Reprinted from DUKE MATHEMATICAL JOURNAL
Vol. 8, No. 2, June, 1941

THE DISTRIBUTION OF THE NUMBER OF SUMMANDS IN THE PARTITIONS OF A POSITIVE INTEGER

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1. It is well known that $p(n)$, the number of unrestricted partitions of a positive integer n, is given by the asymptotic formula [2][1]

$$p(n) \sim \frac{1}{4n3^{\frac{1}{2}}} \exp Cn^{\frac{1}{2}}, \qquad C = \pi(\tfrac{2}{3})^{\frac{1}{2}}. \tag{1.1}$$

In §2 we prove that the "normal" number of summands in the partitions of n is $C^{-1}n^{\frac{1}{2}} \log n$. More precisely, we prove the following

THEOREM 1.1. *Denote by $p_k(n)$ the number of partitions of n which have at most k summands. Then, for*

$$k = C^{-1}n^{\frac{1}{2}} \log n + xn^{\frac{1}{2}}, \tag{1.2}$$

we have

$$\lim_{n\to\infty} \frac{p_k(n)}{p(n)} = \exp\left(-\frac{2}{C} e^{-\frac{1}{2}Cx}\right). \tag{1.3}$$

The right member of (1.3) is strictly monotone and continuous; it tends to 0 as $x \to -\infty$ and to 1 as $x \to +\infty$. Hence, it is a distribution function. Also from (1.3) we clearly obtain the weaker result that if $f(n)$ is any function tending with n to infinity, then the number of summands in "almost all" partitions of n lies between

$$\frac{n^{\frac{1}{2}} \log n}{C} \pm f(n) \cdot n^{\frac{1}{2}}. \tag{1.4}$$

It is easily seen that the number of partitions of n having k or less summands is equal to the number of partitions of n in which no summand exceeds k. Thus the preceding results can be applied to this case also.

In §3 we consider $P(n)$, the number of partitions of n into unequal parts. (By a theorem of Euler, $P(n)$ is also equal to the number of partitions of n into odd summands with repetitions allowed.) We obtain results similar to the above for $p_k(n)$, but we shall not give all details of the proof.

In §4 we derive an asymptotic formula for $p_k(n)$,

$$p_k(n) \sim \frac{\binom{n-1}{k-1}}{k!}, \tag{1.5}$$

valid uniformly in k in the range $k = o(n^{\frac{1}{3}})$.

Received January 9, 1941. The first-mentioned author is a Harrison Research Fellow.

[1] Numbers in brackets refer to the bibliography at the end of this paper.

These matters, to our knowledge, have not been discussed previously. Somewhat similar questions have been suggested by Castelnuovo [1] and treated by Tricomi [5]. The collected works of Sylvester are full of papers dealing with $p_k(n)$, for particular values of k. However, Sylvester did not consider the effect of making k a function of n, i.e., he did not discuss the asymptotic behavior of $p_k(n)$. His attack was entirely algebraic. In their famous paper on partitions Hardy and Ramanujan [2] give an inequality for $p_k(n)$ for finite k. If we use the generating function for $p_k(n)$ and the calculus of residues, it is easy to derive an asymptotic formula (see §5).

In one of his numerous papers on partitions Sylvester ([4], pp. 90–99, esp. p. 93, footnote) remarked that in attempting to work out problems of this sort one meets with another class of partitions in the midst of the problem, so that it is difficult to avoid circularity. It has been possible to do this in our case by using elementary inequalities for the occurring partition function.

2. We start from the following identity

$$
\begin{aligned}
p_k(n) &= p(n) - \sum_{1 \leqq r \leqq n-k} p(n - (k + r)) \\
&\quad + \sum_{\substack{0 < r_1 < r_2 \\ 1 < r_1 + r_2 \leqq n - 2k}} p(n - (k + r_1) - (k + r_2)) \\
&\quad - \sum_{\substack{0 < r_1 < r_2 < r_3 \\ 1 < r_1 + r_2 + r_3 \leqq n - 3k}} p(n - (k + r_1) - (k + r_2) - (k + r_3)) + - \cdots \\
&= p(n)\{1 - S_1 + S_2 - S_3 + - \cdots\}.
\end{aligned} \tag{2.1}
$$

