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ON THE EASIER WARING PROBLEM FOR POWERS OF PRIMES. II

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In a previous paper I proved that the density of the positive integers of the form $p_1^2+p_2^2-q_1^2-q_2^2$, where the letters p, q, and later P, Q, r, denote primes, is positive. As indicated in the Introduction of I, I now give proofs of the following results:

The density of each of the sets of integers

$$p_1^2+p_2^2-p_3^2,\quad \sum_{\nu=1}^{4} p_\nu^3-\sum_{\mu=1}^{4} q_\mu^3,\quad \sum_{\nu=1}^{2^l}\epsilon_\nu p_\nu^l\quad (\epsilon_\nu=\pm 1)$$

is positive.

From a well-known result of Schnirelmann†, it follows that *a constant* c_1 *exists such that every integer is the sum of at most* c_1 *positive and negative lth powers of primes*‡.

Throughout this paper n denotes a sufficiently large positive integer, δ and ϵ sufficiently small positive numbers, $c_1, c_2, \ldots$ and γ positive numbers independent of n. Also $0<\gamma<1$, and γ is used as an exponent of n only in applications of Lemma 2 of I.

I

LEMMA 1. *Let* $f(x,y)=x^\sigma+\ldots+y^\sigma$ *be a polynomial in* x *and* y *of degree* σ *with integer coefficients all less than* n *in absolute value. Suppose that*

$$a<10\log n,\quad \prod_{p\mid a}\left(1+\frac{1}{p}\right)<c_2,$$

and let k *be a positive number independent of* n. *Write*

$$g(x,y)=\begin{cases}\prod_{q\mid f(x,y)}\left(1+\dfrac{k}{q}\right) & \textit{if}\quad f(x,y)\neq 0,\\ 0 & \textit{if}\quad f(x,y)=0.\end{cases}$$

* "On the easier Waring problem for powers of primes. I", *Proc. Cambridge Phil. Soc.* 33 (1937), 6. I shall refer to this paper as I.

† See, for example, E. Landau, *Göttinger Nachr.* (1930), 255–76, Satz 21.

‡ Vinogradov has recently proved that every large number of the form $24k+5$ is the sum of 5 squares of primes. See *C.R. Acad. Sci. U.R.S.S.* 16 (1937), 131–2.

Then a constant c_3 exists such that

$$\Sigma = \sum_{\substack{r_1 \leqslant r_2 \leqslant n \\ r_1^2 \equiv r_2^2 \pmod a}} g(r_1, r_2) < \frac{c_3 n^2}{(\log n)^2} \frac{2^{\nu(a)}}{a},$$

where $\nu(a)$ denotes the number of different prime factors of a.

Proof. From
$$\prod_{p \mid a}\left(1+\frac{k}{p}\right) < \prod_{p \mid a}\left(1+\frac{1}{p}\right)^k < c_2^k$$
we obtain

$$\Sigma \leqslant \prod_{p \mid a}\left(1+\frac{k}{p}\right) \sum_{\substack{r_1 \leqslant r_2 \leqslant n \\ r_1^2 \equiv r_2^2 \pmod a \\ g(r_1 r_2) \neq 0}} \prod_{\substack{q \mid f(r_1, r_2) \\ (q, a)=1}} \left(1+\frac{k}{q}\right) < c_2^k \sum_{\substack{r_1 \leqslant r_2 \leqslant n \\ r_1^2 \equiv r_2^2 \pmod a \\ g(r_1 r_2) \neq 0}} \sum_{\substack{d \mid f(r_1, r_2) \\ (d, a)=1}} \frac{\mu^2(d)\, k^{\nu(d)}}{d}. \tag{1}$$

If $(a, d)=1$, the number of solutions of the congruence $f(x, y) \equiv 0 \pmod d$ with $x^2 \equiv y^2 \pmod a$, $0 \leqslant x,\ y \leqslant n$, does not exceed $2^{\nu(a)+2}\sigma^{\nu(d)} n\left(\frac{n}{ad}+1\right)$. Suppose now that $d \leqslant n^{\frac{1}{2}}$ and $(a, d)=1$; then the number of solutions of the congruence $f(r_1, r_2) \equiv 0 \pmod d$, $r_1^2 \equiv r_2^2 \pmod a$, $r_1 \leqslant r_2 \leqslant n$ is not greater than

$$c_4 \frac{2^{\nu(a)+2}\sigma^{\nu(d)}}{(\log n)^2} n\left(\frac{n}{\phi(ad)}+1\right) < c_5 \frac{2^{\nu(a)}\sigma^{\nu(d)}}{(\log n)^2} \frac{n^2}{\phi(ad)}, \tag{2}$$

since for fixed r_1 we obtain at most $2^{\nu(a)+2}\sigma^{\nu(d)}$ possible residues $(\bmod\, ad)$ for r_2 and, by a result of Brun and Titchmarsh*, the number of primes not exceeding n and congruent to $z \pmod{ad}$ is not greater than

$$c_6 \frac{n}{\phi(ad) \log n}.$$

Thus by inverting the order of summation in (1) and using $|f(x, y)| < c_7 n^{\sigma+1}$ we obtain, by (2) and $\phi(a) = a \prod_{p \mid a}\left(1-\frac{1}{p}\right) \geqslant \frac{a}{\prod_{p \mid a}\left(1+\frac{2}{p}\right)} > \frac{a}{c_2^2}$,

$$\Sigma < \frac{c_8 2^{\nu(a)}}{a}\left(\sum_{d=1}^{n^{\frac{1}{2}}} \frac{\mu^2(d)\,(k\sigma)^{\nu(d)}}{d(\log n)^2} \frac{n^2}{\phi(d)} + \sum_{c_7 n^{\sigma+1} > d > n^{\frac{1}{2}}} \frac{\mu^2(d)\,(k\sigma)^{\nu(d)}}{d} n\left(\frac{n}{ad}+1\right)\right)$$
$$< c_8 \frac{2^{\nu(a)}}{a}\left[\frac{n^2}{(\log n)^2} \prod_q \left(1+\frac{k\sigma}{q(q-1)}\right) + n^2 \sum_{d > n^{\frac{1}{2}}} \frac{\mu^2(d)\,(k\sigma)^{\nu(d)}}{d^2} + n \prod_{q < c_7 n^{\sigma+1}}\left(1+\frac{k\sigma}{q}\right)\right].$$

It is well known that $\nu(d) < c_9 \frac{\log d}{\log \log d}$; thus $(k\sigma)^{\nu(d)} < d^{\epsilon}$ for sufficiently large d. Also $\prod_{q<N}\left(1+\frac{k\sigma}{q}\right) = O(N^{\epsilon})$. Hence

$$\Sigma < c_8 \frac{2^{\nu(a)}}{a}\left[\frac{c_{10} n^2}{(\log n)^2} + n^2 \sum_{d > n^{\frac{1}{2}}} \frac{1}{d^{2-\epsilon}} + n(c_7 n^{\sigma+1})^{\epsilon}\right] < c_3 \frac{n^2}{(\log n)^2} \frac{2^{\nu(a)}}{a},$$

which proves the lemma.

