ON THE SMOOTHNESS PROPERTIES OF A FAMILY OF BERNOULLI CONVOLUTIONS.*

By PAUL ERDÖS.

Let $L(u,\sigma)$, $-\infty < u < +\infty$ denote the Fourier-Stieltjes transform, $\int_{-\infty}^{\infty} e^{iux} d\sigma(x)$, of a distribution function $\sigma(x)$, $-\infty < x < +\infty$. Thus if $\beta(x)$ is the distribution function which is $0, \frac{1}{2}, 1$ according as $x \leqq -1$, $-1 < x \leqq 1$, $1 < x$, then $L(u,\beta) = \cos u$; and so, if b is a positive constant, $\cos(u/b)$ is the transform of the distribution function $\beta(bx)$. Hence, if a is a positive constant, the infinite convolution

$$\sigma_a(x) = \beta(ax) * \beta(a^2x) * \beta(a^3x) * \cdots$$

is convergent if and only if $a > 1$; its Fourier-Stieltjes transform being

$$L(u,\sigma_a) = \prod_{n=1}^{\infty} \cos(u/a^n), \qquad (a > 1). \tag{1}$$

It is known[1] that the distribution function σ_a is continuous for every $a > 1$ and, in fact, is either absolutely continuous or purely singular, depending on the value of a. In this direction it is known[2] that the set of points x in the neighborhood of which $\sigma_a(x)$ is not constant is either the interval $x \leqq a/(a-1)$ or a nowhere dense perfect set of measure zero contained in this interval according as $1 < a \leqq 2$ or $2 < a$. While this implies that $\sigma_a(x)$ is singular if $2 < a$ it does not imply that $\sigma_a(x)$ is absolutely continuous if $a < 2$. In fact it has recently[3] been shown that there exist certain algebraic irrationalities $a < 2$ for which $L(u,\sigma_a)$ does not tend to zero with $1/u$ and so σ_a cannot be absolutely continuous. (It was conjectured, *loc. cit.*[3], that such values of a are clustering at $a = 1 + 0$ which would imply that they lie dense in the interval $1 < a < 2$). On the other hand it is known[4] that those $a < 2$

* Received July 30, 1939.

[1] B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), 48-88, particularly Theorem 11.

[2] R. Kershner and A. Wintner, "On symmetric Bernoulli convolutions," *American Journal of Mathematics*, vol. 57 (1935), 541-548.

[3] P. Erdös, "On a family of symmetric Bernoulli convolutions," *American Journal of Mathematics*, vol. 61 (1939), 974-976.

[4] A. Wintner, "On convergent Poisson convolutions," *American Journal of Mathematics*, vol. 57 (1935), 827-838.

for which σ_a is absolutely continuous are certainly clustering at $a = 1 + 0$, since if $a = 2^{1/m}$, where m is a positive integer, then σ_a has a continuous derivative of order $m - 1$.

The object of the present paper is to show that the successive smoothing of σ_a can be considered as the general case when $a \to 1 + 0$. In fact it will be shown that there exists, for every positive integer m, a positive $\eta(m)$ such that the set of those points a of the interval $1 < a < 1 + \eta(m)$ for which σ_a does not possess a continuous derivative of order $m - 1$ is a set of measure zero. To this end it is sufficient to prove that there exists, for every positive integer m, a positive $\delta(m)$ such that the set of those points a of the interval $1 < a < 1 + \delta(m)$ for which

$$L(u, \sigma_a) = o(|u|^{-m}), \qquad u \to \infty, \tag{2}$$

does not hold is a set of measure zero.

Let $c_1, c_2, \cdots, c_N$ be N positive integers which satisfy the following conditions:

(i) $\quad c_1 \leqq 2;$

(ii) $\quad c_i < c_{i+1}, \qquad (i = 1, 2, \cdots, N-1);$

(iii) $\quad c_{i+1} < 3c_i, \qquad (i = 1, 2, \cdots, N-1);$

(iv) there exists an α such that $2^{\frac{1}{2}} < \alpha < 2$ and $|c_{i+1} - \alpha c_i| < 2$, $\qquad (i = 1, 2, \cdots, N-1)$.

