ON THE DENSITY OF SOME SEQUENCES OF NUMBERS

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The functions $f(m)$, $\phi(m)$ are called additive and mutiplicative respectively if they are defined for non-negative integers m, and if, for $(m_1, m_2) = 1$, that is, for m_1 and m_2 relatively prime,

$$f(m_1 m_2) = f(m_1) + f(m_2),$$

$$\phi(m_1 m_2) = \phi(m_1)\,\phi(m_2).$$

We consider only functions $f(m)$, $\phi(m)$ satisfying the following conditions:

(1) $$f(m) \geqslant 0.$$

(2) $\Sigma f(p)/p$ converges when the summation is extended to all primes p.

(3) $f(p_1) \neq f(p_2)$ if p_1, p_2 are different primes.

(4) $$\phi(m) \geqslant 1.$$

(5) $$\Sigma \frac{\phi(p)-1}{p} \text{ converges.}$$

(6) $$\phi(p_1) \neq \phi(p_2).$$

* Received 25 October, 1934; read 15 November, 1934.

It is shown later on that conditions (2), (5) are superfluous. We shall throughout denote by $N(f;\ c,\ d)$ the number of positive integers not exceeding n, say $m_1,\ m_2,\ \ldots,$ or simply m, for which

$$c \leqslant f(m) \leqslant d,$$

where c, d are given constants; when $d=\infty$, we use simply $N(f;\ c)$. We shall now prove the

Theorem. *If c is a given number, then $N(f,\ c)/n$ and $N(\phi,\ c)/n$ tend to limits when $n \to \infty$.*

We have the case of the abundant numbers on taking $c=2$,

$$\phi(m)=\sigma(m)/m,$$

where $\sigma(m)$ denotes the sum of the divisors of m.

It is sufficient to consider additive functions, since, if $\phi(m)$ is multiplicative, $\log \phi(m)$ is additive. Also if $\Sigma\,\{\phi(p)-1\}/p$ converges, so does $\Sigma\,\{\log \phi(p)\}/p$, since, if $x>1$, $\log x < x-1$.

The method will be more intelligible if we consider first the special case in which $f(p^a)=f(p)$ for any integral exponent a; so that

$$f(m)=\sum_{p|m} f(p).$$

Consider also the function

$$f_k(m)=\sum_{\substack{p|m\\ p\leqslant p_k}} f(p),$$

where p_k denotes the k-th prime. We show that $N(f_k,\ c)/n$ tends to a limit A_k. For if we denote by $a_1,\ a_2,\ \ldots,\ a_l$ the integers whose prime factors are not greater than p_k and for which also $f_k(a_i)\geqslant c$, we find the integers $m \leqslant n$ for which $f_k(m)\geqslant c$ by taking all the multiples of $a_1,\ a_2,\ \ldots,\ a_l$ not exceeding n. Hence $N(f_k,\ c)/n \to A_k$ say.

Since $f_{k+1}(m)\geqslant f_k(m)$, $A_{k+1}\geqslant A_k$, and since $A_k \leqslant 1$, $\lim\limits_{k\to\infty} A_k = A$ exists. We prove that $N(f;\ c)/n \to A$. It is sufficient to prove that, for every $\epsilon>0$, a k exists so great that, for $n>n(\epsilon)$, $\{N(f;\ c)-N(f_k;\ c)\}/n<\epsilon$. This means that the number of integers $m\leqslant n$, for which $f_k(m)<c$, and $f(m)\geqslant c$, is less than ϵn.

We require two lemmas.

Lemma 1. *We can find a number δ such that*

$$N(f;\ c,\ c+\delta) < \tfrac{1}{2}\,\epsilon n.$$

The proof is similar to my proof* for the estimate of the primitive abundant numbers. Let $p_1 < p_2 < \ldots < p_t$ be consecutive primes, not necessarily the first prime, second prime, ..., satisfying the conditions

$$(a) \qquad p_1 > \frac{6}{\epsilon},$$

$$(b) \qquad \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right) < \frac{\epsilon}{6},$$

$$(c) \qquad \sum_{i=1}^{t} \frac{1}{p_i^2} < \frac{\epsilon}{6}.$$

We can satisfy these conditions since $\prod_1^\infty (1 - p_i^{-1})$ diverges and $\sum_1^\infty p_i^{-2}$ converges.

