NOTE ON SOME ELEMENTARY PROPERTIES OF POLYNOMIALS

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In a previous paper T. Grünwald[1] and I proved that if $f(x)$ is a polynomial of degree $n \geqq 2$ and satisfies the following conditions:

$$\text{(1)}\quad \begin{gathered} \text{all roots of } f(x) \text{ are real}, \quad f(-1) = f(+1) = 0, \\ f(x) \neq 0 \quad \text{for} \quad -1 < x < 1, \quad \max_{-1<x<1} f(x) = 1, \end{gathered}$$

then

$$\text{(2)}\qquad \int_{-1}^{+1} f(x) \leqq \frac{4}{3}.$$

Equality occurs only for $f(x) = 1 - x^2$.

This result can be generalized as follows: Suppose $f(x)$ satisfies (1) and let $f(a) = f(b) = d \leqq 1$, $-1 < a < b < 1$; then

$$\text{(3)}\qquad b - a \leqq 2(1 - d)^{1/2}.$$

Again equality occurs only for $f(x) = 1 - x^2$. It is clear that (2) follows from (3) by integration with respect to d.

PROOF. Instead of (3) we prove the following slightly more general result: Let $f(x)$ satisfy (1), and determine the greatest positive constant c_f such that

$$f(a)f(a + c_f) = d^2, \quad -1 < a < a + c_f < 1;$$

then

$$\text{(4)}\qquad c_f \leqq 2(1 - d)^{1/2}.$$

Equality holds only for $f(x) = 1 - x^2$, $a = -(1 - d)^{1/2}$.

Suppose there exists a polynomial of degree $n > 2$ satisfying (1) with $c_f \geqq 2(1-d)^{1/2}$; then we will prove that there exists a polynomial of degree $n-1$ with $c_f > 2(1-d)^{1/2}$; and this proves (4) since it is easy to prove that (4) is satisfied for polynomials of second degree, that is, for $1 - x^2$.

Denote the roots of $f(x)$ by $x_1 = -1$, $x_2 = 1$, $x_3, \cdots, x_n$ and suppose first that for $i > 2$ the x_i are not all of the same sign. Let x_n be the largest positive root and x_{n-1} the smallest negative root, and denote by y the root of $f'(x)$ in $(-1, +1)$. Consider the polynomial of degree n

[1] Annals of Mathematics, (2), vol. 40 (1939), pp. 537–548.

$$\phi(x) = c\frac{f(x)(x-y)^2}{(x-x_n)(x-x_{n-1})},$$

where we choose c so that $\phi(x) \geqq 0$ for $-1 \leqq x \leqq 1$. Then it is easy to see that for large x, $\phi(x)$ and $f(x)$ have opposite signs. Thus their leading coefficients have opposite signs. Hence it is possible to choose c such that the polynomial $F(x)=f(x)+\phi(x)$ is of degree $n-1$. Since $n-2$ of its roots are real it can have only real roots, and since $F'(y)=0$, $F(y)=1$, it follows that $\max_{-1\leqq x\leqq 1} F(x)=1$. Thus $F(x)$ satisfies (1) (obviously $F(x)\neq 0$ for $-1<x<1$) and $F(x)\geqq f(x)$ in $-1, +1$, equality occurring only for $-1, y, +1$. Thus $c_F > c_f$. Hence we may suppose that for $i>2$ all the x_i are of the same sign; without loss of generality we may suppose them negative. Suppose that

$$f(a)f(b) = d^2, \qquad b - a = c_f.$$

We can suppose that $-1<a<y<b<1$. We now prove that

$$(5) \qquad b - y < y - a.$$

For if not then

$$(6) \qquad |f'(b)| > |f'(a)|, \qquad f(b) < f(a),$$

that is,

$$|f'(b)| = \left|(b-y)\prod_{i=1}^{n-2}(b-y_i)\right|, \quad |f'(a)| = \left|(y-a)\prod_{i=1}^{n-2}(y_i-a)\right|,$$

$$y > y_i, \quad i = 1, 2, \cdots, n-2,$$

where $b-y\geqq y-a$ and all other factors in $|f'(b)|$ are greater then the corresponding factors in $f'(a)$. This proves the first inequality of (6). To prove the second inequality we remark that from what has just been said it follows that for $u_1-y=y-u_2$, $-1<u_2<y<u_1<1$, we have

$$|f'(u_1)| > |f'(u_2)|,$$

and since $b-z\geqq y-a$ the second inequality follows by integration.

By simple calculation it follows from (6) that

$$f(b-\epsilon)f(a-\epsilon) > f(a)f(b) = d^2, \qquad \epsilon > 0 \text{ sufficiently small.}$$

Thus $b-a<c_f$. This contradiction proves (5).

Let x_n be the root of $f(x)$ with greatest absolute value. Consider the new polynomial

$$f_1(x) = c\frac{x-x_n'}{x-x_n}f(x), \quad x_n' = x_n - \delta, \ \delta > 0,$$

where c is chosen in such a way that $\max_{-1\leq x\leq 1} f_1(x)=1$. Then we prove

$$c_{f_1} > c_f. \tag{7}$$

To show (7) it will suffice to show that c_f is an increasing function of $|x_n|$. Choose δ so small that if we denote by $y^{(1)}$ the root of $f_1'(x)$ in $(-1, +1)$ we have $b-y_1<y_1-a$ (it is clear that $y_1<y$).