* E. C. Titchmarsh, *Rendiconti Circ. mat. Palermo*, 54 (1930), 416.

LEMMA 2. *The number A of integers $b_1, b_2, \ldots, b_A$ not exceeding n having exactly t different prime factors is not greater than* $c_{11}\dfrac{n}{(\log\log n)^{\frac{1}{2}}}$.

Proof. By a result of Hardy and Ramanujan*, we have

$$A < c_{12}\frac{n}{\log n}\,\frac{(\log\log n + c_{13})^t}{t!} < c_{12}\frac{n}{\log n}\,\frac{(\log\log n + c_{13})^{[\log\log n + c_{13}]}}{[\log\log n + c_{13}]!}$$

$$< c_{14}\frac{n}{\log n}\,\frac{e^{\log\log n + c_{13}}(\log\log n + c_{13})^{[\log\log n + c_{13}]}}{[\log\log n + c_{13}]^{[\log\log n + c_{13}]}}$$

$$< c_{15}\frac{n}{(\log\log n)^{\frac{1}{2}}}\left(1 + \frac{1}{\log\log n + c_{13}}\right)^{\log\log n + c_{13}} < \frac{c_{11}n}{(\log\log n)^{\frac{1}{2}}}.$$

LEMMA 3. *If A_d denotes the number of integers b_i which are multiples of d, then*

$$A_d < c_{16}\frac{\log d}{d}\,\frac{n}{(\log\log n)^{\frac{1}{2}}}.$$

Proof. The integers in question are evidently of the form dx with

$$t - \nu(d) \leqslant \nu(x) \leqslant t.$$

Thus, from Lemma 2 and from $\nu(d) < c_9\dfrac{\log d}{\log\log d}$, we have

$$A_d < c_{11}\,\nu(d)\,\frac{n}{d\left(\log\log\dfrac{n}{d}\right)^{\frac{1}{2}}} < c_{16}\frac{\log d}{d}\,\frac{n}{(\log\log n)^{\frac{1}{2}}}.$$

LEMMA 4.
$$\sum_{b_i, b_j} 2^{\nu(b_i, b_j)} < c_{14}\frac{n^2}{\log\log n}.$$

Proof. Evidently

$$\sum_{b_i, b_j} 2^{\nu(b_i, b_j)} \leqslant \sum_{d=1}^{n} A_d^2\, 2^{\nu(d)} \leqslant \frac{n^2}{\log\log n}\sum_{d=1}^{\infty}\frac{c_{16}^2(\log d)^2\, 2^{\nu(d)}}{d^2} < c_{17}\,\frac{n^2}{\log\log n}.$$

LEMMA 5. *With a suitable c_{18}, there exists a set of positive integers*

$$c_{19}\log n < a_1 < a_2 < \ldots < a_B < 10\log n$$

with $\prod_{p \mid a_i}\left(1 + \dfrac{1}{p}\right) < c_{20}$, $B > c_{21}\log n$, *such that each of the equations* $p_{i_1} - p_{i_2} = a_i$ *has more than* $c_{18}\dfrac{n}{(\log n)^3}$ *solutions in primes* $p_i < \dfrac{n}{60\log n}$.

The proof of this lemma follows immediately from the proof of Lemma 4 of I.

* S. Ramanujan, *Collected papers* (Cambridge, 1927), pp. 262–75.

Lemma 6. *There exists a set of positive integers $a'_1, a'_2, \ldots, a'_C$ forming a subset of the a's of Lemma 5 such that all the a''s have exactly t prime factors, with*

$$\log_3 n - c_{22}(\log_3 n)^{\frac{1}{2}} < t < \log_3 n + c_{22}(\log_3 n)^{\frac{1}{2}},$$

and

$$C > \frac{c_{23}\log n}{(\log_3 n)^{\frac{1}{2}}}.$$

Proof. A result of Hardy and Ramanujan* states that for more than $(1-\epsilon)\,10\log n$ integers not exceeding $10\log n$ the number of prime factors lies between $\log_3 n - c_{22}(\log_3 n)^{\frac{1}{2}}$ and $\log_3 n + c_{22}(\log_3 n)^{\frac{1}{2}}$, $c_{22} = c_{22}(\epsilon)$; thus for more than $B - 10\epsilon\log n$ of the a's the number of prime factors lies between the above limits. Hence a t exists such that the number of a's having exactly t prime factors is greater than

$$\frac{c_{24}B}{(\log_3 n)^{\frac{1}{2}}} > \frac{c_{23}\log n}{(\log_3 n)^{\frac{1}{2}}},$$

which implies the lemma.

Let a'_i be any integer of Lemma 6. Consider the sets of integers defined by

$$p_{j_1}^2 - p_{j_2}^2,$$

where the p_j's are given by

$$p_{j_1} - p_{j_2} = a'_i, \quad p_{j_1} < \frac{n}{60\log n}. \tag{3}$$

Then

$$p_{j_1}^2 - p_{j_2}^2 = a'_i(2p_{j_2} + a'_i) < \frac{\lambda}{3}n + a_i'^2 < \tfrac{1}{2}n.$$

Now we prove

Lemma 7. *Let p_{j_2} be the primes given by* (3); *then the number of different integers D not exceeding n of the form*

$$2a'_i p_{j_2} + a_i'^2 + q^2$$

for any single a'_i is greater than

$$c_{25}\frac{n}{2^{\nu}(a'_i)} = c_{25}\frac{n}{2^t}.$$

(*These integers are evidently of the form* $p^2 - q^2 + r^2$.)

Proof. It is evidently sufficient to prove that the number D' of different integers of the form

$$2a'_i p_{j_2} + q^2, \quad (\log n)^3 < q < \tfrac{1}{2}n^{\frac{1}{2}}, \tag{4}$$

is greater than

$$c_{25}\frac{n}{2^t}.$$

By Lemma 5 there are at least $c_{18}\dfrac{n}{(\log n)^3}$ values of p_{j_2} and $\dfrac{c_{26}n^{\frac{1}{2}}}{\log n}$ values of q; thus there are at least $c_{27}\dfrac{n^{\frac{3}{2}}}{(\log n)^4}$ integers m, not necessarily all different, given by

$$2a'_i p_{j_2} + q^2.$$

* S. Ramanujan, *op. cit.*

Let $\lambda(m)$ denote the number of times m occurs. Clearly $m < n$ and so

$$\sum_{\substack{m=i \\ \lambda(m) \neq 0}}^{n} \lambda(m) \geqslant c_{27} \frac{n^{\frac{3}{2}}}{(\log n)^4}. \tag{5}$$

We now prove that
$$\sum_{m=1}^{n} \lambda^2(m) < c_{28}\, 2^l \frac{n^2}{(\log n)^8}. \tag{6}$$

We denote by E the number of solutions of

$$2a_i' p_{j_2} + q_1^2 = 2a_i' p_{j_2}' + q_2^2, \tag{7}$$

with $q_2 > q_1$. Evidently
$$\sum_{m=1}^{n} \lambda^2(m) = 2E + \sum_{m=1}^{n} \lambda(m). \tag{8}$$

