j^{a_j-k_j} + c_3 t)^{Z_{i-j-1}(a_{j+1},\dots,a_{i-1})} & \text{for } i > j \end{cases}$$

for some non-zero constants c_1, c_1', c_2 and c_3, where c_2 and c_3 are independent of i.

(b) *Let $y_j = \xi$ be one of the roots of $c_2 y_j^{a_j-k_j} + c_3 t = 0$. For each ξ expand $Q_i^{(j)}$ in Taylor series in local variables $(x_j, y_j - \xi)$, and write it as a sum $\sum_{h \geqslant h_\xi} Q_i^{(j)}(h)$ of homogeneous polynomials $Q_i^{(j)}(h)$ of degree h in these local variables, such that $Q_i^{(j)}(h_\xi) \neq 0$. Then we have*

$$h_\xi = Z_{i-j-1}(a_{j+1}, \dots, a_{i-1}).$$

Hence, by (a) *and* (b), *the variable x_j does not divide $Q_i^{(j)}(h_\xi)$.*

Proof. The proof is straightforward and elementary. It uses, for any fixed $j \in \{1, \dots, r\}$, induction over $i \in \{1, \dots, r+1\}$, the 'inductive equations' (8.2.3), the substitution (8.4.4) and inductive formulas relating $Z(\underline{x})$; cf. (2.1.3). For $i = 1$, we find that $z_1 \circ \psi_j$ is given by (8.4.4), which proves Lemma 8.4.8 with $Q_1^{(j)} = 1$. The inductive step is given by (8.2.3), namely,

$$\big(z_{i+1} \circ \psi_j\big) \cdot x_j^{m_{i-1}^{(j)}} y_j^{m_{i-1}^{(j+1)}} Q_{i-1}^{(j)} = \big(x_j^{m_i^{(j)}} \, y_j^{m_i^{(j+1)}} \, Q_i^{(j)}\big)^{a_i} + t\big(x_j^{m_i^{(j)}} \, y_j^{m_i^{(j+1)}} \, Q_i^{(j)}\big)^{k_i}. \tag{8.4.10}$$

Case $1 \leqslant i \leqslant j$. For some $i \leqslant j-1$, assume that Lemma 8.4.8 is satisfied for both i and $i-1$. We will verify it for $i+1$. First we analyse in (8.4.10) the exponents of x_j (the discussion for y_j-exponents is similar). From the right-hand side of (8.4.7) one factors out $k_i m_i^{(j)}$, and the inductive step for these exponents which we need to verify is $m_{i+1}^{(j)} = k_i m_i^{(j)} - m_{i-1}^{(j)}$, which follows from (2.1.3). Next, the inductive formula for $Q_{i+1}^{(j)}$ is as follows:

$$Q_{i+1}^{(j)} \cdot Q_{i-1}^{(j)} = x_j^{(a_i-k_i)Z_{j-i-1}(k_{i+1},\dots,k_{j-1})} y_j^{(a_i-k_i)Z_{j-i}(k_{i+1},\dots,k_j)} (Q_i^{(j)})^{a_i} + t(Q_i^{(j)})^{k_i}.$$

If $a_i > k_i$, then the exponent of x_j is positive, and hence $Q_{i+1}^{(j)}|_{x_j=0} = t(Q_i^{(j)}|_{x_j=0})^{k_i} \cdot (Q_{i-1}^{(j)}|_{x_j=0})^{-1}$ is constant by induction. If $a_i = k_i$, then one has a similar expression.

Case $i = j+1$. This is the first case when the $m_i^{(j)}$-expression changes its shape (cf. (8.4.7)) and $Q_i^{(j)}|_{x_j=0}$ is not constant. Note that $m_j^{(j)} = 0$ and $m_j^{(j+1)} = 1$, and hence the inductive steps for the coordinate exponents can easily be verified. Moreover, $Q_{j+1}^{(j)} \cdot Q_{j-1}^{(j)} = y_j^{a_j-k_j} + t(Q_j^{(j)})^{k_j}$, and hence Lemma 8.4.8(a) and (b) also follows with $h_\xi = 1$.

Case $i > j+1$. The exponents of x_j and y_j can be analysed similarly, while $Q_{i+1}^{(j)} \cdot Q_{i-1}^{(j)} = (Q_i^{(j)})^{a_i} + t(Q_i^{(j)})^{k_i} \cdot M$, where

$$M := x_j^{(a_i-k_i)Z_{i-j-1}(a_{j+1},\dots,a_{i-1})} y_j^{(a_i-k_i)Z_{i-j-2}(a_{j+2},\dots,a_{i-1})}.$$

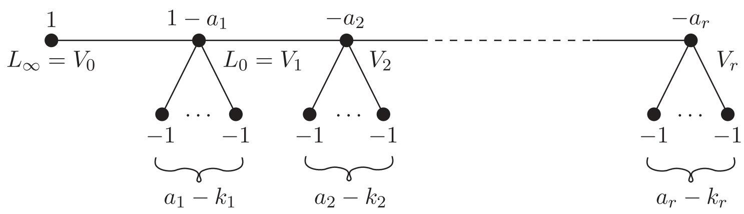


FIGURE 9. *Illustration for Corollary 8.4.11.*

Note that $Z_{i-j-1}(a_{j+1},\dots,a_{i+1})$ is always strictly positive (cf. Lemma 8.1.1(3)). Hence, $M\big|_{x_j=0} = 0$ if $a_i > k_i$ and $=1$ otherwise. Hence Lemma 8.4.8(a) and (b) follows again by (2.1.3). □

