NOTE ON THE PRODUCT OF CONSECUTIVE INTEGERS (II)

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In a previous paper†, I proved that a product of consecutive positive integers is never a square. In part I of the present paper, I show that, for every $l > 2$, there exists a $k_0 = k_0(l)$, such that, for $k \geqslant k_0$,

$$n(n+1)\dots(n+k-1) = y^l \tag{1}$$

is impossible. From the well-known theorem of Thue and Siegel it follows that, for fixed k, equation (1) has only a finite number of solutions; thus there is only a finite number of cases in which a product of consecutive integers is an l-th power.

In the second part of this paper I show that, for $k \geqslant 2^l$,

$$\binom{n}{k} = y^l \quad (n \geqslant 2k) \tag{2}$$

is impossible. The condition $n \geqslant 2k$ involves no loss of generality, since $\binom{n}{k} = \binom{n}{n-k}$. It is obvious that $\binom{n}{2} = y^2$ is possible; for example, $\binom{9}{2} = 6^2$, $\binom{50}{2} = 35^2$; but it is very probable that (2) has no solutions if $l > 2$. I have proved this only for $l = 3$.

I.

We need two lemmas.

Lemma 1. *Let c_1 be a fixed positive number. Let m be sufficiently large, and let $0 < a_1 < a_2 < \dots < a_r \leqslant m$ be a sequence of integers with $r > c_1 m$. Then there exists a positive number c_2, depending only on c_1, such that there are at least $\frac{1}{2}c_1 m$ pairs a_i, a_j for which $(a_i, a_j) > c_2 m$.*

Proof. Denote by $b_1, b_2, \dots, b_s$ all integers greater than $c_2 m$ and not greater than m having every proper divisor less than or equal to $c_2 m$. Obviously every integer lying between $c_2 m$ and m has a divisor among the b's. Hence there are at least

$$r - c_2 m - s > (c_1 - c_2) m - s$$

* Received 24 February, 1939; read 23 March, 1939.

† P. Erdös, *Journal London Math. Soc.*, 4 (1939), 194–198. I refer to this paper as (I).

pairs a_i, a_j for which (a_i, a_j) is divisible by a b, *i.e.* is greater than $c_2 m$. Thus to prove the lemma it is sufficient to show that, for sufficiently small c_2,

$$s < (\tfrac{1}{2}c_1 - c_2)\, m. \tag{3}$$

To prove (3) we split the b's into two classes. In the first class we put the b's less than $c_2^{\frac{1}{2}} m$, and in the second class the other b's. It is evident that every prime factor of any b of the second class is greater than $1/c_2^{\frac{1}{2}}$; thus, if we choose c_2 sufficiently small, the number of b's of the second class is less than $\frac{1}{4}c_1 m$ for sufficiently large m. Also the number of b's of the first class is at most $c_2^{\frac{1}{2}} m$. Hence

$$s < \tfrac{1}{4}c_1 m + c_2^{\frac{1}{2}} m < (\tfrac{1}{2}c_1 - c_2)\, m,$$

for sufficiently small c_2. This proves the lemma.

LEMMA 2. *The number of solutions of*

$$Ax^l - By^l = C,$$

where $l > 2$ and A, B, C are given positive integers, is finite.

Proof. Lemma 2 is a special case of the well-known theorem of Thue and Siegel.

THEOREM I. *For $k > k_0(l)$, (1) has no solutions.*

Proof. First we show that, if (1) has a solution, then* $n > k^l$. We begin by proving that $n > k$. For, if $n \leqslant k$, then, by a theorem of Tchebicheff, there exists a prime p satisfying

$$n+k-1 > p \geqslant \tfrac{1}{2}(n+k) \geqslant n;$$

thus p occurs in the left-hand side of (1) to the first power, which is impossible. Suppose next that $n > k$; then, by a theorem of Sylvester and Schur†, the left-hand side of (1) is divisible by a prime p greater than k. Obviously only one factor, say $n+i$ $(i \leqslant k-1)$, can be divisible by p, and so, if (1) holds, $n+i \equiv 0 \pmod{p^l}$. Thus

$$n+i \geqslant p^l \geqslant (k+1)^l > k^l + 2k + 1, \quad i.e. \quad n > k^l.$$

We now write

$$n+i = a_i x_i^l \quad (i = 0, 1, 2, \ldots, k-1),$$

* The proof is similar to the proof in (I) that $n > k^2$.

† P. Erdös, *Journal London Math. Soc.*, 9 (1934), 282–288.

where the a's are not divisible by any l-th power and have all their prime factors less than k. As in (I), we show that the a's are all different. For otherwise we should have

$$k > a_i x_i^l - a_i x_j^l \geqslant l a_i x_j^{l-1} > l(a_i x_j^l)^{1/l} = l(n+j)^{1/l} > n^{1/l},$$

in obvious contradiction to the inequality proved above.