First we estimate the number of solutions $E(q_1, q_2)$ of (7) for fixed q_1 and q_2. We write (7) in the form

$$p_{j_2}' - p_{j_2} = \frac{q_2^2 - q_1^2}{2a_i'} \quad (q_1^2 \equiv q_2^2 \pmod{2a_i'}), \tag{9}$$

and estimate the number of solutions of (9) by Brun's methods. If $p_{j_2} \leqslant n^\gamma$, there is at most one value of p_{j_2}' for each p_{j_2}, so that these p_{j_2} give a contribution of at most n^γ to $E(q_1, q_2)$. Suppose then that $p_{j_2}' > p_{j_2} > n^\gamma$. Let P be an arbitrary prime not exceeding n^γ, with $P \nmid 2a_i'$; then, by Lemma 5 and (9),

$$p_{j_2} \equiv 0, \quad -a_i', \quad -\frac{q_2^2 - q_1^2}{2a_i'}, \quad -a_i' - \frac{q_2^2 - q_1^2}{2a_i'} \pmod{P},$$

and these four residues are all different if we assume $P \nmid f(q_1, q_2)$, where

$$f(q_1, q_2) = (q_2^2 - q_1^2)(2a_i'^2 + q_2^2 - q_1^2)(2a_i'^2 - q_2^2 + q_1^2).$$

($f(q_1, q_2) \neq 0$, since $q_1 \neq q_2$ and $(q_2^2 - q_1^2) > (\log n)^3 > 2a_i'^2$.) Denote by $E'(q_1, q_2)$ the number of integers $y \leqslant \dfrac{n}{2a_i'}$ for which

$$y \not\equiv 0, \quad -a_i', \quad -\frac{q_2^2 - q_1^2}{2a_i'}, \quad -a_i' - \frac{q_2^2 - q_1^2}{2a_i'} \pmod{P},$$

for any of the primes $P < n^\gamma$ not dividing $2a_i'$. Then, by Lemma 2 of I,

$$E'(q_1, q_2) < \frac{c_{29}\, n}{a_i'} \prod_{\substack{P < n^\gamma \\ P \nmid 2a_i' f(q_1, q_2)}} \left(1 - \frac{4}{P}\right).$$

Using the inequality
$$1 < \left(1 + \frac{5}{P}\right)\left(1 - \frac{4}{P}\right) \tag{10}$$

for the primes $P > 20$ dividing $2a_i' f(q_1, q_2)$, we have

$$E'(q_1, q_2) < \frac{c_{29}\, n}{a_i'} \prod_{P \mid 2a_i' f(q_1, q_2)} \left(1 + \frac{5}{P}\right) \prod_{20 < P < n^\gamma} \left(1 - \frac{4}{P}\right) < \frac{c_{30}\, n}{a_i' (\log n)^4} \prod_{P \mid f(q_1, q_2)} \left(1 + \frac{5}{P}\right),$$

since
$$\prod_{20 < P < n^\gamma} \left(1 - \frac{4}{P}\right) < \frac{c_{31}\, n}{(\log n)^4} \quad \text{and} \quad \prod_{P \mid a} \left(1 + \frac{5}{P}\right) < \prod_{P \mid a_i} \left(1 + \frac{1}{P}\right)^5 < c_{20}^5. \tag{11}$$

Now apply Lemma 1 with $k=5$, $\sigma=6$,

$$f(x,y) = (x^2-y^2)(2a_i'^2+x^2-y^2)(2a_i'^2-x^2+y^2),$$

and replace n by $\frac{1}{2}n^{\frac{1}{2}}$. We obtain

$$\sum_{\substack{q_1^2\equiv q_2^2 \bmod (2a_i')\\ q_1,\, q_2<\frac{1}{2}n^{\frac{1}{2}}}} E'(q_1,q_2) < c_{32}\frac{n^2}{(\log n)^6}\frac{2^{\nu(a_i')}}{a_i'^2} < c_{33}\frac{n^2}{(\log n)^8}2^t.$$

But

$$E(q_1,q_2) \leqslant E'(q_1,q_2)+n^{\gamma},$$

and so

$$\sum_{m=1}^{n}\lambda^2(m) = 2E+\sum_{m=1}^{n}\lambda(m) < \sum_{\substack{q_1^2\equiv q_2^2 \bmod (2a_i')\\ q_1,\, q_2<\frac{1}{2}n^{\frac{1}{2}}}} E(q_1,q_2)+n^{\frac{3}{2}}$$

$$< c_{33}\frac{n^2}{(\log n)^8}2^t+n^{1+\gamma}+n^{\frac{3}{2}} < c_{28}\frac{n^2}{(\log n)^8}2^t$$

which proves (6).

By noting the elementary fact that the sum of the λ's being given the sum of their squares will be a minimum when all the λ's are equal we obtain, by (5),

$$\sum_{\substack{m=1\\ \lambda(m)\neq 0}}^{n}\lambda^2(m) \geqslant D'\left(\frac{c_{27}n^{\frac{3}{2}}}{D'(\log n)^4}\right)^2 = c_{27}^2\frac{n^3}{D'(\log n)^8};$$

thus, by (6),

$$D' > \frac{c_{27}n^3}{(\log n)^8}\frac{(\log n)^8}{c_{28}n^2 2^t} = c_{25}\frac{n}{2^t},$$

which proves the lemma.

Theorem 1. *The density of the integers of the form $p^2+q^2-r^2$ is positive.*

Proof. The number of integers m for which

$$m = 2a_{i_1}'^2 p_{j_1}+a_{i_1}'^2+q_1^2 = 2a_{i_2}'^2 p_{j_2}'+a_{i_2}'^2+q_2^2,$$

for given, unequal, a'_{i_1}, a'_{i_2}, is evidently less than the number of integers not exceeding n which are quadratic residues both of a'_{i_1} and of a'_{i_2}, i.e. less than

$$\frac{n}{2^{\nu[a'_{i_1},\, a'_{i_2}]}}+a'_{i_1}a'_{i_2} < 2\frac{2^{\nu(a'_{i_1},\, a'_{i_2})}n}{2^{\nu(a'_{i_1})+\nu(a'_{i_2})}} = 2\frac{2^{\nu(a'_{i_1},\, a'_{i_2})}}{2^{2t}}n. \tag{12}$$

Consider now $2^{t-\tau}$ of the a's (τ an integer independent of n)

$$a'_{i_1},\ a'_{i_2},\ \ldots,\ a'_{i_{2^{t-\tau}}}.$$

For the number F of different integers of the form

$$2a'_{i_k}p_j+a_{i_k}'^2+q^2, \quad (k=1,2,\ldots,2^{t-\tau})$$

we obtain from Lemma 7 and (12)

$$F > n\left(\frac{c_{25}}{2^{\tau}}-\sum_{k_1<k_2\leqslant 2^{t-\tau}}\frac{2^{\nu(a'_{i_{k_1}},\, a'_{i_{k_2}})}}{2^{2t}}\right).$$

We now estimate
$$\sum_{a'_{i_1}, a'_{i_2}, \ldots, a'_{i_{2^{t-\tau}}}} F = \sum F.$$