(2.1) is a simple application of the Sieve of Eratosthenes; we use also the remark in the paragraph of §1 following (1.4), and the obvious fact[2] that the number of partitions of n into summands which include k is equal to $p(n - k)$. Also, by a well-known principle of Bruns' method ([3], p. 75, (59)),

$$
1 - S_1 + S_2 - + \cdots - S_{2\nu-1} \leqq \frac{p_k(n)}{p(n)} \leqq 1 - S_1 + S_2 - + \cdots + S_{2\nu} \tag{2.2}
$$

$$(\nu = 1, 2, 3, \cdots).$$

Now we estimate $S_1, S_2, \cdots$. Using (1.1), we have, with $k = C^{-1} n^{1/2} \log n + x n^{1/2}$,

$$
S_1 \sim \sum_{1 \leqq r \leqq n-k-1} \frac{n}{n - k - r} \exp\left[C(n - k - r)^{1/2} - C n^{1/2}\right] = \sum_{r \leqq n^{3/4}}{}_1 + \sum_{r > n^{3/4}}{}_2 .
$$

In $\sum_1$, $n(n - k - r)^{-1} \sim 1$ and $(n - k - 1)^{1/2} \sim n^{1/2} - \frac{1}{2} n^{-1/2}(k + r)$; thus

[2] This principle will be used several times in this paper.

$$\textstyle\sum_1 \sim \displaystyle\sum_{r \leqq n^{3/5}} \exp\left[-C \cdot \tfrac{1}{2} n^{-1/2}(k + r)\right] = n^{-1/2} \exp\left[-\tfrac{1}{2}Cx\right] \sum_{1 \leqq r \leqq n^{3/5}} \exp\left[-\tfrac{1}{2}Crn^{-1/2}\right]$$

$$= n^{-1/2} \exp\left[-\tfrac{1}{2}Cx\right] \exp\left[-\tfrac{1}{2}Cn^{-1/2}\right] \frac{1 - \exp\left[-\tfrac{1}{2}Cn^{1/10}\right]}{1 - \exp\left[-\tfrac{1}{2}Cn^{-1/2}\right]}$$

$$\sim n^{-1/2} \exp\left[-\tfrac{1}{2}Cx\right] \cdot \frac{2n^{1/2}}{C};$$

$$\textstyle\sum_2 < n \displaystyle\sum_{r > n^{3/5}} \exp\left[-\tfrac{1}{2}Cn^{-1/2}(k + r)\right] < n \sum_{r > n^{3/5}} \exp\left[-\tfrac{1}{2}Crn^{-1/2}\right]$$

$$< n \exp\left[-\tfrac{1}{2}Cn^{-1/2}n^{3/5}\right] \sum_{r > n^{3/5}} 1 < n^2 \exp\left[-\tfrac{1}{2}Cn^{1/10}\right] = o(1).$$

Therefore,

$$S_1 \sim \frac{2}{C} \exp\left[-\tfrac{1}{2}Cx\right]. \tag{2.3}$$

Next

$$S_2 = \frac{1}{2!\,p(n)} \sum_{1 \leqq r_1, r_2 \leqq n-2k} p(n - 2k - r_1 - r_2) - \frac{1}{p(n)} \sum_{1 \leqq r \leqq n-2k} p(n - 2k - 2r)$$

$$= \frac{1}{2!} \left(\textstyle\sum_1 + \sum_2\right) - \displaystyle\sum_{r \leqq n^{3/5}}{}_3 - \sum_{r > n^{3/5}}{}_4,$$

where $\sum_1$ runs over all pairs (r_1, r_2) in which neither r_1 nor r_2 exceeds $n^{3/5}$; $\sum_2$ over all pairs in which at least one member exceeds $n^{3/5}$. As before, we find

$$\textstyle\sum_1 \sim \dfrac{1}{n} \exp\left[-Cx\right] \displaystyle\sum_{r_1, r_2 \leqq n^{3/5}} \exp\left[-\tfrac{1}{2}Cn^{-1/2}(r_1 + r_2)\right]$$