The function $z_i \circ \psi_j$ for $1 \leqslant i \leqslant j$ is regular, while for $i > j$ it is given by

$$z_i \circ \psi_j = \frac{Q_i^{(j)}}{x_j^{Z_{i-j-1}(a_{j+1},\dots,a_{i-1})}\, y_j^{Z_{i-j-2}(a_{j+2},\dots,a_{i-1})}}.$$

Note that the exponent $Z_{i-j-1}(a_{j+1},\dots,a_{i-1})$ is always strictly positive; cf. Lemma 8.1.1(3). The y_j-coordinates of the indeterminacy points on $\{x_j=0\}$ are given by $Q_i^{(j)}|_{x_j=0}$, which corresponds to the roots ξ introduced in the above technical Lemma 8.4.8. In particular, by this lemma, any of them is eliminated by one blow-up.

All the other statements of Theorem 8.4.6 now follow easily. This ends its proof. □

Theorem 8.4.6 shows that E_π has $(r-1)+\sum_{i=1}^r (a_i-k_i)$ irreducible components and that $C_\pi = \bigcup_{j=2}^r V_j$.

Using the correspondence between the equations relating the u_i in (8.4.1) and the self-intersections of the corresponding curves in the associated toric variety $\mathcal{Z}_{\mathcal{F}_{\underline{k}},N}$, we get the following corollary.

COROLLARY 8.4.11. *Consider the lines L_∞ and L_0 on $\mathbb{P}^2$ as above. Blow up $r-1+\sum_{i=1}^r (a_i-k_i)$ infinitely near points of L_0 in order to get the dual graph in Figure 9 of the configuration of the total transform of $L_\infty \cup L_0$ (this procedure topologically is unique, and its existence is guaranteed by the fact that $\underline{k} \in K_r(\underline{a})$). Denote the space obtained by this modification by $B\mathbb{P}^2$. Then the Milnor fibre $\mathcal{X}^t_{\underline{k}}$ of $S^{\mathrm{CS}}_{\underline{k}}$ is diffeomorphic to $B\mathbb{P}^2 \setminus (\bigcup_{j=0}^r V_j)$.*

Moreover, let T be an open tubular neighbourhood of $\bigcup_{j=0}^r V_j$, and set $F_{p,q}(\underline{k}) = B\mathbb{P}^2 \setminus T$. Then $F_{p,q}(\underline{k})$ is a representative of the Milnor fibre of $S^{\mathrm{CS}}_{\underline{k}}$.

Furthermore, the marking $\{V_i\}_i$, as in the Figure 9, defines on the boundary of $F_{p,q}(\underline{k})$ an order; denote this supplemented space by $F_{p,q}(\underline{k})^*$. Then its ordered boundary is $L(p,q)^*$.

8.5. *The identification of Lisca's fillings with Milnor fibres*

Let $W_{p,q}(\underline{k})^*$ be Lisca's filling endowed with the preferred order on its boundary (cf. (4.4)); and let $F_{p,q}(\underline{k})^*$ be the Milnor fibre as in Corollary 8.4.11.

THEOREM 8.5.1. *The ordered manifold $W_{p,q}(\underline{k})^*$ is orientation-preserving diffeomorphic to $F_{p,q}(\underline{k})^*$ by a diffeomorphism that preserves the orders of the boundaries.*

Proof. We see that $F_{p,q}(\underline{k})^*$ from Corollary 8.4.11 satisfies Proposition 4.4.2. Indeed, $B\mathbb{P}^2$, with V_0 anti-blown-down differentiably, will serve as the differentiable closed 4-manifold V. The homology classes of the spheres V_i are the s_i ($1 \leqslant i \leqslant r$), and the wished homology classes e with $e^2 = -1$ are the classes of the (-1) exceptional curves from Figure 9, multiplied by ± 1. Moreover, using the intersection form on $H_2(B\mathbb{P}^2)$, we see that these are the only classes e with $e^2 = -1$ that intersect non-trivially only one component among $V_1, \ldots, V_r$. In fact, all the homological computations in $H_2(B\mathbb{P}^2)$ fit perfectly with Lisca's computation from [**21**, § 4]. The compatibility of orders is guaranteed by the compatibilities of the constructions, see also §§ 3.5 and 3.6. □

8.6. *Remarks*

(1) Let $\rho_{\underline{k}}$ be the modification introduced above (cf. Theorem 8.4.6 or § 8.3) (as the minimal modification which eliminates the indeterminacy of $\widehat{\pi}|_{\mathbb{C}^2}$). Analysing the proof of Lemma 8.4.8 we realize that $\rho_{\underline{k}}$ serves also as the minimal modification which eliminates the indeterminacy of the last component of π from (8.3.5), namely of the rational function $z_{r+1} = P_{r+1}/z_0^{p-q}$. In particular, we find the following alternative description of the Milnor fibre $F_{p,q}(\underline{k})$.