We evidently have
$$\sum F > n\left[\binom{C}{2^{t-\tau}}\frac{c_{25}}{2^\tau} - 2\sum_{a'_{i_1}, a'_{i_2}, \ldots, a'_{i_{2^{t-\tau}}}}\ \sum_{k_1<k_2\leqslant 2^{t-\tau}} \frac{2^{\nu(a'_{i_{k_1}}, a'_{i_{k_2}})}}{2^{2t}}\right]$$
$$> n\left[\binom{C}{2^{t-\tau}}\frac{c_{25}}{2^\tau} - \frac{1}{2^{2t-1}}\binom{C-2}{2^{t-\tau}-2}\sum_{i<j<C} 2^{\nu(a'_i, a'_j)}\right]^*$$
$$> n\left[\binom{C}{2^{t-\tau}}\frac{c_{25}}{2^\tau} - \frac{1}{2^{2t-1}}\binom{C-2}{2^{t-\tau}-2}\sum_{h_1, h_2} 2^{\nu(b_{h_1}, b_{h_2})}\right],$$
where $b_1, b_2, \ldots$ denote the integers not exceeding $10\log n$ having exactly t different prime factors.

From Lemmas 4 and 6 we obtain
$$\sum F > n\left[\binom{C}{2^{t-\tau}}\frac{c_{25}}{2^\tau} - \frac{1}{2^{2t-1}}\binom{C-2}{2^{t-\tau}-2}c_{34}C^2\right].$$

Thus, finally,
$$\sum F > n\binom{C}{2^{t-\tau}}\left[\frac{c_{25}}{2^\tau} - \frac{2c_{34}2^{t-\tau}(2^{t-\tau}-1)C^2}{2^{2t}C(C-1)}\right]$$
$$> n\binom{C}{2^{t-\tau}}\left(\frac{c_{25}}{2^\tau} - \frac{c_{35}}{2^\tau}\right) > c_{36}n\binom{C}{2^{t-\tau}}$$
for sufficiently large τ.

Hence
$$\operatorname*{Max}_{a'_{i_1}, a'_{i_2}, \ldots, a'_{i_{2^{t-\tau}}}} F \geqslant c_{36}n,$$
which proves the theorem.

We now prove that our theorem is best possible. We prove a stronger result, namely that the density of the positive integers of the form m^2-p^2 is 0. We give only the outlines of the proof.

First we show that the number of integers G not exceeding n (not necessarily all different) of the form m^2-p^2 lies between $c_{37}n$ and $c_{38}n$, which at the same time shows that our result is not quite trivial.

We have
$$m^2-p^2 = (m-p)(m+p) \leqslant n,$$
or, substituting $m-p=e$,
$$p \leqslant \frac{n-e^2}{2e}.$$

Thus†
$$G \geqslant \sum_{e=1}^{\frac{1}{2}n^{\frac{1}{2}}} \pi\left(\frac{n-e^2}{2e}\right) \geqslant \sum_{e=1}^{\frac{1}{2}n^{\frac{1}{2}}} \pi\left(\frac{n}{4e}\right) \geqslant c_{39}\sum_{e=1}^{\frac{1}{2}n^{\frac{1}{2}}} \frac{n}{4e\log n} > c_{37}n. \tag{13}$$

On the other hand,
$$G \leqslant \sum_{e=1}^{n^{\frac{1}{2}}} \pi\left(\frac{n}{e}\right) < c_{40}\sum_{e=1}^{n^{\frac{1}{2}}} \frac{n}{e\log n} < c_{38}n. \tag{14}$$

* a'_i and a'_j here run through the integers of Lemma 6.

† $\pi(x)$ denotes the number of primes not exceeding x.

We now state two lemmas.

LEMMA 8. *Denote by f_1, f_2, ... the integers not exceeding n having less than $\frac{2}{3}\log\log n$ prime factors. Then*

$$\Sigma \frac{1}{f_i} = o(\log n).$$

Sketch of the proof. Evidently

$$\Sigma \frac{1}{f_i} < 1 + \sum_{p \leqslant n} \frac{1}{p} + \frac{\left(\sum_{p \leqslant n} \frac{1}{p}\right)^2}{2!} + \ldots + \frac{\left(\sum_{p \leqslant n} \frac{1}{p}\right)^{[\frac{2}{3}\log\log n]}}{[\frac{2}{3}\log\log n]!}$$

$$< \sum_{k \leqslant \frac{2}{3}\log\log n} \frac{(\log\log n + c_{41})^k}{k!} = o(\log n).$$

LEMMA 9. *The number of primes p not exceeding n for which $2p+e$ ($e < \frac{1}{2}n$) has less than $\frac{2}{3}\log\log n$ prime factors is $o\left(\frac{n}{\log n}\right)$.*

The proof is very similar to one given in the first part of my paper*: "On the normal number of prime factors of $p-1$ and some related problems concerning Euler's ϕ function." We omit the proof of the lemma because in spite of its being rather long and complicated no idea is used not contained in the paper quoted.

THEOREM 2. *The density of the integers of the form $m^2 - p^2$ is 0.*

Proof. We split the integers of the form

$$m^2 - p^2 = (m-p)(m+p) = e(2p+e)$$

into two classes. In the first class are all the integers for which $e < \frac{1}{3}n^{\frac{1}{2}}$, in the second class all the others.

As in (14), the number of integers of the second class is not greater than

$$\sum_{e=\frac{1}{3}n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} \pi\left(\frac{n}{e}\right) < c_{40} \sum_{e=\frac{1}{3}n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} \frac{n}{e\log n} = O\left(\frac{n}{\log n}\right) = o(n),$$

so that it is sufficient to consider the integers of the first class.

Now split the integers of this first class into two groups. For the integers of the first group the number of prime factors of e is less than $\frac{2}{3}\log\log n$, the second group contains all the other integers.

The number of integers in the first group, by (14) and Lemma 8, is less than

$$\sum_{\substack{e=1 \\ \nu(e) \leqslant \frac{2}{3}\log\log n}}^{\frac{1}{3}n^{\frac{1}{2}}} \pi\left(\frac{n}{e}\right) < c_{40} \sum_{\substack{e=1 \\ \nu(e) \leqslant \frac{2}{3}\log\log n}}^{\frac{1}{3}n^{\frac{1}{2}}} \frac{n}{e\log n} = o(n),$$

which means that it is sufficient to consider the integers of the second group.

Finally we split the integers of the second group into two subgroups. The first subgroup contains the integers for which $2p+e$ has less than $\frac{2}{3}\log\log n$ prime factors and the second one all the other integers.

* P. Erdös, *Quart. J. Math.* 6 (1935), 205–13.

Replacing n in Lemma 9 by n/e we obtain for the number of integers of the first subgroup the upper bound

$$\sum_{e=1}^{\frac{1}{3}n^{\frac{1}{2}}} o\left(\frac{n}{e \log n}\right) = o(n).$$

Thus we need only consider the integers of the second subgroup. But the number of prime factors of the integers of this subgroup is not less than $\frac{4}{3}\log\log(n)$; hence by a result of Hardy and Ramanujan*, the number of integers in it is $o(n)$, which completes the proof. This proof is very similar to the proofs used in my papers: "Note on sequences of the integers no one of which is divisible by any other"†, and "On the generalisation of a theorem of Besicovich"‡.

