ON THE SMOOTHNESS OF THE ASYMPTOTIC DISTRIBUTION OF ADDITIVE ARITHMETICAL FUNCTIONS.*

By Paul Erdös.

Introduction. Starting with any given sequence $a_2, a_3, \cdots, a_p, \cdots$ of real numbers, define a sequence $f_1, f_2, f_3, \cdots$ by placing $f_n = \Sigma a_p$, where the summation runs through all prime divisors p of n (in particular, $f_1 = 0$). Clearly, $f_{n+m} = f_n + f_m$ whenever $(n, m) = 1$.

Put $a^+_p = a_p$ or $a^+_p = 1$ according as $-1 < a_p < 1$ does or does not hold. It is known [1] that the additive function f_n of n possesses an asymptotic distribution function $\sigma(x), -\infty < x < +\infty$, if and only if the series

$$(1) \qquad \Sigma \frac{a^+_p}{p} \text{ and } \Sigma \frac{(a^+_p)^2}{p} \text{ are convergent,}$$

in which case the Fourier-Stieltjes transform

$$(2) \qquad L(u) = \int_{-\infty}^{+\infty} e^{iux} d\sigma(x), \qquad -\infty < u < +\infty$$

is represented for every real u by the convergent product

$$(3) \qquad L(u) = \Pi \left(1 - \frac{1 - \exp(ia_p u)}{p}\right).$$

The relation (3) and a general theorem of P. Lévy imply [2] that the distribution function $\sigma(x)$ is continuous for $-\infty < x < +\infty$ if and only if

$$(4) \qquad \sum_{a_p \neq 0} \frac{1}{p} = \infty.$$

It will always be assumed that (1) and (4) are satisfied.

It follows from a general theorem of Jessen and Wintner [3] that the monotone continuous function $\sigma(x)$ either is absolutely continuous or purely singular for $-\infty < x < +\infty$. The object of the present note is to show that either of these cases can actually occur for additive arithmetical functions f_n of simple type.

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* Received March 23, 1939.

[1] P. Erdös and A. Wintner, *American Journal of Mathematics*, vol. 61 (1939), pp. 713-721.

[2] P. Lévy, *Studia Mathematica*, vol. 3 (1931), p. 150; cf. *loc. cit.* [1]

[3] B. Jessen and A. Wintner, *Transactions of the American Mathematical Society*, vol. 39 (1935), p. 86; cf. *loc. cit.* [1].

In particular, the result of § 1 will imply that if

$$a_p = \frac{(-1)^{\frac{1}{2}(p-1)}}{(\log\log p)^{3/4}}, \qquad (p > e^e), \tag{5}$$

then there exists a transcendental entire function $\sigma(z) = \sigma(x+iy)$ which reduces for $y = 0$ to the distribution function of f_n. On the other hand, the result of § 3 will show that if

$$f_n = \log \frac{n}{\phi(n)}, \tag{6}$$

where $\phi(n)$ is Euler's function, then the distribution function of f_n is not absolutely continuous.

1. The method of this § 1 is, in contrast to that of § 3, not of an elementary nature, and consists of an adaptation of a method applied by Wintner to Bernoulli convolutions and the corresponding distribution functions occurring in the theory of the Riemann zeta function.[4] This method consists in estimating the product (3) for large $|u|$ by the following approach: Since each of the factors of (3) has, for every u, an absolute value not exceeding 1, it is clear that

$$L(u) \leqq \prod_{A(u)<p<B(u)} \left| 1 - \frac{1-\exp(ia_p u)}{p} \right| \tag{7}$$

holds for arbitrary positive functions $A(u)$, $B(u)$ of u. And the method consists in choosing $A(u)$, $B(u)$, if possible, in such a way as to assure that

$$L(u) = O(\exp - C\,|u|^c), \qquad u \to \pm\infty, \tag{8}$$

holds for some pair of positive constants c, C. If (2) satisfies (8), then $\sigma(x)$ has for every x derivatives of arbitrary high order; while if (8) holds for $c = 1$ and some $C > 0$, then $\sigma(x)$ is regular analytic and bounded in every strip $|\Im x| <$ const. $< C$ about the real axis of the complex x-plane.[5] In particular, $\sigma(x)$ is an entire function if (8) holds for some $c > 1$ and for some $C > 0$.

Suppose that $|a_p|$ is monotone in p, and define the range of p-values over which the product (7) is extended by

$$A(u) < p < B(u): \quad 3\pi < 4\,|a_p u| < 5\pi. \tag{9}$$

Then each factor of the product on the right of (7) is less than $1 - 1/p$; so that $L(u) < \Pi'(1-1/p)$, where p runs, for every fixed u, over the range (9).

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[4] A. Wintner, *Bulletin of the American Mathematical Society*, vol. 41 (1935), pp. 137-138; *American Journal of Mathematics*, vol. 61 (1939), pp. 231-236.

[5] A. Wintner, *American Journal of Mathematics*, vol. 56 (1934), p. 659.

Hence, by Mertens' elementary result $\prod_{p<t}(1-1/p) \sim e^{-\gamma}/\log t$,

$$L(u) = O\left(\frac{\log A(u)}{\log B(u)}\right), \qquad u \to \pm\infty, \tag{10}$$

where $A(u)$, $B(u)$ are defined by (9).