For each $\underline{k} \in K_r(\underline{a})$, define the polynomial P_{r+1} via the inductive system (8.3.3). Let $\rho_{\underline{k}} : B\mathbb{P}^2 \to \mathbb{P}^2$ be the minimal modification of $\mathbb{P}^2$ which eliminates the indeterminacy points of P_{r+1}/z_0^{p-q} sitting in $\mathbb{C}^2$. Then the dual graph of the total transform of $L_\infty \cup L_0$ has the form indicated in Figure 9, and $F_{p,q}(\underline{k})$ is orientation-preserving diffeomorphic to $B\mathbb{P}^2 \setminus (\bigcup_{j=0}^r V_j)$.

(2) One proves that the irreducible decomposition of P_{r+1} has the following form:

$$P_{r+1} = \prod_{j=1}^{r} \prod_{\ell=1}^{a_j-k_j} [P_j + \xi_{j,\ell} \cdot z_0^{Z_{j-2}(a_2,\ldots,a_{j-1})}]^{Z_{j-1}(k_1,\ldots,k_{j-1})}, \tag{8.6.1}$$

where $\prod_{\ell=1}^{a_j-k_j}(\lambda + \xi_{j,\ell}) = \lambda^{a_j-k_j} + t$. Moreover, the strict transforms by $\rho_{\underline{k}}$ of these irreducible components define 'curvettas' of the -1 curves from Figure 9.

(3) Sections 8.2 and 8.3 contain some common results with Balke's paper [**3**]. In fact, [**3**] convinced us that the identification Theorem 8.5.1 should be guided by a rather straightforward construction.

9. *Invariants of 4-manifolds by closing boundaries*

In this section we present a procedure that provides invariants for 4-manifolds with boundary W, by 'closing' them with another (fixed) 4-manifold U (a '*cap*'). Our main motivation is Lisca's criterion Proposition 4.3.1 and his construction in [**21**, § 7]. A similar 'closing' will appear naturally for the Milnor fibres of sandwiched singularities as well (§ 10.1).

Then we generalize the results of § 4.3: we will not only replace the plumbing 4-manifold $\Pi(\underline{a})$ by an arbitrary 4-manifold U (with the same boundary), but also show that the same criterion works for any such U that satisfies some homological properties.

9.1. *The closing procedure*

Let us fix a 4-manifold with boundary U, which will be used as a 'cap' for other 4-manifolds W.

Assume that, for some 4-manifold with boundary W, we have an (orientation-preserving) diffeomorphism $\phi : \partial W \to \partial \overline{U}$. Then we construct the closed manifold $V = V(W, U, \phi)$ by gluing W and U along their boundaries using ϕ. We say that V is obtained by *closing* the boundary of W by U. Its diffeomorphism type depends only on the isotopy class of ϕ. We write $\mu : U \hookrightarrow V$ for the inclusion, and $\partial_{U,\phi}$ for the composition $\phi_*^{-1} \circ \partial_U : H_2(U, \partial U) \to H_1(\partial U) \to H_1(\partial W)$.

Our goal is to establish some properties of W read from the homology of V.

In what follows, we suppose that $H_1(U)=0$ *and* $H_2(U)$ *is free with a fixed basis* $\underline{c}:=(c_1,\dots,c_r)$. Denote by $\underline{c}^*:=(c_1^*,\dots,c_r^*)$ the dual basis of $H_2(U,\partial U)$ (via the intersection pairing $H_2(U)\otimes H_2(U,\partial U)\to\mathbb{Z}$, which is a perfect pairing under the above assumptions). Let $\mathcal{M}(\underline{c}):=(Q_U(c_i,c_j))_{i,j}\in \mathrm{Mat}_{r,r}(\mathbb{Z})$ be the intersection matrix of U.

Once we close W by U, we concentrate on the following homological objects: $\partial_{U,\phi}(\underline{c}^*):=(\partial_{U,\phi}(c_1^*),\dots,\partial_{U,\phi}(c_r^*))$ in $H_1(\partial W)^r$, and the image $\mu_*(\underline{c})\in H_2(V)^r$ of $\underline{c}$.

Proposition 9.1.1. *Suppose that ∂W is a rational homology sphere. Then, up to an isomorphism (of such triplets), $(H_2(V),Q_V;\mu_*(\underline{c}))$ depends only on the manifold W, on $\partial_{U,\phi}(\underline{c}^*)\in H_1(\partial W)^r$ and on $\mathcal{M}(\underline{c})$, but not on the choice of the particular oriented 4-manifold U (with $H_1(U)=0$ and $H_2(U)$ free) used for the closing.*

Proof. The cohomological Mayer–Vietoris exact sequence, the vanishing $H^1(\partial W)=0$ and Poincaré–Lefschetz duality provide the exact sequence as follows:

$$0\longrightarrow H_2(V)\longrightarrow H_2(W,\partial W)\oplus H_2(U,\partial U)\stackrel{\Delta}{\to} H_1(\partial W).$$

Hence $H_2(V)=\ker\Delta$, where $\Delta(x\oplus y)=\partial_W(x)-\partial_{U,\phi}(y)$. Consider the exact sequence

$$0\longrightarrow H_2(U)\stackrel{i}{\longrightarrow} H_2(U,\partial U)\stackrel{\partial_{U,\phi}}{\longrightarrow} H_1(\partial W). \tag{9.1.2}$$