We could also obtain the stronger result: the number of integers not exceeding n which can be written in the form $m^2 - p^2$ is $O\left(\frac{n}{(\log n)^{c_{42}}}\right)$.

II

The proof used in this section is very similar to that of I. We require five lemmas.

LEMMA 10. *For every positive ϵ and δ there exists a c_{43} such that the number of integers m not exceeding n with*

$$\sum_{\substack{P \mid m \\ P > c_{43}}} \frac{1}{P} > \delta$$

is less than ϵn.

Proof. For sufficiently large c_{43} we have

$$\sum_{m=1}^{n} \sum_{\substack{P \mid m \\ P > c_{43}}} \frac{1}{P} = \sum_{P > c_{43}} \frac{1}{P}\left[\frac{n}{P}\right] \leqslant n \sum_{P > c_{43}} \frac{1}{P^2} < \epsilon\delta n,$$

which establishes the lemma.

LEMMA 11. *Let $a_1 < a_2 < \ldots < a_x \leqslant n$, $b_1 < b_2 < \ldots < b_y \leqslant n$ be two sets of positive integers such that $x, y > c_{44} n$; then a constant c_{45} exists such that, for at least two integers a_i and b_j, $(a_i, b_j) < c_{45}$.*

Proof. The number of pairs such that $(k, l) > c_{45}$, $k, l \leqslant n$ is evidently not greater than

$$\sum_{d > c_{45}} \frac{n^2}{d^2} < \frac{2n^2}{c_{45}} < c_{44}^2 n^2 < xy$$

for sufficiently large c_{45}. This proves the lemma.

* S. Ramanujan, *op. cit.*

† *J. London Math. Soc.* 10 (1935), 126–8.

‡ *J. London Math. Soc.* 11 (1936), 92–8.

LEMMA 12. *There exist two positive integers* α_1 *and* β_1 *such that*

(i)
$$(\alpha_1, \beta_1) < c_{46},$$
$$c_{47} \log n < \alpha_1, \beta_1 < c_{48} \log n,$$
$$\prod_{P \mid \alpha_1} \left(1+\frac{1}{P}\right), \quad \prod_{Q \mid \beta_1} \left(1+\frac{1}{Q}\right) < c_{49},$$
$$\sum_{\substack{P \mid \alpha \\ P > c_{43}}} \frac{1}{P}, \quad \sum_{\substack{Q \mid \alpha \\ Q > c_{43}}} \frac{1}{Q} < \delta;$$

(ii) *each of the equations*
$$\alpha_1 = p_1 - p_2, \quad \beta_1 = q_1 - q_2$$
has more than $c_{50} \frac{n}{(\log n)^5}$ *solutions in primes* p *and* q *satisfying*
$$p, \ q < c_{51} \frac{n}{(\log n)^3}.$$

Proof. From the proof of Lemma 4 of I we see that there exist at least $c_{52} \log n$ integers g_i satisfying the conditions

(i)
$$c_{47} \log n < g_i < c_{48} \log n,$$
$$\prod_{P \mid g_i} \left(1+\frac{1}{P}\right) < c_{49};$$

(ii) the number of solutions of $g_i = p_1 - p_2$ with $p_1 < c_{51} \frac{n}{(\log n)^3}$ is greater than $c_{50} \frac{n}{(\log n)^3}$.

By Lemma 10 it follows that at least $(c_{52} - \epsilon) \log n = c_{53} \log n$ integers satisfy also the condition
$$\sum_{\substack{P \mid g_i \\ P > c_{43}}} \frac{1}{P} < \delta.$$
Hence, as in the proof of Lemma 4 of I, there exist among these integers two, say α_1 and β_1, such that $\alpha_1 - \beta_1 \leqslant \frac{c_{48}}{c_{53}} < c_{46}$, whence
$$(\alpha_1, \beta_1) = (\alpha_1, (\alpha_1 - \beta_1)) \leqslant \alpha_1 - \beta_1 < c_{46},$$
which proves the lemma.

LEMMA 13. *There exist two positive integers* α_2 *and* β_2 *such that*

(i)
$$(\alpha_1, \beta_2), (\alpha_2, \beta_1), (\alpha_2, \ \beta_2) < c_{54} \quad (c_{54} > c_{43}),$$
$$c_{55} (\log n)^2 < \alpha_2, \beta_2 < c_{56} (\log n)^2,$$
$$\prod_{P \mid \alpha_2(\alpha_1+\alpha_2)} \left(1+\frac{1}{P}\right), \quad \prod_{Q \mid \beta_2(\beta_1+\beta_2)} \left(1+\frac{1}{Q}\right) < c_{57};$$

(ii) *each of the equations*

$$\alpha_2 = p_\nu - p_\mu, \quad \beta_2 = q_\nu - q_\mu$$

has more than $c_{58}\dfrac{n}{(\log n)^7}$ *solutions in primes* $p_\nu, p_\mu, q_\nu, q_\mu$ *satisfying*

$$p_\nu - p'_\nu = p_\mu - p'_\mu = \alpha_1; \quad q_\nu - q'_\nu = q_\mu - q'_\mu = \beta_1; \quad p_\nu, q_\nu < c_{51}\frac{n}{(\log n)^3}. \tag{15}$$

Proof. Denote by $\psi(x)$ the number of solutions of

$$x = p_\nu - p_\mu$$

with p_ν, p_μ satisfying (15).

We first find a lower bound for

$$\psi = \sum_{\substack{x=1 \\ x \neq \alpha_1}}^{c_{56}(\log n)^2} \psi(x).$$

Split the interval $\left(0, c_{51}\dfrac{n}{(\log n)^3}\right)$ into $1 + \left[\dfrac{c_{51}n}{c_{56}(\log n)^5}\right]$ intervals each of length $c_{56}(\log n)^2$ (except the first, which may be neglected) and containing $\psi_1, \psi_2, \ldots$ primes p_ν respectively. Then, taking in each subinterval any two primes p_ν whose difference is not equal to α_1, we have

$$2\psi \geqslant \psi_1^2 + \psi_2^2 + \ldots - 2\psi_1 - 2\psi_2 - \ldots.$$

Now by Lemma 12, and Lemma 3 of I, we have*

$$c_{59}\frac{n}{(\log n)^5} > \psi_1 + \psi_2 + \ldots > c_{50}\frac{n}{(\log n)^5}.$$

Further, if the sum of the ψ's is given, the sum of their squares is a minimum when all the ψ's are equal. Hence by taking

$$\psi_i = c_{50}\frac{n}{(\log n)^5}\bigg/\left[\frac{c_{51}n}{c_{56}(\log n)^5}\right] \geqslant \frac{c_{50}c_{56}}{c_{51}},$$

we obtain

$$2\psi \geqslant \frac{c_{50}^2 c_{56}^2}{c_{51}^2}\left[\frac{c_{51}n}{c_{56}(\log n)^5}\right] - 2c_{59}\frac{n}{(\log n)^5} > c_{61}\frac{n}{(\log n)^5} \tag{16}$$

for sufficiently large c_{56}.