Now choose δ so that

$$0 < \delta < \min |f(p_i) - f(p_j)|,$$

where p_i, p_j are any two different primes from $p_1, \ldots, p_t$. This is possible by (3).

We now show that it is sufficient to consider only such of the $N(f;\, c, c+\delta)$ as satisfy the two further conditions

(α) m is divisible by one of the p_i $(i = 1, 2, \ldots, t)$.

(β) m is not divisible by any one of the p_i^2 $(i = 1, 2, \ldots, t)$.

For it follows from (b) and (c) that the number of integers $m \leqslant n$ which do not satisfy either (α) or (β) is less than $\frac{1}{3}\epsilon n$ for $n > n(\epsilon)$, *i.e.* the number of integers $m \leqslant n$ not satisfying (α) is given by

$$n - \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_2}\right] \ldots + \left[\frac{n}{p_1 p_2}\right] + \ldots - \left[\frac{n}{p_1 p_2 p_3}\right] \ldots = n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right) + R,$$

$$|R| < 2^t;$$

while the number not satisfying (β) is less than

$$\left[\frac{n}{p_1^2}\right] + \ldots \left[\frac{n}{p_t^2}\right] < n \sum_{r=1}^{t} \frac{1}{p_r^2} + t.$$

Also t is independent of n.

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* P. Erdös, "On the density of the abundant numbers", *Journal London Math. Soc.*, 9 (1934), 278–282.

From (α), m_λ/p_i is an integer for all λ and appropriate p_i. From (α),

$$\frac{m_\lambda}{p_i}<\frac{\epsilon n}{6},$$

and so the number of the m satisfying (α), (β) is less than $\frac{1}{6}\epsilon n$, since we shall now prove that all these quotients are different. Suppose that

$$\frac{m_\lambda}{p_i}=\frac{m_\mu}{p_j},$$

and so $p_i \neq p_j$ if $\lambda \neq \mu$. Then

$$f\left(\frac{m_\lambda}{p_i}\right)=f\left(\frac{m_\mu}{p_j}\right),$$

$$f(m_\lambda)-f(p_i)=f(m_\mu)-f(p_j),$$

$$f(m_\lambda)-f(m_\mu)=f(p_i)-f(p_j).$$

But

$$f(m_\lambda)-f(m_\mu)\leqslant\delta,\ f(p_i)-f(p_j)>\delta,$$

and this contradiction shows that the quotients are different.

LEMMA 2. *The number of integers* $m\leqslant n$, *for which*

$$f(m)-f_k(m)>\delta,$$

is less than $\frac{1}{2}\epsilon n$ *for sufficiently large* $k=k(\epsilon)$ *say.*

For clearly

$$\sum_{m=1}^{n}\{f(m)-f_k(m)\}=\sum_{p=p_{k+1}}^{n}\left[\frac{n}{p}\right]f(p)<n\sum_{p=p_{k+1}}^{\infty}\frac{f(p)}{p}<\tfrac{1}{2}\epsilon\delta n,$$

from (2), for sufficiently large k. Hence the lemma is proved.

We now proceed to the proof of our theorem. We divide the integers not greater than n satisfying the two conditions

$$f_k(m)<c,\ f(m)\geqslant c,$$

where $k=k(\epsilon)$, into two classes. In the first class, we put the integers m for which $f(m)>c+\delta$. For these $f(m)-f_k(m)>\delta$, and so, from Lemma 2, their number is less than $\frac{1}{2}\epsilon n$. In the second class, we put the integers for which $f(m)\leqslant c+\delta$. Their number is less than $\frac{1}{2}\epsilon n$ from

Lemma 1. Hence our theorem is proved for the special case when

$$f(p^{\alpha}) = f(p).$$

The transition to the general case is now so simple that it will suffice to sketch the proof. We have, for

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h},$$

$$f(m) = \sum_{i=1}^{h} f(p_i^{\alpha_i}).$$

Call $g_1, g_2, \ldots$ the integers all of whose prime-factors occur with an exponent greater than 1. We have

$$\sum_{i=1}^{\infty} \frac{1}{g_i} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \ldots\right),$$

where the product, which refers to all the primes p, converges.