The form Q_U (given by $\mathcal{M}(\underline{c})$) extends to a rational form $Q_{U,\mathbb{Q}}$ on $H_2(U)_\mathbb{Q}$, and identifies $H_2(U,\partial U)$ with the sublattice of elements $x\in H_2(U)_\mathbb{Q}$ satisfying $Q_{U,\mathbb{Q}}(x,y)\in\mathbb{Z}$ for all $y\in H_2(U)$. Hence, the restriction of $Q_{U,\mathbb{Q}}$ provides a rational form $Q_{U,\partial U}:H_2(U,\partial U)^{\otimes 2}\to\mathbb{Q}$. In this way we recover $H_2(U,\partial U)$ with its form $Q_{U,\partial U}$ and the dual base $\underline{c}^*$, and the sublattice $H_2(U)$ in it. These, and the fact that $H_2(U)$ injects by $y\mapsto(0\oplus i(y))$ into $\ker\Delta$, show that $H_2(V)$ and $\mu_*(\underline{c})\in H_2(V)^r$ can be recovered from the input data.

Let us consider now W instead of U. The analogue of sequence (9.1.2) and a similar discussion as above show that the form Q_W extends to a rational form $Q_{W,\partial W}:H_2(W,\partial W)^{\otimes 2}\to\mathbb{Q}$. The point is that the wished Q_V is exactly the restriction of $Q_{W,\partial W}\oplus Q_{U,\partial U}$ to $\ker\Delta$ (which automatically takes only integral values). □

9.2. *The dependence on ϕ*

The following proposition shows that in the presence of an order, in Proposition 9.1.1 the choice of the gluing diffeomorphism ϕ is irrelevant.

Proposition 9.2.1. *Assume that U is a 4-manifold with boundary such that $\partial\bar{U}$ is identified with $L(p,q)^*$, $H_1(U)=0$ and $H_2(U)$ is free with a fixed base $\underline{c}$.*

Let W be a Stein filling of $L(p,q)^*$ (that is, on the boundary of W one can identify the preferred order of the lens space), and let V be obtained from W by closing its boundary with U using a gluing map that preserves the orientations and the orders of the boundaries. Then $(H_2(V),Q_V;\mu_*(\underline{c}))$ (constructed in Proposition 9.1.1) is independent of the choice of ϕ.

Moreover, $(H_2(V),Q_V;\mu_*(-\underline{c}))\simeq(H_2(V),Q_V;\mu_*(\underline{c}))$ too.

Proof. The argument is similar to § 4.3. The ambiguity regarding ϕ stays in the group $\mathrm{Diff}^{+,o}(L(p,q))$. If a gluing ϕ is replaced by $\varphi\circ\phi$, where $\varphi\in\mathrm{Diff}^{+,o}(L(p,q))$ induces on $H_1(L(p,q))$ multiplication by -1, then we can twist W by a self-diffeomorphism which induces on the boundary φ (as in § 4.3), or instead, we can just multiply the homology of W by -1. The last isomorphism can be realized via multiplication by -1 of $H_2(V)$. □

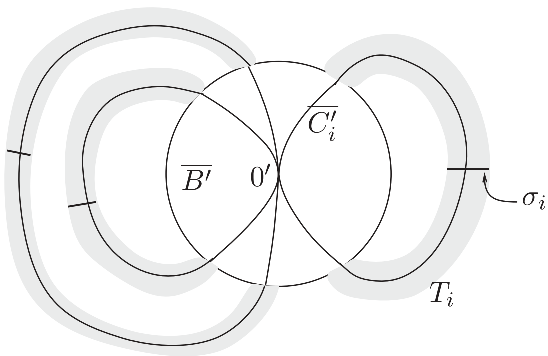


FIGURE 10. *The 4-manifold with boundary U.*

10. *'From de Jong and van Straten to Lisca'*

10.1. *Closing the boundary of the Milnor fibre*

We keep all the notation of § 6. We consider again a decorated germ (C, l) with smooth components C_i and a picture deformation (C_S, l_S).

As the disc configuration D is obtained by deforming C, its boundary $\partial D := \bigcup_{1 \leqslant i \leqslant r} D_i \hookrightarrow \partial B$ is isotopic as an oriented link to $\partial C \hookrightarrow \partial B$. Therefore, we can isotope D outside a compact ball containing all the points P_j till its boundary coincides with the boundary of C. Let (B', C') be a second copy of (B, C), and define

$$(V, \Sigma) := (B, D) \bigcup_{id} (\overline{B}', \overline{C}').$$

Here V is the oriented 4-sphere obtained by gluing the boundaries of B and $\overline{B}'$; and $\Sigma := \bigcup_{i=1}^{r} \Sigma_i$, where Σ_i is obtained by gluing D_i (perturbed by the above isotopy) and $\overline{C}_i'$ along their common boundaries. Moreover, one can also glue $(\overline{B}', \overline{C}')$ with $(\tilde{B}, \tilde{D})$ in such a way that the morphism β of (6.3.3) may be extended by the identity on $\overline{B}'$, yielding

$$(\tilde{V}, \tilde{\Sigma}) \xrightarrow{\beta} (V, \Sigma).$$

Here $\tilde{\Sigma} := \bigcup_{i=1}^{r} \tilde{\Sigma}_i$, where $\tilde{\Sigma}_i$ denotes the strict transform of the sphere Σ_i, that is, $\tilde{\Sigma}_i = \tilde{D}_i \cup \overline{C}_i'$. Write $T := \bigcup_{1 \leqslant i \leqslant r} T_i$ and set also (see Figure 10)

$$U := \overline{B}' \cup T. \tag{10.1.1}$$

Since $W = \tilde{B} \setminus T$ (cf. Proposition 6.3.4), $\tilde{V}$ is obtained by closing the boundary of W by the cap U. Our goal is to recognize W by a combination of Lisca's criterion (Proposition 4.3.1), of Proposition 9.1.1, and of Proposition 9.2.1 applied for this closing. We start the needed preparations for this program.