$$= \frac{1}{n} \exp\left[-Cx\right] \Big(\sum_{r_1 \leqq n^{3/5}} \exp\left[-\tfrac{1}{2}Cr_1 n^{-1/2}\right]\Big)^2 \sim \left(\frac{2}{C} \exp\left[-\tfrac{1}{2}Cx\right]\right)^2,$$

$$\textstyle\sum_2 = o(1),$$

$$\textstyle\sum_3 = \dfrac{1}{n} \exp\left[-Cx\right] \displaystyle\sum_{r \leqq n^{3/5}} \exp\left[-Crn^{-1/2}\right] \sim C^{-1} n^{-1/2} \exp\left[-Cx\right]$$

$$= o(1),$$

$$\textstyle\sum_4 = o(1).$$

Therefore,

$$S_2 \sim \frac{1}{2!} \left(\frac{2}{C} \exp\left[-\tfrac{1}{2}Cx\right]\right)^2. \tag{2.4}$$

Similarly we get for S_ν

$$S_\nu \sim \frac{1}{\nu!} \left(\frac{2}{C} \exp\left[-\tfrac{1}{2}Cx\right]\right)^\nu. \tag{2.5}$$

Hence from (2.2) and the fact that $S_\nu \to 0$ with ν^{-1}, we have

$$\frac{p_k(n)}{p(n)} \sim 1 + \sum_{\nu=1}^{\infty} (-1)^\nu S_\nu = \exp\left(-\frac{2}{C}\, e^{-\frac{1}{2}Cx}\right),$$

which is (1.3).

3. We now consider $P(n)$, the number of partitions of n into unequal summands. Such a partition will be called an "unequal partition"; a partition into odd summands we shall call an "odd partition". We outline the proof of the following

THEOREM 3.1. *For almost all unequal partitions of n, the number of summands in a given partition not exceeding $xn^{\frac{1}{2}}$ lies between*

$$\frac{2n^{\frac{1}{2}}}{D} \log \frac{2}{1 + e^{-Dx}} \pm \epsilon n^{\frac{1}{2}}, \qquad D = \pi(\tfrac{1}{3})^{\frac{1}{2}}. \tag{3.1}$$

To the odd partition

$$n = 1\cdot x_1 + 3\cdot x_3 + \cdots + (2r + 1)x_{2r+1} \tag{3.21}$$

corresponds in a one-to-one way the unequal partition

$$n = \sum_{l=0}^{r} (2l + 1) \sum_{t=1}^{s_{2l+1}} 2^{\alpha_{2l+1,t}} = \sum_{l,t} (2l + 1)2^{\alpha_{2l+1,t}}, \tag{3.22}$$

where[3]

$$x_i = 2^{\alpha_{i,1}} + 2^{\alpha_{i,2}} + \cdots + 2^{\alpha_{i,s_i}}.$$

Denote by $A(x)$ the number of summands not exceeding $xn^{\frac{1}{2}}$ in a given partition of n, and by $\sum_{P(n)}$ a sum which runs over all unequal partitions of n. Then

$$\sum_{P(n)} A(x) = \sum_{1 \leq u < xn^{\frac{1}{2}}} P_u(n), \tag{3.3}$$

where $P_u(n)$ is the number of unequal partitions of n which contain the summand u. Let $u = 2^k(2v + 1)$.

In order to calculate $P_u(n)$, we consider all odd partitions of n (3.21) which contain $(2v + 1)$,

$$n = 1\cdot x_1 + \cdots + (2v + 1)x_{2v+1} + \cdots, \tag{3.41}$$

and in which, moreover, 2^k occurs in the dyadic expansion of x_{2v+1},

$$x_{2v+1} = \cdots + 2^k + \cdots. \tag{3.42}$$

By the correspondence (3.21), (3.22), $P_u(n)$ is equal to the number of such partitions.

In order to count these partitions we let $k = 0, 1, 2, \cdots$ in turn. $k = 0$

[3] This correspondence, of course, furnishes a proof of Euler's theorem.

implies u is odd. Then in (3.41), x_{2v+1} runs through all odd integers, since in (3.42) a $2^0 = 1$ must occur. Hence, we are interested in those odd partitions which contain $2v + 1$ exactly once, exactly three times, etc. Their number is clearly

$$(3.51)\quad \begin{aligned} &P(n - (2v + 1)) - P(n - 2(2v + 1)) \\ &\qquad + P(n - 3(2v + 1)) - P(n - 4(2v + 1)) + \cdots, \end{aligned}$$

and this must be summed on $v = 0, 1, 2, \cdots$, such that $u = 2v + 1 < xn^{\frac{1}{2}}$.

