ON THE DENSITY OF SOME SEQUENCES OF NUMBERS (II)

Paul Erdös*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 12, 1937.]

The functions $f(m)$ and $\phi(m)$ are called additive and multiplicative respectively if they are defined for non-negative integers m, and if, for $(m_1, m_2) = 1$,

$$f(m_1 m_2) = f(m_1) + f(m_2),$$

$$\phi(m_1 m_2) = \phi(m_1)\,\phi(m_2).$$

In my paper "On the density of some sequences of numbers†" I proved the following

Theorem. *Let the additive function $f(m)$ satisfy the following conditions:*

(1) $f(m) \geqslant 0$,

(2) $f(p_1) \neq f(p_2)$ *if* p_1, p_2 *are different primes.*

Further let $N(f;\ c,\ d)$ denote the number of positive integers m not exceeding n, for which

$$c \leqslant f(m) \leqslant d,$$

where c, d are given constants; when $d = \infty$, write $N(f;\ c)$ for $N(f;\ c, \infty)$. Then $N(f;\ c)/n$ tends to a limit as $n \to \infty$.

I shall now prove that condition (2) is superfluous. Just as in (I), it is sufficient to consider the case when f is such that $f(p) = f(p^\alpha)$, for any positive integer α. I use throughout the notation of (I).

The case in which $\sum_p \frac{f(p)}{p}$ diverges may be settled just as in (I).

Suppose then that $\sum_p \frac{f(p)}{p}$ is convergent.

First take the case in which $\sum_{f(p)\neq 0} \frac{1}{p}$ converges. Denote by $a_1, a_2, \ldots$ the integers composed of the primes p for which $f(p) \neq 0$. Evidently

$$\Sigma \frac{1}{a_i} = \prod_{f(p)\neq 0} \frac{1}{1-(1/p)}$$

converges.

---

* Received 6 June, 1936; read 18 June, 1936.

† *Journal London Math. Soc.*, 10 (1935), 120–125.

Let us denote by $a(m)$ the greatest a_i contained in m. Since $\sum_{f(p)\neq 0} \frac{1}{p}$ converges, it easily follows from the sieve of Eratosthenes that the density of integers not divisible by any p, with $f(p)\neq 0$, is equal to $\prod_{f(p)\neq 0}\left(1-\frac{1}{p}\right)$. Hence the density of the integers m for which $a(m)=a_i$ is

$$\frac{1}{a_i}\prod_{f(p)\neq 0}\left(1-\frac{1}{p}\right).$$

Finally, since $\Sigma\, 1/a_i$ converges, the density of the integers for which $f(m)\geqslant c$ is equal to

$$\prod_{f(p)\neq 0}\left(1-\frac{1}{p}\right)\sum_{f(a_i)\geqslant c}\frac{1}{a_i}.$$

And so the theorem holds.

Take next the case in which $\sum_{f(p)\neq 0}\frac{1}{p}$ diverges. The proof is similar to that of (I). We require the same lemmas, and nothing is to be altered except that Lemma 1 of (I) must be proved without using the hypothesis

$$f(p_1)\neq f(p_2).$$

LEMMA 1 of (I). *We can find a positive number* δ *such that, for all sufficiently large* n,

$$N(f;\ c,\ c+\delta)<\epsilon n.$$

The new proof requires two lemmas. The first is the same as Lemma 2 of (I), namely:

LEMMA 1. *Let* $f_k(m)=\sum_{\substack{p\mid m\\ p\leqslant p_k}} f(p)$, *where* p_k *denotes the* k*-th prime.*

Then the number of integers $m\leqslant n$, *for which*

$$f(m)-f_k(m)>\delta,$$

is less than $\frac{1}{2}\epsilon n$ *for sufficiently large* $k=k(\epsilon)$.

The proof of this did not involve the hypothesis $f(p_1)\neq f(p_2)$.