LEMMA 10.1.2. *The manifold U is independent of the chosen picture deformation (therefore one may close all the different Milnor fibres using the same U). In fact, each T_i is a 4-dimensional handle of index 2 glued to $\overline{B}'$ along the knot $\partial C_i \hookrightarrow \partial \overline{B}'$ endowed with the $(-l_i)$-framing.*

Proof. As $H_2(V) = 0$ (because $V \simeq \mathbb{S}^4$), we get $\Sigma_i^2 = 0$. As $\tilde{\Sigma}_i$ is obtained from Σ_i by blowing up (positively) l_i points on it and taking its strict transform, we deduce that $\tilde{\Sigma}_i^2 = -l_i$;

But this self-intersection is also equal to the self-linking number of the attaching circle $\partial \overline{C}_i'$ of the handle T_i with respect to the attaching framing. □

Assume now that the decorated curve (C, l) satisfying $X(C, l) = \mathcal{X}_{p,q}$ is chosen as in § 7. In particular, l is defined by (7.1.3) or (7.1.4). We assume that the components of C are marked as in Figure 4. We write $W(\underline{a}, \underline{k})$ for W and $\tilde{V}(\underline{a}, \underline{k}) = W(\underline{a}, \underline{k}) \cup U(\underline{a})$ for its closing by $U = U(\underline{a})$.

LEMMA 10.1.3. *The intersection numbers of the oriented spheres $(\tilde{\Sigma}_i)_{1 \leqslant i \leqslant r}$ inside the oriented 4-manifold $\tilde{V}(\underline{a}, \underline{k})$ are the following:*

$$\begin{aligned} \tilde{\Sigma}_i^2 &= -l_i \quad \text{for all } i \in \{1, \dots, r\}, \\ \tilde{\Sigma}_i \cdot \tilde{\Sigma}_j &= 1 - l_i \quad \text{for all } i < j. \end{aligned}$$

Proof. The first equalities were obtained for arbitrary decorated germs (C, l) with smooth components C_i during the proof of Lemma 10.1.2. For $i < j$, the surfaces $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_j$ meet at the origin of $\overline{B}'$. Therefore, $\tilde{\Sigma}_i \cdot_{\tilde{V}(\underline{a}, \underline{k})} \tilde{\Sigma}_j = \overline{C}_i \cdot_{\overline{B}} \overline{C}_j = -\overline{C}_i \cdot_B \overline{C}_j = -C_i \cdot C_j$. Then apply (7.1.2). □

Consider the following homology classes in $H_2(U(\underline{a}))$:

$$\begin{aligned} c_1 &:= [\tilde{\Sigma}_1] \\ c_i &:= [\tilde{\Sigma}_i] - [\tilde{\Sigma}_{i-1}] \quad \text{for all } i \in \{2, \dots, r\}. \end{aligned} \tag{10.1.4}$$

A direct consequence of Lemma 10.0.2 and of formula (7.1.4) is the following.

LEMMA 10.1.5. *One has the following intersection numbers of the homology classes c_i:*

$$c_i^2 = -a_i \quad \text{for all } i \in \{1, \dots, r\},$$
$$c_i \cdot c_j = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

Next, we wish to identify $\partial_U(\underline{c}^*)$.

PROPOSITION 10.1.6. *Using the notations (3.4.1), one has a canonical identification $\partial \overline{U}(\underline{a}) \simeq L(p, q)^*$ (that is, which also identifies the preferred order on the boundary). Moreover (using the notation of § 3.4), in $H_1(\partial U)^r$ the following equality holds:*

$$(\partial_U(c_1^*), \dots, \partial_U(c_r^*)) = \pm (\alpha_1^{\partial U}, \dots, \alpha_r^{\partial U}).$$

Proof. For each $i \in \{1, \dots, r\}$ denote by σ_i a co-core of the handle T_i (see Figure 10) and orient it such that its intersection number with $\tilde{\Sigma}_i$ is $+1$. Therefore $([\sigma_1], \dots, [\sigma_r])$ is the dual basis of $([\tilde{\Sigma}_1], \dots, [\tilde{\Sigma}_r])$ in $H_2(U, \partial U)$. Hence, from equations (10.1.4), we get

$$\begin{aligned} \partial_U[\sigma_i] &= \partial_U(c_i^*) - \partial_U(c_{i+1}^*) \quad \text{for all } i \in \{1, \dots, r-1\}, \\ \partial_U[\sigma_r] &= \partial_U(c_r^*). \end{aligned} \tag{10.1.7}$$

Now look at the oriented 4-manifold $\overline{U}$. By relation (10.1.1), we see that $\overline{U} := B' \cup (\bigcup_{1 \leqslant i \leqslant r} \overline{T}_i)$. We use the complex structure of B' to do blow-ups. Denote by $\tilde{\overline{U}} \xrightarrow{\pi} \overline{U}$ the composition of blow-ups of points above $0' \in \overline{U}$, such that the dual graph of the preimage