In the same way we count, for a general k, those odd partitions which contain $2v + 1$ exactly $2^k, 2^k + 1, \cdots, 2^{k+1} - 1$ times; $2^{k+1} + 2^k, 2^{k+1} + 2^k + 1, \cdots, 2^{k+2}$ times; etc. The number of such partitions is seen to be

$$(3.52)\quad \begin{aligned} &P(n - 2^k(2v + 1)) - P(n - 2\cdot 2^k(2v + 1)) \\ &\qquad + P(n - 3\cdot 2^k(2v + 1)) - P(n - 4\cdot 2^k(2v + 1)) + \cdots, \end{aligned}$$

this to be summed on $v = 0, 1, \cdots$, such that $u = 2^k(2v + 1) < xn^{\frac{1}{2}}$.

To these sums we can apply the method of §2, using the asymptotic expression for $P(n)$ given by Hardy-Ramanujan ([2], p. 113),[4]

$$(3.61)\quad P(n) \sim \frac{\exp [Dn^{\frac{1}{2}}]}{4\cdot 3^{\frac{1}{4}}\cdot n^{\frac{3}{4}}}.$$

In this way we obtain the asymptotic value of $\sum_{P(n)} A(x)$ as

$$(3.62)\quad \sum_{P(n)} A(x) \sim \frac{2n^{\frac{1}{2}}}{D} P(n) \log \frac{2}{1 + e^{-Dx}}.$$

Next we consider

$$(3.71)\quad \begin{aligned} \Delta(x) &= \sum_{P(n)} [A(x) - n^{\frac{1}{2}}F(x)]^2 \\ &\sim \sum_{P(n)} A^2(x) - nP(n)F^2(x), \end{aligned}$$

where we have written for abbreviation

$$F(x) = \frac{2}{D} \log \frac{2}{1 + e^{-Dx}}.$$

Now

$$(3.72)\quad \sum_{P(n)} A^2(x) = \sum_{1 \leq u_1, u_2 < xn^{\frac{1}{2}}} P_{u_1, u_2}(n),$$

where $P_{u_1,u_2}(n)$ denotes the number of unequal partitions of n containing both

[4] See also L. K. Hua, *On the number of partitions of a number into unequal parts*, Bulletin of the American Mathematical Society, vol. 46(1940), p. 419, abstract no. 279.

u_1 and u_2 ; $P_{u_1,u_1}(n) = P_{u_1}(n)$. We calculate $P_{u_1,u_2}(n)$ by the same methods used to find $P_u(n)$. It turns out that[5]

$$(3.73) \qquad P_{u_1,u_2}(n) \sim E_1 \cdot E_2 \cdot P(n), \qquad E_1 = \frac{P_{u_1}(n)}{P(n)}, \qquad E_2 = \frac{P_{u_2}(n)}{P(n)};$$

thus

$$(3.74) \qquad \sum_{P(n)} A^2(x) \sim nF^2(x)P(n)$$

and

$$(3.75) \qquad \Delta(x) = o(nF^2(x)P(n)).$$

For a fixed $\epsilon > 0$, let $N(x, \epsilon)$ be the number of unequal partitions of n for which

$$| A(x) - n^{\frac{1}{2}}F(x) | > \epsilon n^{\frac{1}{2}}.$$

Then

$$\Delta(x) > N(x, \epsilon) \cdot \epsilon^2 n,$$

and by (3.75),

$$N(x, \epsilon) = o(P(n)).$$

This is equivalent to Theorem 3.1.

This leads to the following $(x \to \infty)$

THEOREM 3.2. *For almost all unequal partitions of n the number of summands in a given partition lies between*

$$\frac{2n^{\frac{1}{2}}}{D} \log 2 \pm \epsilon n^{\frac{1}{2}}.$$

By sharper arguments we can obtain

THEOREM 3.3. *The number of unequal partitions of n in which the number of summands in a given partition is less than*

$$\frac{2n^{\frac{1}{2}}}{D} \log 2 + yn^{\frac{1}{4}}$$

is given by a Gaussian integral.

