On sequences of positive integers.

By

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1. Let $a_1, a_2, \dots$ be any sequence of (different) positive integers, and let $b_1, b_2, \dots$ be the sequence consisting of all positive integers which are divisible by at least one a. We define

$$A_1 = \frac{1}{a_1},$$

$$A_2 = \frac{1}{a_2} - \frac{1}{[a_1, a_2]},$$

$$\dots\dots\dots\dots\dots$$

$$A_\nu = \frac{1}{a_\nu} - \sum_{\mu<\nu} \frac{1}{[a_\mu, a_\nu]} + \sum_{\lambda<\mu<\nu} \frac{1}{[a_\lambda, a_\mu, a_\nu]} - \dots,$$

where $[a, b, c, \dots]$ denotes the least common multiple of $a, b, c, \dots$ Then A_ν is easily seen to be the density of those integers which are divisible by a_ν but not by any one of $a_1 \dots a_{\nu-1}$. Hence $A_\nu \geqq 0$, and $\sum_1^m A_\nu$, being the density of those integers which are divisible by at least one of $a_1, \dots, a_m$, is less than 1. If we define

$$A = \sum_1^\infty A_\nu$$

then $0 < A \leqq 1$, and it is reasonable to expect that A is the density in some sense of the sequence $\{b_i\}$. It was proved by Besicovitch[1]) that the sequence $\{b_i\}$ may have different upper and lower densities. We shall prove (§ 2) that the "logarithmic density" of $\{b_i\}$ exists and has the value A, and also that the *lower* density of $\{b_i\}$ has the value A.

In § 3 we use the former of these results to prove that if a sequence $a_1, a_2, \ldots$ of positive integers has the property

$$\overline{\lim_{x=\infty}} (\log x)^{-1} \sum_{a_m \leqq x} a_m^{-1} > 0,$$

then it a has subsequence $a_{i_1}, a_{i_2}, \ldots$ in which $a_{i_k} \mid a_{i_{k+1}}$ $(k = 1, 2, \ldots)$. Naturally every sequence of positive lower density satisfies the condition.

2. Let $\theta(n)$ be 1 if n is a b_i (i. e. if there is an $a_j \mid n$) and 0 otherwise. Let

$$F(s) = \sum_1^\infty \theta(n) n^{-s} \qquad (s > 1).$$

Let

$$A_\nu(s) = \frac{1}{a_\nu^s} - \sum_{\mu<\nu} \frac{1}{[a_\mu, a_\nu]^s} + \sum_{\lambda<\mu<\nu} \frac{1}{[a_\lambda, a_\mu, a_\nu]^s} - \ldots,$$

so that $A_\nu(1) = A_\nu$, and

$$A(s) = \sum_1^\infty A_\nu(s).$$

Then it is easily seen that

$$F(s) = \zeta(s) A(s)$$

for $s > 1$.

L e m m a 1: *If* $1 < s_1 < s_2$, *then for any* m,

$$\sum_1^m A_\nu(s_2) \leqq \sum_1^m A_\nu(s_1).$$

P r o o f: Let $\theta_m(n)$ be 1 if n is divisible by any one of $a_1, \ldots, a_m$ and 0 otherwise, and let $F_m(s) = \sum_{n=1}^\infty \theta_m(n) n^{-s}$. As before

$$F_m(s) = \zeta(s) \sum_1^m A_\nu(s).$$

We have the inequality

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[1]) Math. Annalen 110 (1934), 336 — 341.

$$(1) \qquad \theta_m(n)\log n \geqq \sum_{d|n} \theta_m(d)\,\Lambda\left(\frac{n}{d}\right)$$

for all n. For if $\theta_m(n)=0$ then $\theta_m(d)=0$ for all $d \mid n$, and if $\theta_m(n)=1$ then

$$\log n = \sum_{d|n} \Lambda\left(\frac{n}{d}\right) \geqq \sum_{d|n} \theta_m(d)\,\Lambda\left(\frac{n}{d}\right).$$

From (1):

$$\sum_{n=1}^{\infty} \theta_m(n)\log n\, n^{-s} \geqq \left(\sum_{n=1}^{\infty} \theta_m(n)\, n^{-s}\right)\left(\sum_{n=1}^{\infty} \Lambda(n)\, n^{-s}\right)$$

for $s>1$, i. e.

$$-F_m'(s) \geqq F_m(s)\left(-\frac{\zeta'(s)}{\zeta(s)}\right),$$

hence

$$\frac{d}{ds}\left(\sum_1^m A_\nu(s)\right) \leqq 0$$

for $s>1$, which proves the Lemma.