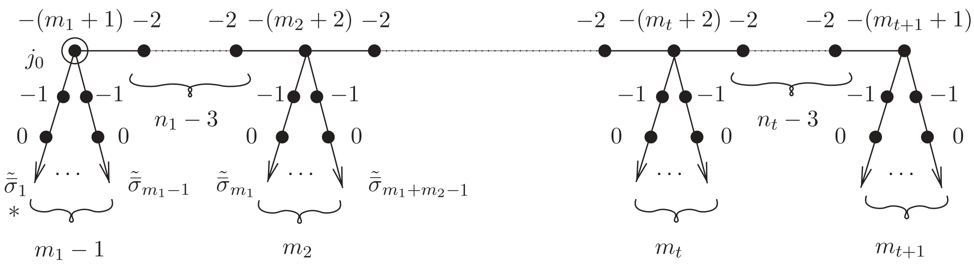


FIGURE 11. *The plumbed 4-manifold* $\tilde{\tilde{U}}$.

$\pi^{-1}(C')$ is isomorphic to the one from Figure 4. Then $\tilde{\tilde{U}}$ is a 4-manifold obtained by plumbing according to the graph of Figure 11 (this is equivalent to Lemma 10.1.2).

Note that its boundary can be canonically identified with $L(p,q)$ (via plumbing calculus). Indeed, first blowing down the (-1)-curves and then by anti-blowing down (in the differential category) the $(+1)$-curves arising from the (0)-curves, we get a plumbing graph which without arrowheads is exactly the graph $G(\underline{b})$. Considering in both graphs the preferred order (cf. § 3.5), we get the proof of the first statement. Note that this also appoints the preferred order to $\partial U(\underline{a})$.

In Figure 11, the arrowheads denote again the preimages of the co-cores $\overline{\sigma}_i$ (that is, σ_i with opposite orientation; they are analogues of the R_i of § 3.5). Therefore, by Proposition 3.6.1, up to a simultaneous change of sign, one has

$$\partial_U[\overline{\sigma}_i] = \nu_*(w_l) \quad \text{whenever} \quad m_1 + \ldots + m_{l-1} \leqslant i \leqslant m_1 + \ldots + m_l - 1,$$

where the vectors w_l were defined in Theorem 2.2.8. Then from the ‘duality relation’ stated in Theorem 2.2.8 we have

$$\partial_U[\sigma_i] = \alpha_i^{\partial U} - \alpha_{i+1}^{\partial U},$$

where $\alpha_{r+1}^{\partial U} := \nu_*(\overline{v}_{r+1}) = 0$. This combined with (10.1.7) ends the proof. □

If we sum up the results of this and the previous section, we get the following corollary.

COROLLARY 10.1.8. *Consider* $\underline{s} \in H_2(\Pi(\underline{a}))^r$ *as in Proposition* 4.3.1, *and* $\underline{c} \in H_2(U(\underline{a}))^r$ *defined in* (10.1.4). *Then the following facts hold.*

(I) (i) *We have* $Q_{\Pi(\underline{a})}(s_i, s_j) = Q_{U(\underline{a})}(c_i, c_j)$ *for all* i, j;
(ii) $\partial_{\Pi(\underline{a})}(\underline{s}^*) = \pm\partial_{U(\underline{a})}(\underline{c}^*)$,

(II) *Let* W *be a Stein filling of* $L(p,q)^*$ (*that is, on the boundary of* W *one can identify the preferred order of the lens space*). *Close its boundary* (*using a diffeomorphism which preserves the orientations and the order of the boundaries*) *by* $\Pi(\underline{a})$ *and* $U(\underline{a})$ *obtaining* V^{Π} *and* V^U, *respectively. Then we have*

$$\big(H_2(V^{\Pi}), Q_{V^{\Pi}}; \mu_*(\underline{s})\big) = \big(H_2(V^U), Q_{V^U}; \mu_*(\underline{c})\big).$$

This says that Lisca’s criterion (in order to recognize W), expressed originally in $\big(H_2(V^{\Pi}), Q_{V^{\Pi}}; \mu_*(\underline{s})\big)$ can be reinterpreted in $\big(H_2(V^U), Q_{V^U}; \mu_*(\underline{c})\big)$ too. Let us apply this for the closing $\tilde{V}(\underline{a},\underline{k}) = W(\underline{a},\underline{k}) \cup U(\underline{a})$, and search for the corresponding (-1) curves. Set:

$$E_j := \beta^{-1}(P_j) \quad \text{for all } j \in \{1, \ldots, n\}, \tag{10.1.9}$$

where the number n is defined in Theorem 7.2.2.