We add the following two theorems, which may be of some interest. They can be proved very easily by using the methods of this section.

THEOREM 3.4. *Let*

$$(3.81) \qquad n = a_1 + a_2 + \cdots + a_k$$

[5] (3.73) expresses the independence, in the sense of probability, of the function $P_u(n)/(P(n))^2$. This holds, however, only for the values considered, i.e., u_1, $u_2 < xn^{\frac{1}{2}}$.

be any partition of n. Define

$$f(n; a_1, a_2, \cdots, a_k) = f(n) = \sum_i A_i,$$

where A_i runs over the different summands in the given partition. Then for almost all partitions $f(n)$ lies between

$$\frac{6n}{\pi^2}(1 \pm \epsilon). \tag{3.82}$$

THEOREM 3.5. *Let $\varphi(n; a_1, a_2, \cdots, a_k) = \varphi(n)$ denote the number of different summands in the partition* (3.81). *Then for almost all partitions $\varphi(n)$ lies between*

$$\frac{2n^{\frac{1}{2}}}{C}(1 \pm \epsilon), \qquad C = \pi(\tfrac{2}{3})^{\frac{1}{2}}. \tag{3.9}$$

4. We now discuss the asymptotic behavior of $p_k(n)$ for $k = o(n^{\frac{1}{3}})$ and prove the following

THEOREM 4.1.

$$p_k(n) \sim \frac{\binom{n-1}{k-1}}{k!}, \tag{4.1}$$

this formula being valid uniformly in k for $k = o(n^{\frac{1}{3}})$.[6]

LEMMA 4.2. *Let $k = o(n^{\frac{1}{3}})$. Then*[7]

$$p_k(n) > \frac{1}{2}\cdot\frac{n}{k^2}\cdot p_{k-1}(n). \tag{4.2}$$

In the proof of this lemma, we shall consider partitions into exactly k summands some of which may be zero. This is equivalent to the case of partitions into k or fewer summands.

First we show that

$$p_k(n) > \frac{n}{k^3}\, p_{k-1}(n). \tag{4.31}$$

Let

$$n = a_1 + a_2 + \cdots + a_{k-1}, \qquad 0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_{k-1}, \tag{4.32}$$

be any partition of n into $k - 1$ parts. Clearly $a_{k-1} > n/k$. Now if we write $a_{k-1} = x + y$, $0 \leqq x \leqq y$, we obtain from each partition (4.32) at least

[6] I.e., for every $\epsilon > 0$, and $0 < k^3n^{-1} < \epsilon$, the ratio of $p_k(n)$ to $\binom{n-1}{k-1}/k!$ remains between $1 \pm \epsilon$ as $n \to \infty$.

[7] This result no doubt holds for $k = O(n^{\frac{1}{3}})$.

$a_{k-1}/2 > n/2k$ partitions of n into k parts. Hence, from all partitions (4.32) we get at least $p_{k-1}(n) \cdot n/2k$ partitions of n into k parts,

$$n = b_1 + b_2 + \cdots + b_k, \qquad 0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_k. \tag{4.33}$$

In the set (4.33) no partition is duplicated more than $\binom{k}{2}$ times; therefore

$$p_{k-1}(n) \cdot \frac{n}{2k} \leqq p_k(n) \cdot \binom{k}{2},$$

and (4.31) follows.

Next, in (4.32), let $A_1, A_2, \cdots, A_r$ be the distinct positive summands, $0 < A_1 < \cdots < A_r$. If we break up each A_i into two parts as in the preceding paragraph, we obtain at least

$$\tfrac{1}{2}(A_1 + A_2 + \cdots + A_r) \tag{4.41}$$

partitions in (4.33).

In the following we denote by $\sum_{p_k(n)}$ a sum which runs over all partitions of n into k parts some of which may be zero. We shall estimate $\sum_{p_{k-1}(n)} \sum_{i=1}^{r} A_i$.