On the other hand, we now prove that, for $x \neq \alpha_1$,

$$\psi(x) < c_{62}\frac{n}{(\log n)^7}\prod_{P \mid x(x^2-\alpha_1^2)}\left(1 + \frac{5}{P}\right). \tag{17}$$

* For it follows from Lemma 3 of I that the number of solutions of $\alpha_1 = p'_\nu - p_\nu$ with $p_1 < c_{51}\dfrac{n}{(\log n)^3}$ is less than

$$c_{60}\frac{n}{(\log n)^5}\prod_{P \mid \alpha_1}\left(1 + \frac{1}{P}\right) < c_{59}\frac{n}{(\log n)^5}.$$

We estimate $\psi(x)$ by Brun's method. Suppose that p_ν in (15) is greater than n^γ, and let R be an arbitrary prime not exceeding n^γ with $R \nmid \alpha_1$. Then evidently, by (15),

$$p_\nu \not\equiv 0,\ x,\ \alpha_1,\ x+\alpha_1 \pmod R.$$

These four residues are all different if we assume that $P \nmid x(x^2-\alpha_1^2)$. Denote by $\psi'(x)$ the number of integers $m < c_{51}\dfrac{n}{(\log n)^3}$, for which

$$m \not\equiv 0,\ x,\ \alpha_1,\ x+\alpha_1 \pmod R$$

for any of the primes $R < n^\gamma$ not dividing α_1. Then, by Lemma 2 of I,

$$\psi'(x) < c_{63}\frac{n}{(\log n)^3}\prod_{\substack{R<n^\gamma \\ R \nmid \alpha_1 x(x^2-\alpha_1^2)}}\left(1-\frac{4}{R}\right);$$

further, by (11) and Lemma 12, we have

$$\psi'(x) < c_{64}\frac{n}{(\log n)^7}\prod_{P \mid x(x^2-\alpha_1^2)}\left(1+\frac{5}{P}\right).$$

But evidently $\quad \psi(x) < \psi'(x) + n^\gamma < c_{62}\dfrac{n}{(\log n)^7}\displaystyle\prod_{P \mid x(x^2-\alpha_1^2)}\left(1+\frac{5}{P}\right),$

which proves (17).

Consider first the values of x for which $\displaystyle\prod_{P \mid x(x^2-\alpha_1^2)}\left(1+\frac{5}{P}\right) > c_{57}$. Then from Lemma 1 of I, on putting $r = 3$, $k = 5$, $x_1 = 0$, $x_2 = \alpha$, $x_3 = -\alpha$ and replacing n by $c_{56}(\log n)^2$, we obtain

$$\sum_{\substack{x \leqslant c_{56}(\log n)^2 \\ x \neq \alpha_1 \\ \prod_{P \mid x(x^2-\alpha_1^2)}(1+5/P) > c_{57}}} \psi(x) < \epsilon\frac{n}{(\log n)^5}.$$

Hence, from (16),

$$\sum_{\substack{x \leqslant c_{56}(\log n)^2 \\ x \neq \alpha_1 \\ \prod_{P \mid x(x^2-\alpha_1^2)}(1+5/P) \leqslant c_{57}}} \psi(x) > (c_{61}-\epsilon)\frac{n}{(\log n)^5} = c_{65}\frac{n}{(\log n)^5}. \tag{18}$$

Noting from (17) that, for these x, $\psi(x) < c_{66}\dfrac{n}{(\log n)^7}$, we have

$$\sum \psi(x) > (c_{65}-c_{55}c_{66}-c_{56}c_{58})\frac{n}{(\log n)^5} = c_{67}\frac{n}{(\log n)^7} \quad (c_{67}>0), \tag{19}$$

for sufficiently small c_{55} and c_{58}, where the summation is extended over the x satisfying

$$c_{55}(\log n)^2 < x < c_{56}(\log n)^2, \quad \psi(x) > c_{58}(\log n)^2, \quad \prod_{P \mid x(x^2-\alpha_1^2)}\left(1+\frac{5}{P}\right) < c_{57}. \tag{20}$$

Thus there must clearly be at least $\frac{c_{67}}{c_{66}}(\log n)^2 = c_{68}(\log n)^2$ terms in the sum (19). But by Lemma 12, for $c_{54} > c_{43}$,

$$\sum_{\substack{P \mid \alpha_1 \\ P < c_{54}}} \frac{1}{P} < \delta,$$

so that the number of integers x not exceeding $c_{56}(\log n)^2$ for which $(x, \alpha_1) > c_{54}$ does not exceed

$$c_{56} \sum_{\substack{P \mid \alpha_1 \\ P > c_{54}}} \left[\frac{(\log n)^2}{P}\right] < \delta c_{56}(\log n)^2.$$

Thus by Lemma 12 there exist at least

$$(c_{68} - \delta c_{56})(\log n)^2 = c_{69}(\log n)^2$$

integers x satisfying (20) and also $(\alpha_1, x) < c_{54}$.

Similarly, denoting by $\phi(y)$ the number of solutions of

$$y = q_\nu - q_\mu$$

satisfying (15), we find that there are at least $c_{69}(\log n)^2$ integers y with $(y, \beta_1) < c_{54}$ such that

$$c_{55}(\log n)^2 < y < c_{56}(\log n)^2, \quad \prod_{P \mid y(y^2 - \beta_1^2)} \left(1 + \frac{5}{P}\right) < c_{57}, \quad \phi(y) > c_{58} \frac{n}{(\log n)^7}. \tag{21}$$

Thus by Lemma (11) there exist two integers α_2 and β_2 satisfying (20), (21) and (α_1, β_2), (α_2, β_1), $(\alpha_2, \beta_2) < c_{54}$, which proves Lemma 13.

LEMMA 14. *There exist four positive integers α_1, α_2, β_1, β_2 such that*

(i) $c_{47} \log n < \alpha_1, \beta_1 < c_{48} \log n, \quad c_{55}(\log n)^2 < \alpha_2, \beta_2 < c_{56}(\log n)^2,$

$$(\alpha_1 \alpha_2, \beta_1 \beta_2) < c_{70},$$

$$\prod_{P \mid \alpha_1 \alpha_2 (\alpha_1 + \alpha_2)} \left(1 + \frac{1}{P}\right), \quad \prod_{Q \mid \beta_1 \beta_2 (\beta_1 + \beta_2)} \left(1 + \frac{1}{Q}\right) < c_{71};$$

(ii) *the number of solutions of the equations*

$$\left.\begin{aligned} p_1 - p_2 &= \alpha_1, & p_1 - p_3 &= \alpha_2, & p_1 - p_4 &= \alpha_1 + \alpha_2, \\ q_1 - q_2 &= \beta_1, & q_1 - q_3 &= \beta_2, & q_1 - q_4 &= \beta_1 + \beta_2, \end{aligned}\right\} \tag{22}$$

is not less than $c_{58} \frac{n}{(\log n)^4}$ *in primes p and q satisfying*

$$p, q < c_{51} \frac{n}{(\log n)^3}.$$

The lemma follows immediately by choosing for α_1, α_2, β_1, β_2 the integers of Lemmas 12 and 13.

