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ON THE ASYMPTOTIC DENSITY OF THE SUM OF TWO SEQUENCES

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Let $a_1 < a_2 < \cdots$ be an infinite sequence, A, of positive integers. Denote the number of a's not exceeding n by $f(n)$. Schnirelmann has defined the density of A as G.L.B. $f(n)/n$.[1] Now let $a_1 < a_2 < \cdots ; b_1 < b_2 < \cdots$ be two sequences. We define the sum $A + B$ of these two sequences as the set of integers of the form a_i or b_j or $\{a_i + b_j\}$. Schnirelmann proved that if the density of A is α and that of B is β then the density of $A + B$ is $\geqq \alpha + \beta - \alpha\beta$.

Khintchine[2] proved that, provided that $\alpha = \beta \leqq \frac{1}{2}$, the density of $A + B$ is $\geqq 2\alpha$. He conjectured more generally that if $\alpha + \beta \leqq 1$ the density of $A + B$ is $\geqq \alpha + \beta$. It is easy to see that if $\alpha + \beta \geqq 1$ then every integer is in $A + B$, so the density of $A + B$ is 1. Khintchine's conjecture seems very deep.

Besicovitch[3] defined $\beta' = \text{G.L.B.}\ \varphi(n)/(n + 1)$ where $\varphi(n)$ denotes the number of the b's not exceeding n, and proved that the Schnirelmann density of the sequence of numbers $\{a_i, a_i + b_j\}$ is $\geqq \alpha + \beta'$. An example of Rado showed that this result is the best possible.

Define the asymptotic density of A as $\varliminf f(n)/n$. Then if $\alpha \leqq \frac{1}{2}$ and $a_1 = 1$ I have proved that the asymptotic density of $A + B$ is $\geqq \frac{3}{2}\alpha$.[4] The following simple example of Heilbronn shows that this result is the best possible: Let the a's be the integers $\equiv 0, 1 \pmod 4$. Then $A + A$ contains the integers $\equiv 0, 1, 2 \pmod 4$. In the present note we prove the following

THEOREM: *Let the asymptotic density of A be α and that of B be β, where $\alpha + \beta \leqq 1$, $\beta \leqq \alpha$, $b_1 = 1$. Then the asymptotic density of $A + B$ is not less than $\alpha + \frac{1}{2}\beta$, and, in fact, one of the sequences $\{a_i, a_i + 1\}$ or $\{a_i + b_j\}$ has asymptotic density $\geqq \alpha + \frac{1}{2}\beta$.*

It is easy to see that if $\alpha + \beta > 1$ then all large integers are in $A + B$. For if not then, none of the integers $n - a_i$ belong to B, and the asymptotic density of B would be not greater than $1 - \alpha < \beta$.

To prove our theorem we first need a slight sharpening of the theorem of Besicovitch; in fact, we prove the following

LEMMA: *Define the modified density of B as follows:*

[1] Schnirelmann, *Über additive Eigenschaften der Zahlen*, Math. Annalen 107 (1933), pp. 649–690.

[2] Khintchine, *Zur additiven Zahlentheorie*, Recueil math. de la soc. Moscow 39 (1932), pp. 27–34.

[3] Besicovitch, *On the density of the sum of two sequences of integers*, Journ. of the London math. soc. 10 (1935), pp. 246–248.

[4] Erdös, *On the asymptotic density of the sum of two sequences one of which forms a basis for the integers. ii.*, Travaux de l'institut math. de Tblissi 3 (1938), pp. 217–223.

1) $$\beta_1 = \underset{n>k}{\text{G.L.B.}} \frac{\varphi(n)}{n+1},$$

where the integers $1, 2, \cdots, k$ *belong to* B, *but* $k + 1$ *does not belong to* B. *Clearly* $\beta_1 \geqq \beta'$. *Then the Schnirelmann density of the sequence* $\{a_i, a_i + b_j\}$ *is not less than* $\alpha + \beta_1$.

The proof of this lemma follows closely the proof of Besicovitch. Denote by $f(u, v)$, $\varphi(u, v)$, $\psi(u, v)$ respectively the number of a's, b's, and terms of the sequence $\{a_i, a_i + b_j\}$ in the interval (u, v)—that is, among the integers $u + 1, u + 2, \cdots, v$. We first observe that if $r + 1$ is any integer which does not belong to the sequence $\{a_i, a_i + b_j\}$ then

2) $$f(u, v) + \varphi(r - v, r - u) \leqq v - u.$$

For as t runs through (u, v), $r + 1 - t$ runs through $(r - v, r - u)$, and if t belongs to A then $r + 1 - t$ does not belong to B.

We may assume that the Schnirelmann density of the sequence $\{a_i, a_i + b_j\}$ is less than 1, and that $\alpha > 0$, so that $a_1 = 1$. Define $m_0 = 0$, define $r_0 + 1$ as the least positive integer not belonging to $\{a_i, a_i + b_j\}$, define $m_1 + 1$ as the least integer greater than r_0 belonging to A, define $r_1 + 1$ as the least integer greater than m_1 not belonging to $\{a_i, a_i + b_j\}$, and so on.