We have

$$\sum_{p_{k-1}(n)} \sum_{i=1}^{r} A_i = \sum_{s=1}^{n} s p_{k-2}(n - s), \tag{4.42}$$

since a given integer s appears in the left member as many times as there are partitions of n into $k - 1$ parts one of which is s, i.e., just $p_{k-2}(n - s)$ times. By an extension of the same reasoning we get

$$\begin{aligned} \sum_{p_{k-1}(n)} \sum_{i=1}^{k-1} a_i &= n p_{k-1}(n) \\ &= \sum_{s=1}^{n} s\{p_{k-2}(n - s) + p_{k-3}(n - 2s) + p_{k-4}(n - 3s) + \cdots\}, \end{aligned} \tag{4.43}$$

the series in the braces terminating of its own accord. Now

$$p_{k-3}(n - 2s) + p_{k-4}(n - 3s) + \cdots < 3p_{k-2}(n - s). \tag{4.44}$$

For, clearly,

$$\begin{aligned} &p_{k-3}(n - 2s) \leqq p_{k-2}(n - s), \\ &p_{k-4}(n - 3s) \leqq p_{k-3}(n - s), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &p_{k-u}(n - (u - 1)s) \leqq p_{k-u+1}(n - s) \end{aligned}$$

(see footnote 2); hence

$$p_{k-3}(n - 2s) + p_{k-4}(n - 3s) + \cdots \leqq p_{k-2}(n - s) + p_{k-3}(n - s) + \cdots.$$

Applying (4.31) to the last inequality, we see that the left member does not exceed

$$p_{k-2}(n-s)\left\{1+\frac{(k-2)^3}{n-s}+\frac{(k-3)^3(k-2)^3}{(n-s)^2}+\cdots\right\}.$$

We remark that we need only consider $s < \frac{1}{2}n$, for otherwise the right member of (4.43) reduces to the first term. For $s < \frac{1}{2}n$, the above expression in braces is less than

$$1+\frac{2k^3}{n}+\left(\frac{2k^3}{n}\right)^2+\cdots<1+\tfrac{1}{2}+\tfrac{1}{4}+\cdots=2,$$

since $k = o(n^{\frac{1}{3}})$. This proves (4.44).

Finally, (4.42), (4.43), (4.44) give

$$\begin{aligned}\sum_{p_{k-1}(n)}\sum_{i=1}^{r}A_i &= \sum_{s=1}^{n} s p_{k-2}(n-s)\\ &> \frac{1}{4}\sum_{s=1}^{n} s\{p_{k-2}(n-s)+p_{k-3}(n-2s)+\cdots\}\\ &= \tfrac{1}{4}np_{k-1}(n).\end{aligned}\tag{4.45}$$

(4.41) and (4.45) mean that by the process of breaking up each A_i into two parts we obtain from the set (4.32) at least $\frac{1}{8}np_{k-1}(n)$ partitions in (4.33). Moreover, no partition is duplicated more than $\binom{k}{2}$ times. Hence

$$\tfrac{1}{4}np_{k-1}(n) < p_k(n)\cdot\binom{k}{2},$$

and this proves Lemma 4.2.

Corollary 4.3. *If $k = o(n^{\frac{1}{3}})$, then the number of partitions of n into exactly k positive summands is asymptotically equal to the number of partitions of n into k or fewer positive summands.*

For by a t-fold application of Lemma 4.2, we have

$$p_{k-t}(n) < \left(\frac{2k^2}{n}\right)^t p_k(n);$$

hence

$$\sum_{t=1}^{k-1} p_{k-t}(n) < p_k(n)\sum_{t=1}^{k-1}\left(\frac{2k^2}{n}\right)^t = o(p_k(n)),$$

since $k = o(n^{\frac{1}{3}})$.