Just as in the special case, we can prove that, if we denote by $N^{(i)}$ the number of integers $m^{(i)} \leqslant n$ whose quadratic part is g_i, *i.e.* the greatest g by which $m^{(i)}$ is divisible, and for which $f(m^{(i)}) \geqslant c$, then $N^{(i)}/n$ tends to a limit A_i. It is evident that $\Sigma A_i = A$ is a convergent series since $A_i \leqslant 1/g_i$. It is also easily seen that A is the density of the integers m for which $f(m) \geqslant c$.

We can discuss similarly the slightly more general case when we replace condition (2) by (2′), which includes (2).

(2′) The primes 2, 3, 5, ... can be divided into two classes q and r so that both $\Sigma \{f(q)/q\}$ and $\Sigma (1/r)$ converge.

The proof runs just as above, the only difference being that now we define $g_1, g_2, \ldots$ as the integers which are the product of an integer composed of the r_i and of an integer whose prime factors all occur with an exponent greater than 1.

If, however, the additive function $f(m)$ satisfies (1) and does not satisfy (2′), so that now condition (3) is omitted, then $N(f; c)/n \to 1$. We consider the special case when $f(n) = \sum_{p|n} f(p)$. The general case can be dealt with similarly.

The proof is deducible from the following theorem of P. Turán, which he has communicated to me.

Let $\chi(m)$ be an additive function which is bounded and not negative for all primes $m = p_i$ and suppose that

$$\sum_{p \leqslant n} \frac{\chi(p)}{p} = \psi(n) \to \infty \quad \text{as} \quad n \to \infty.$$

Then the number of integers $m \leqslant n$ for which

$$|\chi(m) - \psi(n)| > \epsilon \psi(n)$$

is $o(n)$.

Turán's proof is similar to his proof* of the Hardy-Ramanujan theorem that almost all integers not greater than n have $\log \log n$ prime factors. Their theorem is, in fact, given by taking $\chi(p) = 1$.

We now divide the primes 2, 3, 5, ... into two classes q' and r' such that $f(q') \geqslant 1$ for the q', and $f(r') < 1$ for the r' if $\Sigma (1/q')$ diverges; we denote $\sum_{q' \leqslant n} (1/q')$ by $\psi_1(n)$. It immediately follows from Turán's theorem, on putting $\chi(q') = 1$, $\chi(r') = 0$, that the number of integers not greater than n which are divisible by more than $\psi_1(n)(1+\epsilon)$ or less than $\psi_1(n)(1-\epsilon)$ of the q' is $o(n)$. Since $f(q') \geqslant 1$, we have, for almost all integers not greater than n,

$$f(m) > (1-\epsilon)\psi_1(n),$$

and so $N(f, c)/n \to 1$.

If $\Sigma (1/q')$ converges, then, from (2'), $\Sigma (1/r')$ diverges.

Put $\chi(r') = f(r')$ and $\chi(q') = 0$. Since (2') is not satisfied,

$$\sum_{p<n} \frac{\chi(p)}{p} = \psi_2(n) \to \infty;$$

also $\chi(p) \leqslant 1$. Hence, by Turán's theorem, we have, for almost all integers not greater than n,

$$\chi(m) > (1-\epsilon)\psi_2(n).$$

Since $f(m) \geqslant \chi(m)$, $N(f, c)/n \to 1$ also; thus our theorem is proved.

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* P. Turán, "On a theorem of Hardy and Ramanujan", *Journal London Math. Soc.*, 9 (1934), 274–276.

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