THEOREM 3. *The density of the positive integers of the form* $\sum_{i=1}^{4} p_i^3 - \sum_{j=1}^{4} q_j^3$ *is positive.*

Proof. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the integers of Lemma 14. Consider the two sets of integers defined by

$$p_1^3 - p_2^3 - p_3^3 + p_4^3, \quad q_1^3 - q_2^3 - q_3^3 + q_4^3,$$

where the p's and q's are given by Lemma 14.

Then

$$p_1^3 - p_2^3 - p_3^3 + p_4^3 = 6\alpha_1\alpha_2 p_4 + 3\alpha_1\alpha_2(\alpha_1+\alpha_2) < 6c_{48}c_{56}c_{51}n + c_{72}(\log n)^4 < \tfrac{1}{2}n,$$

and similarly

$$q_1^3 - q_2^3 - q_3^3 + q_4^3 < \tfrac{1}{2}n$$

for sufficiently small c_{51}.

Since there are at least $c_{58}\dfrac{n}{(\log n)^7}$ values for each of p_4 and q_4, there are at least $c_{58}^2\dfrac{n^2}{(\log n)^{14}}$ integers m, not necessarily all different, given by

$$6\alpha_1\alpha_2 p_4 + 6\beta_1\beta_2 q_4 = m. \tag{23}$$

Let $\lambda_1(m)$ denote the number of times which m occurs. Clearly $m < n$, and so

$$\sum_{m=1}^{n} \lambda_1(m) \geqslant c_{58}^2 \frac{n^2}{(\log n)^{14}}. \tag{24}$$

We estimate $\lambda_1(m)$, as in the proof of Theorem 1 of I, by Brun's method. If $p_4 < m^\gamma$ there is at most one value of q_4 for each p_4, and similarly if $q_4 < m^\gamma$. Suppose then that $p_4, q_4 > m^\gamma$. Let R be an arbitrary prime not exceeding m^γ with

$$R \nmid 6\alpha_1\alpha_2\beta_1\beta_2(\alpha_1+\alpha_2)(\beta_1+\beta_2).$$

Then

$$p_4 \not\equiv 0,\quad -\alpha_1,\quad -\alpha_2,\quad -\alpha_1-\alpha_2,\quad \frac{m}{6\alpha_1\alpha_2},\quad \frac{m}{6\alpha_1\alpha_2}+\frac{\beta_1^2\beta_2}{\alpha_1\alpha_2},$$

$$\frac{m}{6\alpha_1\alpha_2}+\frac{\beta_1\beta_2^2}{\alpha_1\alpha_2},\quad \frac{m}{6\alpha_1\alpha_2}+\frac{\beta_1\beta_2(\beta_1+\beta_2)}{\alpha_1\alpha_2} \pmod{R},$$

by (23) and (24). These eight residues are all different if we assume that $R \nmid f(m)$, where

$$\begin{aligned} f(m) = {} & m(m+6\alpha_1^2\alpha_2)(m+6\alpha_1\alpha_2^2)[(m+6\alpha_1\alpha_2(\alpha_1+\alpha_2)](m+6\beta_1^2\beta_2) \\ & \times (m+6\beta_1^2\beta_2+6\alpha_1^2\alpha_2)(m+6\beta_1^2\beta_2+6\alpha_1\alpha_2^2)[(m+6\beta_1^2\beta_2+6\alpha_1\alpha_2(\alpha_1+\alpha_2)] \\ & \times (m+6\beta_1\beta_2^2)(m+6\beta_1\beta_2^2+6\alpha_1^2\alpha_2)(m+6\beta_1\beta_2^2+6\alpha_1\alpha_2^2) \\ & \times [m+6\beta_1\beta_2^2+6\alpha_1\alpha_2(\alpha_1+\alpha_2)][m+6\beta_1\beta_2(\beta_1+\beta_2)] \\ & \times [m+6\beta_1\beta_2(\beta_1+\beta_2)+6\alpha_1^2\alpha_2][m+6\beta_1\beta_2(\beta_1+\beta_2)+6\alpha_1\alpha_2^2] \\ & \times [m+6\beta_1\beta_2(\beta_1+\beta_2)+6\alpha_1\alpha_2(\alpha_1+\alpha_2)]. \end{aligned}$$

(Evidently $f(m) \neq 0$.)

Denote by $\lambda_2(m)$ the number of integers $x < \dfrac{m}{6\alpha_1\alpha_2}$ for which

$$x \not\equiv 0, \quad -\alpha_1, \quad -\alpha_2, \quad -\alpha_1-\alpha_2, \; \frac{m}{6\alpha_1\alpha_2}, \; \frac{m}{6\alpha_1\alpha_2}+\frac{\beta_1^2\beta_2}{\alpha_1\alpha_2},$$
$$\frac{m}{6\alpha_1\alpha_2}+\frac{\beta_1\beta_2^2}{\alpha_1\alpha_2}, \; \frac{m}{6\alpha_1\alpha_2}+\frac{\beta_1\beta_2(\beta_1+\beta_2)}{\alpha_1\alpha_2} \pmod{R}$$

for any of the primes $R < m^\gamma$ not dividing $6\alpha_1\alpha_2\beta_1\beta_2(\alpha_1+\alpha_2)(\beta_1+\beta_2)f(m)$, and also for which

$$m - 6\alpha_1\alpha_2 \equiv 0 \pmod{6\beta_1\beta_2},$$

so that x may be one of $(\alpha_1\alpha_2, \beta_1\beta_2)$ different residues $(\operatorname{mod} \beta_1\beta_2)$. By Lemma 2 of I,

$$\lambda_2(m) < c_{73}\, m \frac{(\alpha_1\alpha_2, \beta_1\beta_2)}{6\alpha_1\alpha_2\beta_1\beta_2} \prod_{\substack{R<m^\gamma \\ R \nmid 6\alpha_1\alpha_2(\alpha_1+\alpha_2)(\beta_1+\beta_2)f(m)}} \left(1-\frac{8}{R}\right).$$

On using the inequality

$$1 < \left(1+\frac{9}{R}\right)\left(1-\frac{8}{R}\right), \quad R > 72,$$

for the primes $R > 72$ dividing $6\alpha_1\alpha_2\beta_1\beta_2(\alpha_1+\alpha_2)(\beta_1+\beta_2)f(m)$ we have, by Lemma 14,

$$\lambda_2(m) < \frac{c_{77}\, m}{\alpha_1\alpha_2\beta_1\beta_2} \prod_{R \mid f(m)} \left(1+\frac{9}{R}\right) \prod_{72<R<m^\gamma} \left(1-\frac{8}{R}\right) < c_{75} \frac{m}{(\log m)^{14}} \prod_{R \mid f(m)} \left(1+\frac{9}{R}\right).$$