LEMMA 1. *There exist two positive absolute constants γ_1, γ_2 such that if M is any fixed number $> \gamma_2$, there are less than $[M^{1/4}]$ different sequences $c_1, c_2, \cdots, c_N$ satisfying the requirements (i)-(iv), the inequality $c_N \leqq M$, and the condition that the number of those indices i $(i = 1, 2, \cdots, N)$ which satisfy $|c_{i+1} - \alpha c_i| > 1/10$ is less than $\gamma_1 \log M$.*

Proof. Suppose that $|c_{i+1} - \alpha c_i| \leqq 1/10$ and $|c_{i+2} - \alpha c_{i+1}| \leqq 1/10$ for a fixed i. Then

$$\left|\frac{c_{i+1}}{c_i} - \alpha\right| < \frac{1}{10c_i},$$

hence

$$\left|\frac{c^2_{i+1}}{c_i} - \alpha c_{i+1}\right| < \frac{c_{i+1}}{10c_i} < \frac{3}{10}$$

by (iii). Consequently, since $|c_{i+2} - \alpha c_{i+1}| < 1/10$ by assumption,

$$\left|\frac{c^2_{i+1}}{c_i} - c_{i+2}\right| < \frac{3}{10} + \frac{1}{10} < \frac{1}{2}$$

and so c_{i+2} is uniquely determined as the nearest integer[5] to c^2_{i+1}/c_i.

[5] The above considerations are suggested by the investigations of Ch. Pisot, "La répartition modulo un et les nombres algébriques," *Annali d. R. Sc. Norm. Sup. di Pisa*, ser. II, vol. VII, p. 238.

Consequently if $i_1, i_2, \cdots, i_l$ denote all those among the N indices i which satisfy the inequality $| c_{i+1} - \alpha c_i | > 1/10$ then all indices i which are not of the form $i_r + 1$ or $i_r + 2$ for some $r = 1, 2, \cdots, l$, are such that c_i is uniquely determined by c_{i-1} and c_{i-2}. On the other hand, even if j is of the form $i_r + 1$ or $i_r + 2$, so that c_j is not uniquely determined by c_{j-1} and c_{j-2}, then there are, by (iv), (or (i)), at most 4 choices for c_j after c_{j-1} has been determined. Hence there are at most 4^{2l} different sequences $c_1, c_2, \cdots, c_N$ which have a given set of exceptional indices $i_1, i_2, \cdots, i_l$.

Finally (ii) and (iv) together with the assumption $a_N \leqq M$ clearly imply that $N < 5 \log M$ for sufficiently large M, say for $M > \gamma_2$. Since the number of exceptional indices $i_1, i_2, \cdots, i_l$ is less than $\gamma_1 \log M$, by the hypothesis of Lemma 1, it is seen that the number of distinct possible choices for a set of exceptional indices cannot exceed

$$\binom{[5 \log M]}{0} + \binom{[5 \log M]}{1} + \cdots + \binom{[5 \log M]}{[\gamma_1 \log M]}$$

and is therefore less than $M^{1/8}$ if γ_1 is chosen sufficiently small. Since it was shown above that there are at most 4^{2l} sequences $c_1, c_2, \cdots, c_N$ with a given set of exceptional indices, it follows that the number of distinct sequences $c_1, c_2, \cdots, c_N$ which satisfy the requirements of Lemma 1 for a fixed $M > \gamma_2$ is less than

$$M^{1/8} \cdot 4^{2l} < M^{1/8} \cdot 4^{2\gamma_1 \log M} < M^{1/4}$$

if γ_1 is sufficiently small. This completes the proof of Lemma 1.