Lemma 2: *$A(s) \to A$ as $s \to 1$ $(s>1)$.*

Proof: By Lemma 1, we have for $s>1$ and any m,

$$\sum_1^m A_\nu(s) \leqq \lim_{s=1} \sum_1^m A_\nu(s) = \sum_1^m A_\nu \leqq A,$$

hence $A(s) \leqq A$. But

$$\underline{\lim}_{s=1} A(s) \geqq \lim_{s=1} \sum_1^m A_\nu(s) = \sum_1^m A_\nu,$$

and so

$$\underline{\lim}_{s=1}\, A(s) \geqq A,$$

which proves the Lemma.

Theorem 1: *(a) $\lim_{x=\infty} (\log x)^{-1} \sum_{n=1}^{x} \theta(n)\, n^{-1}$ exists and has the value A.*

$$(b)\ \underline{\lim}_{x=\infty}\, x^{-1} \sum_{n=1}^{x} \theta(n) = A.$$

Proof: By lemma 2,

$$(2) \qquad F(s) = \sum_1^{\infty} \theta(n)\, n^{-s} \sim \frac{A}{s-1}$$

as $s \to 1$ $(s > 1)$. Part (a) of the Theorem follows from this by a Tauberian theorem due to Hardy and Littlewood. [2])

As regards (b), it is obvious from the meaning of $\sum_{1}^{m} A_\nu$ as a density that the lower limit in (b) is $\geqq A$, and if equality did not hold we should have

$$s_n = \sum_{l=1}^{n} \theta(l) > (A + \delta) n$$

for some $\delta > 0$ and all $n \geqq N$, and so

$$F(s) = \sum_{1}^{\infty} s_n (n^{-s} - (n+1)^{-s}) > (A + \delta) \sum_{N+1}^{\infty} n^{-s},$$

which on making $s \to 1$ contradicts (2).

3. Theorem 2: *If $a_1, a_2, \ldots$ is a sequence of (different) positive integers, and*

$$\alpha = \overline{\lim_{x=\infty}} (\log x)^{-1} \sum_{a_n \leqq x} a_n^{-1} > 0,$$

then there exists a subsequence $a_{i_1}, a_{i_2}, \ldots$ such that $a_{i_k} \mid a_{i_{k+1}}$ $(k = 1, 2, \ldots)$.

Proof: It suffices to prove that there exists an a_i such that

$$\overline{\lim_{x=\infty}} (\log x)^{-1} \sum_{\substack{a_n \leqq x \\ a_i \mid a_n}} a_n^{-1} > 0. \tag{3}$$

We take r so large that

$$\sum_{\nu > r} A_\nu < \alpha, \tag{4}$$

and we shall prove that there exists an a_i with $i \leqq r$ satisfying (3). If the left side of (3) were zero for $i \leqq r$, we should have

$$\alpha = \overline{\lim_{x=\infty}} (\log x)^{-1} \sum_{\substack{a_n \leqq x \\ a_1 \nmid a_n, \ldots, a_r \nmid a_n}} a_n^{-1}$$

[2]) Proc. London Math. Soc. (2) 13 (1914), 174 — 191, Theorem 16.

$$\leqq \overline{\lim_{x=\infty}} (\log x)^{-1} \sum_{\substack{n=1 \\ a_1 \nmid n, \ldots, a_r \nmid n}}^{x} \theta(n)\, n^{-1}.$$

By Theorem 1 (a) the last expression has the value

$$A - \sum_{\tau=1}^{r} A_{\nu}.$$

From (4) we have a contradiction.

The condition in Theorem 2 is easily seen to be best possible of its kind, i. e. one can construct sequences $\{a_i\}$ for which

$$(\log x)^{-1} \sum_{a_n \leqq x} a_n^{-1}$$

tends to zero arbitrarily slowly, but in which no subsequence with the desired property exists.

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