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ON THE DISTRIBUTION OF NORMAL POINT GROUPS

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Let $-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ be n real numbers, and let us write $\omega_n(x) = \prod_{k=1}^{n} (x - x_k)$. Consider the unique polynominal $f_n(x)$ of degree not exceeding $2n - 1$ such that

$$f(x_k) = y_k, f'(x_k) = 0.$$

$f_n(x)$ is called the step parabola. It is well known[1] that

$$f_n(x) = \sum_{k=1}^{n} y_k l_k^2(x) v_k(x) = \sum_{k=1}^{n} y_k h_k(x) \tag{1}$$

where

$$l_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)} \text{ and } v_k(x) = 1 - 2l_k'(x_k)(x - x_k).$$

Thus the linear function $v_k(x)$ is 1 at x_k and 0 at

$$X_k = x_k + \frac{\omega'(x_k)}{\omega''(x_k)} = x_k + \frac{1/2}{\sum\limits_{k \neq v} \frac{1}{x_k - x_v}}.$$

The system $X_1, X_2, \ldots, X_n$ is called by Fejér[2] the conjugate point system of $x_1, x_2, \ldots, x_n$. If all the X_i are outside $-1, +1$, Fejér calls the point group normal. He pointed out that the roots of many of the classical polynomials are normal, e.g., the roots of the Tchebicheff and Legendre polynomials.

Fejér[3] proved that if $x_1, x_2, \ldots x_n$ is a normal point group then $\lim\limits_{n \to \infty}(x_{i+1} - x_i) = 0$. Turán and I[4] improved this to

$$x_{i+1} - x_i < \frac{c_1}{n\sqrt{1 - x_i^2}}.$$

Recently I proved that for the x_i satisfying $-1 + c_2 < x < 1 - c_2$

$$x_{i+1} - x_i = \frac{\pi}{n\sqrt{1 - x_i^2}} + 0\left(\frac{1}{n^{3/2}}\right). \tag{2}$$

It seems likely that if $-1 + c_2 < x < 1 - c_2$ then

$$x_{i+1} - x_i = \frac{\pi}{n\sqrt{1 - x_i^2}} + 0\left(\frac{1}{n^2}\right).$$

Let now $-1 = z_1 < z_2 < \ldots < z_n < 1$ be the roots of the polynomial $P_n(z) + P_{n-1}(z)$ (where $P_n(z)$ denotes the nth Legendre polynomial). It is known[5] that $Z_2 = Z_3 = \ldots = Z_n = 1$, and $Z_1 < -1$ (the Z_i are the conjugate points); i.e., this polynomial is barely normal. It is also easy to show that if for a certain point group $z_1 = -1$ and $Z_2 = \ldots = Z_n = 1$, then

$$P_n(z) + P_{n-1}(z) = \prod_{i=1}^{n} (z - z_i).$$

Thus $P_n(z) + P_{n-1}(z)$ is characterized by this property. Now we prove the following THEOREM 1: *Let* $-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ *be a normal point group.* Then

$$z_i \leqslant x_i \leqslant -z_{n-i}.$$

It is easy to see that these limits are the best possible since the point group $-z_i$ is also normal.

Proof. It will be sufficient to prove that

$$z_i \leqslant x_i.$$

We prove the following stronger result: Suppose $-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ is such that X_i does not fall in the interval $(x_i, 1)$ (we will refer to this property as A). Then

$$z_i \leqslant x_i.$$

Let us investigate the point group satisfying A and for which x_i is as small as possible. It is easy to see that such a point group exists.[6] Now we prove that this point group is $z_1, z_2, \ldots z_n$, and this will complete our proof. Suppose that this is not true. Then either $x_1 \neq -1$ or there exists an x, say x_j, such that $X_j \neq 1$. Suppose first $x_1 \neq -1$. Consider the point group $-1 < x_1 - \epsilon_1 < x_2 < \ldots x_n \leqslant 1$ (ϵ_1 sufficiently small). A simple calculation shows that the new point group also satisfies A, and in fact the conjugate points which were not less than 1 increased in absolute value. Thus if we denote by $X_1', X_2', \ldots X_n'$ the conjugate points of $x_1 - \epsilon_1, x_2 \ldots x_n$, we have $X_i' \neq 1$. Consider now the point group $x_1 - \epsilon_1, x_2 \ldots x_i - \epsilon_i \ldots x_n$ (ϵ_i sufficiently small). A simple calculation shows that this point group also satisfies A, which contradicts the minimum property of x_i. In the second case we consider the point group $x_1, x_2 \ldots$

$x_j - \epsilon_j, \ldots x_n$ (ϵ_j sufficiently small) which also satisfies A and the whole proof goes through as before.

Let $\varphi(x)$ be any continuous function in $(-1, +1)$, and let

$$\begin{array}{ccccccc} & & x_1^{(2)} & x_1^{(1)} & x_2^{(2)} & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{array}$$

be any infinite sequence of normal point systems. (Fejér[7] calls a sequence of point groups strongly normal if for any k and n, $v_k(x) \geqslant c_3, -1 \leqslant x \geqslant 1$ (c_3 independent of k and n). Consider the polynomial $f_n(x)$ of degree not greater than $2n - 1$ for which

$$f_n(x_i^{(n)}) = \varphi(x_i^{(n)}) \text{ and } f_n'(x_i^{(n)}) = 0.$$

By (1),

$$f_n(x) = \sum_{k=1}^{n} y_k h_k(x).$$

Fejér[8] conjectured that

$$\lim_{n \to \infty} f_n(x) = \varphi(x)$$

uniformly in $(-1, +1)$. Recently I succeeded in proving that

$$\lim_{n \to \infty} f_n(x) = \varphi(x), \; -1 + \epsilon < x < 1 - \epsilon$$

uniformly for every $\epsilon > 0$. In fact it suffices to suppose that the sequence of point groups is normal.

The proof of this result is not quite simple (it uses (2)) so that we do not give it here.

In a previous paper[9] Turán and I proved that if $x_1, x_2 \ldots x_n$ is a normal point group, then

$$\max_{-1+\epsilon < x < 1-\epsilon} \prod_{i=1}^{n} (x - x_i) < \frac{c_4\sqrt{n}}{2^n}.$$

By using Theorem 1, I can prove that

$$\max_{-1 \leq x \leq 1} \prod_{i=1}^{n} (x - x_i) < \frac{c_4\sqrt{n}}{2^n}.$$

[1] L. Fejér, "Lagrangesche Interpolation und die zugehőrigen konjugierten Punkte," *Math. Ann.*, **106**, 1–55 (1932). See also "On the Characterization of Some . . .," *Amer. Math. Monthly*, **41**, 1–14 (1934).

[2] L. Fejér, *Ibid.*, p. 3.

[3] L. Fejér, *Ibid.*, p. 27.

[4] P. Erdős and P. Turán, "On Interpolation, II," *Ann. Math.*, **39**, 702–724 (1938).

[5] L. Fejér, *Ibid.* p. 32.

[6] Denote by a_i the lower limit of the x_i. To prove the existence of the point group in question it suffices to show that there exists a δ such that if $x_1, x_2 \ldots x_n$ is a point group satisfying A and for which $x_i - a_i < \epsilon_i$ (ϵ_i sufficiently small), then $x_{r+1} - x_r > \delta, r = 1, 2, \ldots n - 1$. This is not difficult.

[7] L. Fejér, *Amer. Math. Monthly*, **41**, 8 (1934).

[8] L. Fejér, "On the Characterization of Some . . .," *Amer. Math. Monthly*, **41**, 13 (1934).

[9] P. Erdős and P. Turán, *Ibid.*