If a, λ are positive numbers let $A_k = A_k(a, \lambda)$ and $\epsilon_k = \epsilon_k(a, \lambda)$ be defined, for $k = 1, 2, \cdots$, by placing

$$(3) \qquad \lambda a^k = A_k + \epsilon_k, \qquad A_k \text{ integer}, \ -\tfrac{1}{2} < \epsilon_k \leqq \tfrac{1}{2}.$$

LEMMA 2. *There exists an absolute constant γ_3, which shall be chosen to be $> \gamma_2$, such that if M has a fixed value greater than γ_3, then the measure of the set Γ of those values a in the interval*

$$(4) \qquad 2^{\frac{1}{2}} < a < 2$$

for which there exists in the interval

$$(5) \qquad 1 < \lambda < 2$$

a $\lambda = \lambda(a)$ such that the inequalities

$$(6.1) \quad \lambda a^k < M; \qquad (6.2) \quad | \epsilon_k(a, \lambda) | > 1/30$$

hold for at most $\frac{1}{2}\gamma_1 \log M$ distinct values of k, is less than $M^{-\frac{1}{4}}$. It is under-

stood that $\epsilon_k = \epsilon_k(a, \lambda)$ is defined as in (3), *and that γ_1, γ_2 are the absolute constants occurring in Lemma* 1.

Proof. Suppose, if possible, that Lemma 2 is false. Then there exist at least $[M^{1/4}]$ values of a in (4), say

$$a_j, \qquad (j = 1, 2, \cdots, [M^{1/4}]),$$

which are in Γ and which are separated by $[M^{1/4}] - 1$ intervals each of which has a length not less than $M^{-3/4}$; so that

$$(7) \qquad | a_j - a_k | \geqq M^{-3/4}.$$

Since a_j is in Γ, there exists a $\lambda = \lambda(a_j)$ in (5) such that

$$\epsilon_k(a_j, \lambda(a_j)) < 1/30$$

holds for all but $\frac{1}{2}\gamma_1 \log M$ values of k satisfying

$$a_j^k \lambda(a_j) < M,$$

where, according to (3)

$$(8) \qquad a_j^k \lambda(a_j) = A_k(a_j, \lambda(a_j)) + \epsilon_k(a_j, \lambda(a_j)) = A_k^{(j)} + \epsilon_k^{(j)}, \text{ say.}$$

It will be shown that

(I) The finite sequence of integers $A_k^{(j)}$ belonging to a fixed j $(= 1, 2, \cdots, [M^{1/4}])$ satisfies the hypotheses of Lemma 1 if this sequence of integers is identified with the sequence of integers $c_1, c_2, \cdots, c_N$ occurring there; and that

(II) The sequences $A_k^{(j)}$ corresponding to different values of j are distinct. Since there are $[M^{1/4}]$ such sequences this will contradict Lemma 1 and so complete the proof of Lemma 2.

In order to prove (I) notice first that (i), (ii), (iii) are obviously satisfied for $c_i = A_i^{(j)}$. Furthermore, by (8)

$$A_{i+1}^{(j)} + \epsilon_{i+1}^{(j)} = a_j(A_i^{(j)} + \epsilon_i^j)$$

and so, by (3) and (4)

$$| A_{i+1}^{(j)} - a_j A_i^{(j)} | = | a_j \epsilon_i^{(j)} - \epsilon_{i+1}^{(j)} | < 2;$$

so that (iv) is also satisfied, with $\alpha = a_j$. The hypothesis (6.1) assures that the assumption $c_N \leqq M$ of Lemma 1 is satisfied. In order to verify the remaining assumption of Lemma 1 recall that there are at most $\frac{1}{2}\gamma_1 \log M$ values of k satisfying (6.1), (6.2). Thus there are at most $\gamma_1 \log M$ values of i such that (6.1), (6.2) are satisfied either for $k = i$ or for $k = i + 1$. But if i has a value distinct from one of these $\gamma_1 \log M$ values, so that

$$| \epsilon_i^{(j)} | < 1/30 \text{ and } \epsilon_{i+1}^{(j)} < 1/30,$$

then, by (4),

$$| A_{i+1}^{(j)} - a_i A_i^{(j)} | = | a_j \epsilon_i^{(j)} - \epsilon_{i+1}^{(j)} | < 1/10.$$

Thus there are at most $\gamma_1 \log M$ indices i for which

$$| A_{i+1}^{(j)} - a_j A_i^{(j)} | > 1/10.$$

This completes the proof of (I).