PROPOSITION 10.1.10. *One has the following equalities of matrices:*

$$\big(\tilde{\Sigma}_i \cdot E_j\big)_{1\leqslant i\leqslant r, 1\leqslant j\leqslant n} = \int D(\underline{a};\underline{k}),$$
$$\big(c_i \cdot E_j\big)_{1\leqslant i\leqslant r, 1\leqslant j\leqslant n} = D(\underline{a};\underline{k}),$$

where the entries of the left-hand side matrices are intersection numbers in $H_2(\tilde{V}(\underline{a},\underline{k}))$. In particular, for any fixed $i \in \{1,\ldots,r\}$ we have

$$\#\{j \in \{1,\ldots,n\} \mid c_i \cdot E_j \neq 0 \quad \textit{but} \quad c_k \cdot E_j = 0 \quad \textit{for all} \quad k \neq i\} = a_i - k_i.$$

Proof. By Definition 6.3.1 the matrix $(\tilde{\Sigma}_i \cdot E_j)_{i,j}$ is equal to the incidence matrix of the picture deformation corresponding to $\underline{k} \in K_r(\underline{a})$. Theorem 7.2.2 implies the first equality of matrices. The second one follows from the construction of $\int D(\underline{a};\underline{k})$ and from Definition 10.1.4. For the last statement we search for columns of $D(\underline{a};\underline{k})$ with only one non-zero entry. For fixed i they correspond exactly to the block $M_{r,a_i-k_i}(i)$ of (7.2.1). □

Finally we get the searched isomorphism between the Milnor fibres of the cyclic quotient singularity $\mathcal{X}_{p,q}$ and the Stein fillings of the standard contact structure on $L(p,q)$.

THEOREM 10.1.11. *Let $W(\underline{a},\underline{k})^*$ be the Milnor fibre $W(\underline{a},\underline{k})$ which has a boundary that is endowed with the preferred order induced by the graph from Figure 11 (which agrees with the order of $\partial\overline{U(\underline{a})}$ via the gluing $\tilde{V}(\underline{a},\underline{k}) := W(\underline{a},\underline{k}) \cup U(\underline{a})$). Then there is an orientation-preserving diffeomorphism that preserves the orders of the boundaries*

$$W(\underline{a},\underline{k})^* \simeq W_{p,q}(\underline{k})^*.$$

Proof. The statement follows from Propositions 4.3.1 and 4.4.2 combined with Corollary 10.1.8 once we check that

$$\#\{e \in H_2(\tilde{V}(\underline{a},\underline{k})) \mid e^2 = -1,\ c_i \cdot e \neq 0 \quad \text{but} \quad c_k \cdot e = 0 \text{ for all } k \neq i\} = 2(a_i - k_i).$$

For this, first note that $\tilde{V}(\underline{a},\underline{k})$ is obtained from $\mathbb{S}^4$ by n blow-ups, and hence $\{[E_i]\}_{i=1}^n$ forms a basis in its second homology group H_2, and the associated intersection matrix is diagonal with all entries -1. This shows the equality of the sets $\{e \in H_2 \ :\ e^2 = -1\} = \{\pm[E_1],\ldots,\pm[E_n]\}$. Then use Proposition 10.1.10. □

11. *Final conclusions*

11.1. *The two most important consequences of the previous sections*

COROLLARY 11.1.1. *Once the order of the links (or equivalently, the order of the coordinates in the two constructions) are choosen in a compatible way, Christophersen and Stevens on one side and de Jong and van Straten on the other side parametrize in the same way the components of the miniversal base space of $\mathcal{X}_{p,q}$ by the elements $\underline{k}$ of $K_r(\underline{a})$ as follows:*

$$S_{\underline{k}}^{\mathrm{CS}} = S_{\underline{k}}^{\mathrm{JS}}.$$

COROLLARY 11.1.2. *All the Milnor fibres $F_{p,q}(\underline{k})^*$ associated with different smoothing components and endowed with the preferred order on their ordered boundaries are different: their boundaries $L(p,q)^*$ and $\underline{k} \in K_r(\underline{a})$ determine uniquely all the Milnor fibres up to orientation-preserving diffeomorphisms which preserve the order of the boundary.*

11.2. *Further research problems*

The bijection (Corollary 11.1.1) is not realized by a direct correspondence (sitting in the theory of singularities); it goes through Lisca's classification. It would be interesting/important to find a construction inside algebraic geometry that would provide a direct identification.

Also, there exists another subtle open problem (which is not touched in the present paper), pointed out by the referee. This aims to provide a new description/characterization of the Milnor fibre of a component in terms of the topology/homology of the corresponding M-resolution, in the sense of Behnke and Christophersen [**5**] (for example, see the end of their Introduction).

Acknowledgements. The second-named author is grateful to Paolo Lisca for having attracted his attention on his conjecture. We are grateful to Jan Stevens for acquainting us with the manuscript [**3**] and to Duco van Straten for several helpful discussions. We thank the referee for the careful reading of the manuscript and for several historical information and mathematical suggestions, which definitely contributed to the improvement of the present paper and might serve as starting points for further research as well (for example, see § 11.2).