Put now

$$c = \left| \frac{1}{f(y_1)} \frac{x_n - y_1}{x_n - \delta - y_1} \right|.$$

(Evidently $cf_1(x)$ satisfies (1).)

Now

$$c^2 f_1(a) f_1(b) = c^2 \frac{a - x_n + \delta}{a - x_n} \frac{b - x_n + \delta}{b - x_n} f(a)f(b)$$

$$> \left(1 + \frac{\delta}{a - x_n}\right)\left(1 + \frac{\delta}{b - x_n}\right)\left(\frac{1}{1 + \delta/y_1 - x_n}\right)^2 f(a)f(b)$$

(that is, $f(y_1)<1$). But from (5) we have

$$\delta\left(\frac{1}{a - x_n} + \frac{1}{b - x_n}\right) > \frac{2\delta}{\left(\frac{a+b}{2} - x_n\right)} > \frac{2\delta}{y_1 - x_n}$$

and

$$\frac{\delta^2}{(a - x_n)(b - x_n)} > \frac{\delta^2}{\left(\frac{a+b}{2} - x_n\right)^2} > \frac{\delta^2}{(y_1 - x_n)^2}.$$

Thus

$$c^2 f_1(a) f_1(b) > f(a)f(b) = d^2.$$

Hence (7) is proved.

If $|x_n|$ tends to infinity $f(x)$ tends to $F(x)=f(x)/(x-x_n)$, which is of degree $n-1$. From (7) it follows that $c_F>c_f$, which proves the theorem.

Let $f(x)$ be a polynomial of degree n all the roots of which are in the interval $(-1,+1)$; and further let $\max_{-1\leq x\leq 1}|f(x)|=1$. For which polynomial is

$$\int_{-1}^{+1} |f(x)|$$

maximal? I was not able to answer this question but it seems very likely that the maximum is reached for $f(x)=T_n(x/c)$, where $c=1/x_n$, and x_n is the greatest root of $T_n(x)$ (the nth Tchebicheff polynomial). Hence $T_n(1/c)=T_n(-1/c)=0$ and all other roots of $T_n(x/c)$ are in $(-1, +1)$. It is easy to see that $T_n(x)$ satisfies the following condition: Let x_i and x_{i+1} be two consecutive roots of $T_n(x)$; then

$$\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \left| T_n(x) \right| = d_n,$$

where d_n is independent of i, and $\lim d_n = 2/\pi$.

This fact suggests the following conjecture which is a generalization of the previous one: Let $f(x)$ be a polynomial of degree n all the roots of which are in $(-1, +1)$, such that $\max_{-1\leqq x\leqq 1} |f(x)| = 1$ and let x_i and x_{i+1} be two consecutive roots of $f(x)$; then

$$\int_{x_i}^{x_{i+1}} \left| f(x) \right| \leqq d_n(x_{i+1}-x_i).$$

Equality holds only for $T_n(cx)$.

It seems very likely that the following result holds: Let $\phi(\theta)$ be a trigonometric polynomial all the roots of which are real, further let $\max_{0\leqq\theta\leqq 2\pi} |\phi(\theta)| = 1$. Then

$$\int_0^{2\pi} \left| \phi(\theta) \right| \leqq 4.$$

Let $f(x)$ be a polynomial of degree n with leading coefficient 1 and all roots in $(-1, +1)$; then the sum of the intervals in $(-1, +1)$ for which $|f(x)| \geqq 1$ does not exceed 1. The proof is quite simple. Evidently

$$f(x)f(-x) = \prod_{i=1}^{n} (x_i^2 - x^2) \leqq 1 \qquad \text{for } |x| \leqq 1,$$

equality occurring only for $x=0$, $|x_i|=1$. Thus one of the numbers $f(x)$ or $f(-x)$ is less than 1, which establishes the result. It is also easy to see that if the sum of the intervals in question is exactly 1 then $f(x)=(1\pm x)^n$. It would not be difficult to prove the following slightly more general result: Let $f(x)$ have leading coefficient 1 and all roots in $(-1, +1)$; then if $-1<a<0<b<1$ at least one of the numbers $|f(a)|$ or $|f(b)|$ is less than 1. These problems become very much more difficult if instead of the interval $-1, +1$ we consider the unit circle. The question would be to determine the polynomial (or polynomials) of degree not greater than n with leading coefficient 1 and all roots in the unit circle such that the area of the set of points for which $|f(x)| \geqq 1$ shall be as big as possible. A first guess would be $f(x)=(x-a)^n$, $|a|=1$, but it can be shown that for sufficiently large n this is not the case. The complete solution of this problem seems difficult.

Mr. Eröd[2] proved that there exists a constant c independent of n such that for a polynomial of degree n satisfying the above conditions the area of the set of points for which $|f(x)| \leqq 1$ is not less than c. The best value of c is not known.

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[2] Oral communication.