In order to prove (II), suppose, if possible, that (II) is false. Then there exists a pair of distinct indices j and k such that

$$A_i^{(j)} = A_i^{(k)}$$

for all $i = 1, 2, \cdots, N$. Thus, by (3),

$$(9) \qquad | a_k{}^l \lambda(a_k) - a_j{}^l \lambda(a_j) | < 2$$

holds, for all l such that $a_k{}^l \lambda(a_k) \leqq M$. In particular (9) holds if l is an index for which

$$(10) \qquad \frac{1}{4} M > a_k{}^l > \frac{1}{10} M.$$

Now it may be assumed that $a_k > a_j$ so that, by (7), $a_k \geqq a_j + M^{-3/4}$. Then

$$a_k{}^{l+1} \lambda(a_k) \geqq a_k{}^l \lambda(a_k)(a_j + M^{-3/4})$$

and so, by (9),

$$a_k{}^{l+1} \lambda(a_k) \geqq (a_j{}^l \lambda(a_j) - 2)(a_j + M^{-3/4}) = a_j{}^{l+1} \lambda(a_j) + a_j{}^l \lambda(a_j) M^{-3/4} - 2(a_j + M^{-3/4}).$$

Hence, by (5) and (10),

$$a_k{}^{l+1} \lambda(a_k) \geqq a_j{}^{l+1} \lambda(a_j) + \frac{1}{10} M^{1/4} - 2 - 2(a_j + M^{-3/4}) \geqq a_j{}^{l+1} \lambda(a_j) + 3$$

if M is sufficiently large, say $M > \gamma_3$. Thus

$$| a_k{}^{l+1} \lambda(a_k) - a_j{}^{l+1} \lambda(a_j) | \geqq 3.$$

This contradicts (9) (since by (10) $a_k{}^{l+1} \lambda(a_k) < M$) where one could write $l + 1$ for l. This contradiction proves (II).

The proof of Lemma 2 is now complete.

LEMMA 3. *There exists, on the interval* (4) *a zero set Z which has the following property: if a is a point of* (4) *not contained in Z then there is a positive $\beta = \beta(a)$ such that if M is any fixed number larger than β and if λ is any number in* (5), *then there are at least $\frac{1}{4}\gamma_1 \log M$ values of k which satisfy both conditions* (6.1), (6.2).

Proof. For any positive integer h let Γ_h denote the set of points a on the interval (4) such that (6.1), (6.2) hold (for some $\lambda = \lambda(a)$ in (5)) for less than $\frac{1}{2}\gamma_1 \log M$ values of k if $M = 2^h$. Then, by Lemma 2,

$$\text{meas}\, \Gamma_h < 2^{-\frac{1}{2}h} \text{ if } 2^h > \gamma_3.$$

Thus if Γ_μ denotes for any fixed $\mu > \gamma_3$ the a-set

(11) $$\Gamma \equiv \Gamma_\mu = \sum_{2^h > \mu} \Gamma_h \text{ then meas } \Gamma_\mu < 4\gamma\mu^{-\frac{1}{2}}.$$

It is clear from the definition of Γ, that if a is not in Γ_μ and if $M > \mu$, then, even if M is not of the form 2^h for some h, there are still at least $\frac{1}{4}\gamma_1 \log M$ values of k satisfying (6.1), (6.2) for any value of λ in (5). Thus if a is not in Γ_μ then there is a $\beta = \beta(a)$ satisfying the requirements of Lemma 3; in fact one can choose $\beta = \mu$. Then the set of points a in (4) such that there does not exist a $\beta = \beta(a)$ satisfying the requirements of Lemma 3 is contained in Γ_μ for every positive μ. Thus by (11), Z is a zero set. This completes the proof of Lemma 3.