References

1. J. Arndt, 'Verselle Deformationen zyklischer Quotientensingularitäten', Dissertation Universität Hamburg, 1988.
2. M. Artin, 'Algebraic construction of Brieskorn's resolutions', *J. Algebra* 29 (1974) 330–348.
3. L. Balke, 'Smoothings of cyclic quotient singularities from a topological point of view', Preprint, 1999, arXiv:math/9911070.
4. W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact complex surfaces*, 2nd enlarged edn (Springer, Berlin, 2004).
5. K. Behnke and J. A. Christophersen, 'M-resolutions and deformations of quotient singularities', *Amer. J. Math.* 116 (1994) 881–903.
6. K. Behnke and O. Riemenschneider, 'Quotient surface singularities and their deformations', *Singularity theory* (eds D. T. Lê, K. Saito and B. Teissier; World Scientific, Singapore, 1995), 1–54.
7. F. A. Bogomolov, and B. de Oliveira, 'Stein small deformations of strictly pseudoconvex surfaces', *Contemp. Math.* 207 (1997) 25–41.
8. F. Bonahon, 'Difféotopies des espaces lenticulaires', *Topology* 22 (1983) 305–314.
9. S. Brohme, 'Monodromieüberlagerung der versellen Deformation zyklischer Quotientensingularitäten', Dissertation Universität Hamburg, 2002. http://deposit.ddb.de/cgi-bin/dokserv?idn=964609487.
10. C. Caubel, A. Némethi and P. Popescu-Pampu, 'Milnor open books and Milnor fillable contact 3-manifolds', *Topology* 45 (2006) 673–689.
11. C. Caubel and P. Popescu-Pampu, 'On the contact boundaries of normal surface singularities', *C. R. Acad. Sci. Paris, Ser. I* 339 (2004) 43–48.
12. J. A. Christophersen, 'On the components and discriminant of the versal base space of cyclic quotient singularities', *Singularity theory and its applications*, Part I, (eds D. Mond, J. Montaldi), Lecture Notes in Mathematics 1462 (Springer, Berlin, 1991) 81–92.
13. Y. Eliashberg, 'Filling by holomorphic discs and its applications', *Geometry of low-dimensional manifolds*, vol. 2 London Mathematical Society, Lecture Note Series, 151 (Cambridge University Press, Cambridge, 1990), 45–67.
14. R. E. Gompf and A. I. Stipsicz, '4-manifolds and Kirby calculus', Graduate Studies in Mathematics 20, (American Mathematical Society, Providence, RI, 1999).
15. H. Grauert, 'Über die Deformationen isolierter Singularitäten analytische Mengen', *Invent. Math.* 15 (1972) 171–198.
16. T. de Jong and D. van Straten, 'Deformation theory of sandwiched singularities', *Duke Math. J.* 95 (1998) 451–522.
17. J. Kollár, 'Toward moduli of singular varieties', *Compos. Math.* 56 (1985) 369–398.
18. J. Kollár and N. I. Shepherd-Barron, 'Threefolds and deformations of surface singularities', *Invent. Math.* 91 (1988) 299–338.
19. D. T. Lê, 'Les singularités sandwich', Resolution of singularities: a research textbook in tribute to Oscar Zariski (eds H. Hauser, J. Lipman and F.Oort; Birkhäuser, Basel, 2000), 457–483.
20. P. Lisca, 'On lens spaces and their symplectic fillings', *Math. Res. Lett.* 11 (2004) 13–22.
21. P. Lisca, 'On symplectic fillings of lens spaces', *Trans. Amer. Math. Soc.* 360 (2008) 765–799.
22. E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society Lecture Note Series 77 (Cambridge University Press, Cambridge, 1984).

23. D. McDuff, 'The structure of rational and ruled symplectic 4-manifolds', *J. Amer. Math. Soc.* 3 (1990) 679–712.
24. K. Möhring, 'On sandwiched singularities', Thesis, Universität in Mainz, 2003.
25. W. Neumann, 'A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves', *Trans. Amer. Math. Soc.* 268 (1981) 299–344.
26. T. Oda, *Convex bodies and algebraic geometry* (Springer, Berlin, 1988).
27. H. Ohta and K. Ono, 'Symplectic fillings of the link of simple elliptic singularities', *J. Reine Angew. Math.* 565 (2003) 183–205.
28. H. Ohta and K. Ono, 'Simple singularities and symplectic fillings', *J. Differential Geom.* 69 (2005) 1–42.
29. P. Orlik and P. Wagreich, 'Algebraic surfaces with k^*-action', *Acta Math.* 138 (1977) 43–81.
30. P. Popescu-Pampu, 'The geometry of continued fractions and the topology of surface singularities', *Adv. Stud. Pure Math.* 46 (2007) 119–195.
31. O. Riemenschneider, 'Deformationen von Quotientensingularitäten (nach zyklischen Gruppen)', *Math. Ann.* 209 (1974) 211–248.
32. O. Riemenschneider, 'Zweidimensionale Quotientensingularitäten: Gleichungen und Syzygien', *Arch. Math.* 37 (1981) 406–417.
33. M. Schlessinger, 'Functors of Artin rings', *Trans. Amer. Math. Soc.* 130 (1968) 208–222.
34. M. Spivakovsky, 'Sandwiched singularities and desingularization of surfaces by normalized Nash transformation', *Ann. of Math.* (2) 131 (1990) 411–491.
35. J. Stevens, 'On the versal deformation of cyclic quotient singularities', *Singularity theory and its applications*, Part I (eds D. Mond and J. Montaldi), Lecture Notes in Mathematics 1462 (Springer, Berlin, 1991) 312–319.
36. J. Stevens, *Deformations of singularities*, Lecture Notes in Mathematics 1811 (Springer, Berlin, 2003).
37. J. Wahl, 'Smoothings of normal surface singularities', *Topology* 20 (1981) 219–246.

András Némethi
Rényi Institute of Mathematics
POB 127
H-1364 Budapest
Hungary

nemethi@renyi.hu

Patrick Popescu-Pampu
Université Paris 7 Denis Diderot
Institut de Mathématiques–UMR CNRS 7586
Équipe "Géométrie et dynamique"
Site Chevaleret
Case 7012, 75205 Paris Cedex 13
France

ppopescu@math.jussieu.fr