Now we split the integers $m\leqslant n$ for which $c\leqslant f(m)\leqslant c+\delta$ into two classes, putting in the first class those for which $f(m)-f_k(m)>\delta$, and in the second class the others. By Lemma 1, the number of integers of the first class is less than $\frac{1}{2}\epsilon n$. For the integers of the second class,

$$c-\delta\leqslant f_k(m)\leqslant c+\delta;$$

hence we see that Lemma 1 of (I) will be proved if we can show that the number of integers $m \leqslant n$ for which $c-\delta \leqslant f_k(m) \leqslant c+\delta$ is less than $\frac{1}{2}\epsilon n$ for sufficiently large $k = k(\epsilon)$.

We now denote

(1) by q_i the primes less than or equal to k for which $f(q_i) > 2\delta$,

(2) by r_i the other primes less than or equal to k,

(3) by a_i the squarefree integers composed of primes less than or equal to k for which $c-\delta \leqslant f(a) \leqslant c+\delta$,

(4) by $\beta_1, \beta_2, \ldots$ the squarefree integers composed of the q_i,

(5) by $\gamma_1, \gamma_2, \ldots$ the squarefree integers composed of the r_i,

(6) by $d_a(m)$ the number of divisors of m among the a_i,

(7) by $d_\gamma(m)$ the number of divisors of m among the γ_i,

(8) by $d_k(m)$ the number of divisors of m among the squarefree integers composed of primes less than or equal to k,

(9) by c_1, c_2, c_3 absolute constants.

Now choose δ so small and k so great that

$$\Sigma \frac{1}{q_i} > A = A(\epsilon),$$

where A is sufficiently large. This is possible since $\sum_{f(p)\neq 0} \frac{1}{p}$ diverges.

We then prove*

LEMMA 2. $$\Sigma \frac{1}{a_i} \leqslant \epsilon^2 \log k.$$

We evidently have

$$\sum_{l=1}^{M} d_a(l) = \sum_{a_i} \left[\frac{M}{a_i}\right] > \sum_{a_i} \frac{M}{a_i} - M. \tag{1}$$

We write $$\sum_{l=1}^{M} d_a(l) = \Sigma_1 + \Sigma_2,$$

* The proof runs similarly to that of Behrend, "On sequences of numbers not divisible one by another", *Journal London Math. Soc.*, 10 (1935), 42–44.

where Σ_1 contains the l's having less than A divisors among the q_i, and Σ_2 all the other l's. Then

$$\Sigma_1 < 2^A \sum_{l=1}^{M} d_\gamma(l) = 2^A \sum_{\gamma_i} \left[\frac{M}{\gamma_i}\right] \leqslant M\, 2^A \prod_{r_i}\left(1+\frac{1}{r_i}\right) = M\, 2^A \frac{\prod_{p\leqslant k}\left(1+\frac{1}{p}\right)}{\prod_{q_i}\left(1+\frac{1}{q_i}\right)}$$

$$\leqslant \frac{c_1 M\, 2^A \log k}{e^A} < \epsilon^3 M \log k,$$

for sufficiently large $A = A(\epsilon)$.

We now estimate Σ_2. Let l be an integer of Σ_2, then, if $\beta = q_1 q_2 \ldots q_x$, $\gamma = r_1 r_2 \ldots r_y$, we have

$$l = \beta\gamma t,$$

where $x \geqslant A$ and t is composed of primes greater than k and the factors of $\beta\gamma$.

We estimate $d_\alpha(l)$ as follows. Any $\alpha \mid l$ is of the form $\alpha = \beta_i \gamma_j$, where $\beta_i \mid \beta$, $\gamma_r \mid \gamma$. The β_i's belonging to the same γ_r cannot divide one another, for if we had $\alpha_1 = \beta_1\gamma_1$, $\alpha_2 = \beta_2\gamma_1$, and $\beta_1 \mid \beta_2$, then

$$2\delta \geqslant f(\alpha_2) - f(\alpha_1) = f(\beta_2) - f(\beta_1) > 2\delta,$$

an evident contradiction. From a theorem of Sperner* it follows immediately that a set of divisors of the product $q_1 q_2 \ldots q_x$, of which no one is divisible by any other, has at most $\binom{x}{[\frac{1}{2}x]}$ elements.