LEMMA 4. *For every $q > 0$ there exists a $\rho = \rho(q) > 1$ and a zero set $Z = Z_q$ of a-values contained in the interval*

(12) $$1 < a < \rho(q)$$

with the following properties: if a is a point of (12) *not contained in Z_q then there exists an $\alpha = \alpha(a) > 0$ such that if M is any fixed number greater than α, and if λ is any point of the interval* (5), *then there are at least $q \log M$ values of k satisfying* (6.1), (6.2).

Proof. Let a be a point in the interval $1 < a < 2^{\frac{1}{2}}$ such that no integral power of a is a point of the zero set Z occurring in Lemma 3. Let $p_1, p_2, \cdots, p_r$ be those prime numbers such that

$$2^{\frac{1}{2}} < a^{p_1} < a^{p_2} < \cdots < a^{p_r} < 2.$$

Now if x is such that $a^x = 2$ then, by the elementary inequalities of Chebyshev, there are two absolute constants γ_4, γ_5 such that

(13) $$\gamma_4 \frac{x}{\log x} > r > \gamma_5 \frac{x}{\log x}.$$

Since a^{p_j} $(j = 1, 2, \cdots, r)$ is in the interval (4) and not a point of Z, there are, by Lemma 3, for every λ in (5), at least $\frac{1}{4}\gamma_1 \log M$ values of k satisfying

(14.1) $$|\lambda_a{}^{p_i k}| < M,$$ (14.2) $$|\epsilon_k(a^{p_j})\lambda)| > 1/30$$

provided $M > \beta(a^{p_i})$. Thus, if $M > \max_{1 \leq i \leq r} \beta(a^{p_i})$, there are at least $\frac{1}{4}\gamma_1 \log M$ values of k satisfying (14.1), (14.2) for each i $(= 1, 2, \cdots, r)$. But there are at most $\dfrac{x \log M}{p_i p_j \log 2}$ values of k such that

$$(a^{p_i p_j})^k = (2^{p_i p_j / x})^k < \frac{1}{\lambda} M < M.$$

Thus there are at least

$$\tfrac{1}{4}r\gamma_1 \log M - \sum_{1 \leqq i \leqq j \leqq r} \frac{\log M}{p_i p_j \log 2}$$

values of k satisfying (6.1) and (6.2). Then by (13) the number of values k which satisfy (6.1) and (6.2) is not less than

$$\tfrac{1}{4}\gamma_1\gamma_5 \frac{x}{\log x} \log M - 4\gamma_6 \frac{x}{(\log x)^2} \log M.$$

But this expression can be made greater than $q \log M$ if x is chosen sufficiently large, i. e., if a is chosen sufficiently small, say $a < \rho(q)$. This completes the proof of Lemma 4 since Z_q may be defined to be the zero set of points a in the interval (12), some integral power of which is a point of Z.

THEOREM. *For every positive integer m, there exists a positive $\delta = \delta(m)$ such that the set of points a of the interval $1 < a < 1 + \delta(n)$ for which*

$$L(u, \sigma_a) = o(|u|^{-m}), \qquad u \to \infty,$$

does not hold is a set of measure zero.

Proof. According to (1)

$$L(u, \sigma_a) = \prod_{n=1}^{\infty} \cos(u/a^n), \qquad (a > 1).$$

Thus, if u is in the interval $a^k < u \leqq a^{k+1}$

$$L(u, \sigma_a) < \prod_{r=1}^{k} \cos(a^r(u/a^k)).$$

Now let $\lambda = u/a^k$ so that $1 < \lambda < 2$. Then

$$L(u, \sigma_a) < \prod_{r=1}^{k} |\cos(\lambda a^r)| = \prod_{\lambda a^r \leqq u} |\cos(\lambda a^r)|.$$

By Lemma 4, with $M = u$, if a is chosen in the interval (12) and not in Z_q and if $u > \alpha(a)$ there are at least $q \log u$ factors in this last product which are less than $\cos \pi/30$ so that

$$|L(u, \sigma_a)| < (\cos \pi/30)^{q \log u}, \qquad u > \alpha(a).$$

Since, according to Lemma 4, q (> 0) can be chosen arbitrarily this completes the proof of the theorem.

INSTITUTE FOR ADVANCED STUDY.