Since there are at most $[k/p^u]+1$ multiples of p^u on the left side of (1) and since the a's are not divisible by l-th powers, it follows that

$$(4) \qquad a_0 a_1 \ldots a_{k-1} \leqslant \prod_{p<k} p^{[k/p]+[k/p^2]+\ldots+[k/p^{l-1}]+l-1} \leqslant \Big(\prod_{p<k} p\Big)^{l-1} k! < (4^k)^{l-1} k!,$$

since*

$$\prod_{p<k} p < 4^k.$$

From (4) it follows that at least $\frac{1}{2}k$ of the a's do not exceed $4^{2l-2}k$, for otherwise we should have

$$a_0 a_1 \ldots a_{k-1} \geqslant 1 \,.\, 2 \ldots [\tfrac{1}{2}k](4^{2l-2}k)^{k-[\frac{1}{2}k]} > 4^{k(l-1)} k!.$$

Next we show that, for sufficiently large k, the x_i corresponding to those a_i which do not exceed $4^{2l-2}k$ are all different. For otherwise we should have

$$k > a_i x_i^l - a_j x_i^l \geqslant x_i^l \geqslant \frac{n}{a_i} \geqslant \frac{k^3}{k 4^{2l-2}} \geqslant k$$

when $k \geqslant 4^{l-1}$.

Now, by applying Lemma 1 with $m = 4^{2l-2}k$, $c_1 = 1/2\,4^{2l-2}$, we deduce that there exist at least $\frac{1}{4}k$ pairs a_i, a_j with $a_i < k4^{2l-2}$, $a_j < k4^{2l-2}$, such that $(a_i, a_j) > c_2 k$, where c_2 depends on l but not on k. For each of these pairs we have

$$(5) \qquad \frac{a_i}{(a_i, a_j)} x_i^l - \frac{a_j}{(a_i, a_j)} x_j^l < \frac{k}{(a_i, a_j)}.$$

The equations (5) are all of the form

$$(6) \qquad Ax^l - By^l = C, \quad A < \frac{1}{c_2} 4^{2l-2}, \quad B < \frac{1}{c_2} 4^{2l-2}, \quad C < \frac{1}{c_2}.$$

Thus the number of different equations (5) is less than

$$4^{4l-4} c_2^{-3}.$$

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* P. Erdös, *loc. cit.*

Hence there is an equation which occurs at least

$$D = kc_2^3/4^{4l-3}$$

times; and, since the x_i's belonging to different a's are all different, this equation has at least D solutions. But for sufficiently large k this contradicts Lemma 2. This completes the proof of Theorem 1.

II.

THEOREM 2. *Suppose that $n \geqslant 2k$; then, for $k \geqslant 2^l$,*

$$\binom{n}{k} = y^l \tag{7}$$

is impossible.

Proof. Write

$$n-i = a_i x_i^l \quad (i = 1, 2, \ldots, k-1),$$

where the a's are not divisible by any l-th power and have only prime factors not exceeding k. We can show just as in the first part of the paper that the a's are all different. If (7) holds, we evidently have

$$\frac{a_0 a_1 \ldots a_{k-1}}{k!} = \frac{u^l}{v^l}. \tag{8}$$

We now show that

$$a_0 a_1 \ldots a_{k-1} \leqslant k!,$$

and in fact

$$a_0 a_1 \ldots a_{k-1} \mid k!.$$

Let p be any prime; let ν_p and μ_p be defined by*

$$p^{\nu_p} \| k!, \quad p^{\mu_p} \| a_0 a_1 \ldots a_{k-1}.$$

It is sufficient to show that $\nu_p \geqslant \mu_p$ for every p. Evidently

$$\nu_p = \sum_{t=1}^{\infty} \left[\frac{k}{p^t}\right], \quad \mu_p \leqslant \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \ldots + \left[\frac{k}{p^{l-1}}\right] + l - 1.$$

Thus

$$\mu_p - \nu_p \leqslant l-1.$$

On the other hand it follows from (8) that

$$\mu_p - \nu_p \equiv 0 \pmod{l};$$

this proves that $\nu_p \geqslant \mu_p$.

For $k \geqslant 2^l$, we have

$$a_0 a_1 \ldots a_{k-1} \geqslant 1 . 2 \ldots (2^l-1)(2^l+1) \ldots (k+1) > k!,$$

an obvious contradiction.

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* $p^\alpha \| m$ stands for $p^\alpha | m$, $p^{\alpha+1} \nmid m$.

THEOREM 3. *Suppose that $n \geqslant 2k$; then*

$$\binom{n}{k} = x^3 \tag{9}$$

is impossible.

Proof. We use the same notation as in Theorem 2. In the previous proof we showed that

$$a_0 a_1 \dots a_{k-1} \leqslant k!,$$

and, since the a's are all different, this means that the a's are the integers 1, 2, ..., k in some order. Suppose first that k is even. Consider

$$n-i = \tfrac{1}{2}kx^3 \quad \text{and} \quad n-j = ky^3 \quad (i,\ j < k);$$

then

$$2(j-i)/k = x^3 - 2y^3 = \pm 1,$$

which is impossible*.

Suppose next that k is odd. Here we obtain

$$2(j-i)/(k-1) = x^3 - 2y^3 = \pm 1 \text{ or } \pm 2.$$

The first case is impossible. The second leads to

$$x^3 = 2(y^3 \pm 1),$$

i.e.

$$y^3 \pm 1 = 4u^3 \quad (u = \tfrac{1}{2}x),$$

which is also impossible†. This proves Theorem 3. If we could show that the equations

$$x^l \pm 1 = 2y^l \quad \text{and} \quad x^l \pm 1 = 2^{l-1} y^l$$

are both impossible for every $l \geqslant 3$, we could immediately deduce that

$$\binom{n}{k} = y^l, \quad l \geqslant 3, \quad n \geqslant 2k$$

is impossible.

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* Dickson, *History of the theory of numbers*, 2 (1920), 574–575.

† Dickson, *ibid.*

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