Further, from Stirling's formula

$$(2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n} < n! \leqslant (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{4}n},$$

we easily deduce that

$$\binom{x}{[\frac{1}{2}x]} \leqslant \frac{2^x}{x^{\frac{1}{2}}} \leqslant \frac{2^x}{A^{\frac{1}{2}}},$$

so that
$$d_\alpha(l) \leqslant \frac{2^{x+y}}{A^{\frac{1}{2}}} \leqslant \frac{d_k(l)}{A^{\frac{1}{2}}}.$$

Hence

$$\Sigma_2 < \sum_{l=1}^{M} d_\alpha(l) \leqslant \frac{\sum_{l=1}^{M} d_k(l)}{A^{\frac{1}{2}}} \leqslant \frac{M}{A^{\frac{1}{2}}} \prod_{p\leqslant k}\left(1+\frac{1}{p}\right) \leqslant \frac{c_2 M \log k}{A^{\frac{1}{2}}} < \epsilon^3 M \log k$$

for sufficiently large A.

---

* Sperner, "Ein Satz über Untermengen einer endlichen Menge", *Math. Zeitschrift*, 27 (1928), 544–548.

Finally, from (1), we have

$$\Sigma \frac{1}{a_i} < 2\epsilon^3 \log k + 1 < \epsilon^2 \log k,$$

and so Lemma 2 is proved.

We now prove our main theorem.

We split the integers $m \leqslant n$ for which $c-\delta \leqslant f_k(m) \leqslant c+\delta$ into two classes. In the first class are the integers for which m is divisible by a square greater than $1/\epsilon^4$, and in the second class the other integers. The number of integers of the first class is evidently less than or equal to

$$\sum_{r<1/\epsilon^2} \frac{n}{r^2} < c_1 \epsilon^2 n.$$

The number of integers of the second class we estimate as follows. We write $K(m) = \prod_{\substack{p \leqslant k \\ p|m}} p$. Since $c-\delta \leqslant f_k(m) = f[K(m)] \leqslant c+\delta$, $K(m)$ is evidently an a. The integers m of the second class for which $K(m) = a_i$ are of the form $a_i \mu t$, where μ is composed of the prime factors of a_i and t is composed of primes greater than k; m is divisible by a square greater than or equal to μ, for, if $\mu = p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_1'^{2\beta_1+1} \dots$, m is divisible by

$$p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_1'^{2\beta_1+2} \dots.$$

Thus $\mu < 1/\epsilon^4$. Hence it easily follows from the sieve of Eratosthenes that the number of integers m of the second class for which $K(m) = a_i$ is less than or equal to

$$\frac{1}{a_i} \left\{ c_2 n \prod_{p<k} \left(1-\frac{1}{p}\right) \sum_{\mu<1/\epsilon^4} \frac{1}{\mu} \right\}.$$

Hence the number of the integers of the second class is less than or equal to

$$c_2 n \prod_{p \leqslant k} \left(1-\frac{1}{p}\right) \Sigma \frac{1}{a_i} \sum_{\mu<1/\epsilon^4} \frac{1}{\mu} < c_3 n \epsilon^2 \log \frac{1}{\epsilon^4} < \tfrac{1}{4}\epsilon n;$$

hence the result.

Similar results hold for multiplicative functions, since, if $\phi(m)$ is multiplicative, $\log \phi(m)$ is additive. Hence we find that, if $\phi(m) \geqslant 1$, $N(\phi;\ c)/n$ tends to a limit as $n \to \infty$.

The University,
Manchester.

Printed by C. F. Hodgson & Son, Ltd., Newton St., London, W.C.2.