For instance, if a_p is given by (5), then (8) is satisfied by a $c > 1$ and a $C > 0$. In fact, if the exponent $\frac{3}{4}$ of the denominator of (5) is replaced by an arbitrary $\alpha > \frac{1}{2}$, then (10) and (9) clearly imply that (5) is satisfied by $c = 1/\alpha$ and $C > 0$. Hence, $\sigma(x)$ is an entire function if $\frac{1}{2} < \alpha < 1$; it is regular analytic at least in a strip $|\Im x| <$ const. if $\alpha = 1$; and it has, at least, derivatives of arbitrarily high order for every u if $\alpha > 1$. It may be mentioned that if $\alpha > 1$, the distribution function $\sigma(x)$ has derivatives of arbitrarily high order also when one omits in (5) the factor $(-1)^{\frac{1}{2}(p-1)}$; in which case $\sigma(x)$ cannot be regular analytic along the real axis, since $\sigma(x) = 0$ for every $x < 0$.

2. Let $a_p = 2^{-p}$. Then it is readily verified from (3) that $L(2^m\pi)$ tends, as $m \to \infty$, to a positive limit; so that

$$L(u) \to 0, \qquad u \to \pm\infty, \tag{11}$$

does not hold. It follows, therefore, from the extension of the Riemann-Lebesgue lemma to (2), that the distribution function $\sigma(x)$ is singular.

Of course, (11) only is a necessary condition for the absolute continuity of $\sigma(x)$; in fact,[6] not even $L(u) = O(u^{-\frac{1}{2}+\epsilon})$, where $\epsilon > 0$ is arbitrarily small, is capable of assuring the absolute continuity of $\sigma(x)$.

On the other hand, it is clear from Plancherel's theorem that if $L(u) = O(u^{-\frac{1}{2}-\epsilon})$ holds for some $\epsilon > 0$, then $\sigma(x)$ must be absolutely continuous. This estimate of $L(u)$ is satisfied in case $a_p = 1/\log p$. In order to see this, one merely needs a slight improvement on the crude step (7) and repeated application of Mertens' asymptotic formula, used in § 1.

3. In contrast to the result of § 1, it will now be shown that if

$$a_p = O(p^{-c}), \qquad p \to \infty, \tag{12}$$

holds for some $c > 0$, then $\sigma(x)$ is singular.

In the proof two elementary facts will be needed:

(I) Choosing a fixed large N, write every positive integer m in the form $m = m'm''$, where m' is composed of primes $\leqq N$, and m'' of primes $> N$. Then the density of those m which satisfy the inequality $m' < N^{c/4}$ exceeds a positive lower bound a which depends on $c > 0$ but not on N.

In fact, the density of the positive integers which are not divisible by

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[6] Cf. N. Wiener, and A. Wintner, *American Journal of Mathematics*, vol. 60 (1938), pp. 513-522.

any prime $\leqq N$ is $\Pi(1-1/p)$, where $p \leqq N$ (sieve of Eratosthenes). Thus, it is readily seen that the density of the integers $m = m'm''$ for which $m' < N^{c/4}$ is

$$\sum_{m' < N^{c/4}} \frac{1}{m'} \prod_{p \leqq N} \left(1 - \frac{1}{p}\right).$$

Hence, (I) follows from $\prod_{p \leqq N} (1 - 1/p) \sim e^{-\gamma}/\log N$.

(II) For a fixed large N and for $k = 1, 2, \cdots$, put $g_k = \Sigma a_p$, where the summation runs through those prime divisors p of k which do not exceed N, and the a_p satisfy (12). Let f_k be defined, as in the Introduction, by the sum Σa_q, where q runs through the prime divisors q of k. Then there exists a $b > 0$ which is independent of N and has the property that the density of those positive integers k which satisfy the inequality $|f_k - g_k| > N^{-c/2}$ cannot exceed $bN^{-c/2}$.

In fact, it is clear that, for an arbitrary n,

$$\sum_{k=1}^{n} |f_k - g_k| \leqq \sum_{p > N} |a_p| = O\left(n \sum_{p > N} \frac{1}{p^{1+c}}\right) < \frac{bn}{N^c},$$

where b is a constant. Thus, the density of those positive integers k which satisfy both inequalities $k \leqq n$, $|f_k - g_k| > N^{c/2}$ cannot be greater than $bnN^{-c/2}$. This clearly implies (II).

It is now easy to show [7] that $\sigma(x)$ is singular on the assumption (12). In fact, let N be large. Consider the x-intervals

$$f_m - N^{-c/2} < x < f_m + N^{c/2}, \text{ where } m = 1, 2, \cdots, [N^{c/4}].$$

It follows from (I) and (II) that the density of those positive integers k for which $x = f_k$ lies in one of these $[N^{c/4}]$ intervals cannot be less than $a - bN^{-c/2}$ and exceeds, therefore, a fixed lower bound $C > 0$ for all sufficiently large N. And the sum of the lengths of these $[N^{c/4}]$ intervals is majorized by $N^{-c/4}$. Hence, on letting $N \to \infty$, one sees from the definition of $\sigma(x)$ as the asymptotic distribution function of f_n, that $\sigma(x)$ cannot be absolutely continuous.

Since (12) implies (4), it follows from the general theorem of Jessen and Wintner, referred to in the Introduction, that $\sigma(x)$ is purely singular. It may be mentioned that, in the particular case (12), a direct and elementary discussion could also assure that $\sigma(x)$ does not possess an absolutely continuous component.

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[7] For a similar argument, cf. E. R. van Kampen and A. Wintner, *Journal of the London Mathematical Society*, vol. 12 (1937), pp. 243-244; also P. Hartman and R. Kershner, *American Journal of Mathematics*, vol. 59 (1937), pp. 809-822.