Now apply Lemma 1 of I with $k \equiv 9$, $r = 16$, $f(x) = f(m)$. We obtain

$$\sum_{\substack{m=1 \\ \prod_{R \mid f(m)} (1+9/R) > c_{76}}}^{n} \lambda_2(m) < \epsilon \frac{n^2}{(\log n)^{14}}.$$

But

$$\lambda_1(m) < \lambda_2(m) + 2m^\gamma,$$

and so

$$\sum_{\substack{m=1 \\ \prod_{R \mid f(m)} (1+9/R) > c_{76}}}^{n} \lambda_1(m) < \frac{\epsilon n^2}{(\log n)^{14}} + 2n^{1+\gamma} < 2\epsilon \frac{n^2}{(\log n)^{14}}. \tag{25}$$

Hence from (24), (25) and by choice of a suitable ϵ,

$$\sum_{\substack{m=1 \\ \prod_{R \mid f(m)} (1+9/R) \leqslant c_{76}}}^{n} \lambda_1(m) \geqslant (c_{58}^2 - 2\epsilon) \frac{n^2}{(\log n)^{14}} > c_{77} \frac{n^2}{(\log n)^{14}}. \tag{26}$$

But for the m in (26) we have

$$\lambda_2(m) < c_{75} c_{76} \frac{n}{(\log n)^{14}}, \quad \lambda_1(m) < c_{78} \frac{n}{(\log n)^{14}};$$

hence in (26) at least $\quad c_{77} \dfrac{n}{(\log n)^{14}} \Big/ c_{78} \dfrac{n}{(\log n)^{14}} = c_{79}\, n$

of the $\lambda_1(m)$ are not zero. Hence also there must be at least $c_{79}n$ integers of the form
$$6\alpha_1\alpha_2p_4+6\beta_1\beta_2q_4+3\alpha_1\alpha_2(\alpha_1+\alpha_2)+3\beta_1\beta_2(\beta_1+\beta_2).$$
These integers are obviously less than n and are of the form
$$p_1^3-p_2^3-p_3^3+p_4^3+q_1^3-q_2^3-q_3^3+q_4^3,$$
which proves Theorem 3.

III

The proof of Theorem 4 is very much more complicated than that of Theorem 3, but since no new idea is applied we do not give the details.

We require here the following analogue of Lemma 14:

Lemma 15. *There exist* $2l-2$ *integers* $\alpha_1, \alpha_2, \ldots, \alpha_{l-1}, \beta_1, \beta_2, \ldots, \beta_{l-1}$ *such that*

(i) $c_{80}\log n<\alpha_1,\ \beta_1<c_{81}\log n,\quad c_{80}(\log n)^2<\alpha_2,\ \beta_2<c_{81}(\log n)^2,\ \ldots,$
$$c_{80}(\log n)^{2^{l-2}}<\alpha_{l-1},\ \beta_{l-1}<c_{81}(\log n)^{2^{l-2}}$$
$$\prod_{P\,|\,\alpha_1\alpha_2\ldots\alpha_{l-1}\prod\limits_{1\leqslant i<j\leqslant l-1}(\alpha_i+\alpha_j)\prod\limits_{1\leqslant i<j<k\leqslant l-1}(\alpha_i+\alpha_j+\alpha_k)\ldots(\alpha_1+\alpha_2+\ldots+\alpha_{l-1})}\left(1+\frac{1}{P}\right)<c_{82},$$
and
$$\prod_{Q\,|\,\beta_1\beta_2\ldots\beta_{l-1}\prod\limits_{1\leqslant i<j\leqslant l-1}(\beta_i+\beta_j)\prod\limits_{1\leqslant i<j<k\leqslant l-1}(\beta_i+\beta_j+\beta_k)\ldots(\beta_1+\beta_2+\ldots+\beta_{l-1})}\left(1+\frac{1}{Q}\right)<c_{82};$$

(ii) *the number of primes* p *and* q *not exceeding* $c_{83}\dfrac{n}{(\log n)^{2^{l-1}-1}}$ *for which the integers*
$$p,\ p+\alpha_i,\ p+\alpha_i+\alpha_j\ (i,j=1,2,\ldots,l-1),\ \ldots,\ p+\alpha_1+\alpha_2+\ldots+\alpha_{l-1},$$
$$q,\ q+\beta_i,\ q+\beta_i+\beta_j\ (i,j=1,2,\ldots,l-1),\ \ldots,\ q+\beta_1+\beta_2+\ldots+\beta_{l-1}$$
are all primes is greater than $c_{84}\dfrac{n}{(\log n)^{2^{l}-1}}$.

We omit the proof since it is similar to that of Lemma 14.

Theorem 4. *The density of the positive integers of the form*
$$\sum_{i=1}^{2^{l-1}}\epsilon_i p_i^l+\sum_{j=1}^{2^{l-1}}\epsilon_j q_j^l$$
is positive.

Consider the integers of the form

$$\left.\begin{aligned}
&(p+\alpha_1+\alpha_2+\ldots+\alpha_{l-1})^l-\sum_{1\leqslant i_1<i_2<\ldots<i_{l-2}\leqslant l-1}(p+\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_{l-2}})^l\\
&\qquad+\sum_{1\leqslant i_1<i_2<\ldots<i_{l-3}\leqslant l-1}(p+\alpha_{i_1}+\ldots+\alpha_{i_{l-3}})^l-\ldots+(-1)^l p^l\\
&=l!\,\alpha_1\alpha_2\ldots\alpha_{l-1}p+f(\alpha_1,\alpha_2,\ldots,\alpha_{l-1})<l!\,c_{81}^{l-1}c_{83}n+f(\alpha_1,\alpha_2,\ldots,\alpha_{l-1})<\tfrac{1}{2}n,\\
&(q+\beta_1+\beta_2+\ldots+\beta_{l-1})^l-\sum_{1\leqslant i_1<i_2<\ldots<i_{l-2}\leqslant l-1}(q+\beta_{i_1}+\beta_{i_2}+\ldots+\beta_{i_{l-2}})^l\\
&\qquad+\sum_{1\leqslant i_1<i_2<\ldots<i_{l-3}\leqslant l-1}(q+\beta_{i_1}+\ldots+\beta_{i_{l-3}})^l-\ldots+(-1)^l q^l\\
&=l!\,\beta_1\beta_2\ldots\beta_{l-1}q+f(\beta_1,\beta_2,\ldots,\beta_{l-1})<l!\,c_{81}^{l-1}c_{83}n+f(\beta_1,\beta_2,\ldots,\beta_{l-1})<\tfrac{1}{2}n,
\end{aligned}\right\}\quad(27)$$

for sufficiently small c_{83}. (p, q, α_i, β_j are the integers of Lemma 15, which implies that the 2^{l-1} terms on the left side of (27) are all primes and that $f(\alpha_1,\alpha_2,\ldots,\alpha_{l-1})$ is a polynomial.)

We now obtain by the method used in § 2 (Brun's method) that the number of integers not exceeding n of the form

$$\alpha_1\alpha_2\ldots\alpha_{l-1}p+\beta_1\beta_2\ldots\beta_{l-1}q$$

is greater than $c_{85}n$. This proves Theorem 4.

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