Lemma 4.4. *The number of partitions of n into exactly k positive summands not all of which are different is $o(p_k(n))$*

Let any such partition be given by

$$n = t_1b_1 + t_2b_2 + \cdots + t_xb_x, \qquad \sum_{i=1}^{x} t_i = k, \tag{4.51}$$

and $t_i > 1$ for some i, i.e., $x < k$. To this we make correspond

$$n = c_1 + c_2 + \cdots + c_x, \qquad c_i = t_ib_i. \tag{4.52}$$

This furnishes a single-valued mapping of (4.51) into a subset of the set of partitions of n with fewer than k summands. This inverse mapping is far from being single-valued, however. In fact, given a fixed partition of (4.52),

$$n = d_1 + d_2 + \cdots + d_{k-t}, \qquad t > 0, \tag{4.53}$$

we inquire in how many ways it can be mapped into (4.51). The inverse mapping exhausts the set (4.51).

For this purpose we select v of the d's, say $d_{i_1}, d_{i_2}, \cdots, d_{i_v}$, and split d_{i_1} into w_1 equal parts, d_{i_2} into w_2 equal parts, $\cdots, d_{i_v}$ into w_v equal parts ($w_1 \geqq 2, \cdots, w_v \geqq 2$).[8] We must evidently have

$$w_1 + w_2 + \cdots + w_v = v + t. \tag{4.54}$$

Since in a given decomposition $v \leqq t$, we get

$$w_1 + w_2 + \cdots + w_v \leqq 2t. \tag{4.55}$$

Hence, the total number of decompositions obtainable from all possible choices of v and $w_1, w_2, \cdots, w_v$ is less than[9]

$$p(1) + p(2) + \cdots + p(2t) < 4^t. \tag{4.56}$$

From a given decomposition (4.54) we obtain at most $(k - t)^v \leqq (k - t)^t < k^t$ partitions in (4.51), so that, all in all, we get at most 4^tk^t partitions in (4.51) from our fixed partition (4.53). But for each t there are $p_{k-t}(n)$ partitions of the form (4.53); hence the total number of partitions of n into k positive summands not all of which are different is less than

$$\sum_{t=1}^{k-1} 4^t k^t p_{k-t}(n),$$

and by Lemma 4.2 this is less than

$$p_k(n) \sum_{t=1}^{k-1} 4^t k^t \left(\frac{2k^2}{n}\right)^t = p_k(n) \sum_{t=1}^{k-1} \left(\frac{8k^3}{n}\right)^t = o(p_k(n)),$$

by virtue of the condition $k = o(n^{\frac{1}{3}})$. Thus Lemma 4.4 is proved.

[8] We assume here $w_1 \mid d_{i_1}, \cdots, w_v \mid d_{i_v}$. This assumption only strengthens the inequalities which follow.

[9] This estimate follows from an elementary inequality for $p(n)$, $p(n) < 2^{n-1}$. For the proof of the latter, see footnote 11.

LEMMA 4.5.[10] *The number, $p_k'(n)$, of partitions of n into k positive summands whose order is considered (i.e., two partitions are counted as different if they differ only in the order of their summands) is* $\binom{n-1}{k-1}$.

Let

$$n = a_1 + a_2 + \cdots + a_k, \qquad a_i > 0. \tag{4.61}$$

To this *partition* we make correspond the *combination*

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots, a_1 + a_2 + \cdots + a_{k-1}, \tag{4.62}$$

and this correspondence is clearly one to one. But each of the $k - 1$ integers in (4.62) is not greater than $n - 1$, since $a_k \geqq 1$.[11]

Now we can prove Theorem 4.1. From Corollary 4.3, it is clear that we need consider only partitions having exactly k positive summands. Moreover, from Lemma 4.4, we see that we may assume all summands in a given partition to be different. But from a partition in which all k summands are different we obtain $k!$ partitions of the type considered in Lemma 4.5. Thus the theorem follows.

5. By the application of the Hardy-Littlewood method we can obtain a second proof of Theorem 4.1. But it hardly seems worth while to use this elaborate method unless something more results. It is easily seen that the essential contribution is furnished by the neighborhood of $x = 1$. Hence what we need is information about the asymptotic character of the generating function

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = 1 + \sum_1^\infty p_k(n)x^n$$

around $x = 1$. The possibility of obtaining a suitably sharp asymptotic representation remains to be investigated.

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[10] This lemma and proof are well known.

[11] The estimate for $p(n)$ given in footnote 9 follows easily from Lemma 4.5. For

$$p(n) < p_1'(n) + p_2'(n) + \cdots + p_n'(n) = 2^{n-1}.$$