It suffices to prove that for each x in (r_{i-1}, m_i) we have

3) $$\psi(0, x) \geqq (\alpha + \beta_1)x,$$

for if (3) holds, suppose that for some y in (m_j, r_j) we had

$$\psi(0, y) < (\alpha + \beta_1)y.$$

(We may suppose $j > 0$; else $y \leqq r_0$, so that $\psi(0, y) = y$). Then since all the integers $m_j + 1, \cdots, y$ belong to $\{a_i, a_i + b_j\}$ and $\alpha + \beta_1 \leqq 1$ we should have

$$\psi(m_j) < (\alpha + \beta_1)m_j,$$

which contradicts (3).

It follows from the definition of k and the definition of m_i and r_i that

4) $$r_i - m_i > k \qquad (i = 0, 1, 2 \cdots).$$

Let $r_{i-1} < x \leqq m_i$; we have

5) $$\psi(r_{i-1}, x) \geqq \varphi(r_{i-1} - m_{i-1} - 1, x - m_{i-1} - 1),$$

since any number $m_{i-1} + 1 + u$, where u belongs to B, is in $\{a_i, a_i + b_j\}$. Also

6) $$\psi(m_{i-1}, r_{i-1}) = r_{i-1} - m_{i-1} \geqq f(m_{i-1}, r_{i-1}) + \varphi(0, r_{i-1} - m_{i-1})$$

by (2). Clearly by the definition of the numbers r_j, m_j we have for $r_{i-1} < x \leqq m_i$, $f(m_{i-1}, x) = f(m_{i-1}, r_{i-1})$. Hence by adding (5) and (6)

7) $$\psi(m_{i-1}, x) \geqq f(m_{i-1}, x) + \varphi(0, x - m_{i-1} - 1) \geqq f(m_{i-1}, x) + \beta_1(x - m_{i-1}),$$

since by (4) $x - m_{i-1} - 1 \geqq r_{i-1} - m_{i-1} > k$. In particular

$$\psi(m_j, m_{j+1}) \geqq f(m_j, m_{j+1}) + \beta_1(m_{j+1} - m_j) \qquad (j = 0, 1, \cdots). \tag{8}$$

Summing (8) for $j = 0, 1, \cdots i - 1$ and adding (7) we have

$$\psi(0, x) \geqq f(0, x) + \beta_1 x \geqq (\alpha + \beta_1)x,$$

which completes the proof of the Lemma.

Now we can prove our theorem. We may assume $\beta > 0$. Suppose first that there exists an x belonging to A, such that the modified density of (the positive terms of) $a_i - x$ is $\geqq \alpha - \frac{1}{2}\beta$. Clearly $x + 1$ has to be in A since $\alpha - \frac{1}{2}\beta > 0$. It follows that there exists for every positive real ϵ a y such that the Schnirelmann density of the positive terms of the sequence $\{b_j - y\}$ is $\geqq \beta - \epsilon$. To see this choose y to be the greatest integer with

$$\frac{\varphi(y)}{y} \leqq \beta - \epsilon.$$

(Since $\underline{\lim}\ \varphi(y)/y = \beta$ such a y exists, unless $\varphi(y)/y > \beta - \epsilon$ for all positive y; in this case we have $y = 0$). Then by the definition of y it is clear that $\varphi(y, z)$ i.e. the number of $\{b_j - y\}$'s in $(0, z - y)$, is not less than $(\beta - \epsilon)(z - y)$, which proves our assertion.

Now consider the sequence $\{b_j - y, b_j - y + a_i - x\}$. By our lemma its Schnirelmann density is $\geqq \alpha + \frac{1}{2}\beta - \epsilon$; hence by adding $x + y$ to its members we obtain the sequence $\{b_j + x, a_i + b_j\}$ whose asymptotic density is clearly $\geqq \alpha + \frac{1}{2}\beta - \epsilon$ for every $\epsilon > 0$. But since x is in A, $b_j + x$ is in $\{a_i + b_j\}$. Hence the asymptotic density of the sequence $\{a_i + b_j\}$ is $\geqq \alpha + \frac{1}{2}\beta$, which proves our theorem in the first case.

Suppose next that Case 1 is not satisfied. We may suppose that there exist arbitrarily large values of i such that a_i and $a_i + 1$ are both in A; otherwise $\{a_i, a_i + 1\}$ has asymptotic density $2\alpha > \alpha + \frac{1}{2}\beta$. Let a_{k_1} be the first a_i such that $a_{k_1} + 1$ is also in A. Then since Case 1 is not satisfied and since $\alpha = \underline{\lim}\ f(n)/n$, there exists a largest integer m_1 such that $f(a_{k_1}, m_1) < (\alpha - \frac{1}{2}\beta)(m_1 - a_{k_1} + 1)$. Again let a_{k_2} be the least a_i greater than m_1 such that $a_{k_2} + 1$ is also in A; there exists as before a largest m_2 such that $f(a_{k_2}, m_2) < (\alpha - \frac{1}{2}\beta)(m_2 - a_{k_2} + 1)$ and so on. Take n large and let m_r be the least $m \geqq n$. It is clear that the intervals $(a_{k_i} - 1, m_i)$, $i = 1, 2 \cdots r$ do not overlap; thus

$$\sum_{i=1}^{r} f(a_{k_i}, m_i) \leqq m_r\left(\alpha - \frac{\beta}{2}\right).$$

Now since the asymptotic density of A is α, we have $f(0, m_r) > (\alpha - \epsilon)m_r$, if n is large enough, and therefore the number of a_i's in $(0, n)$ outside the intervals (a_{k_i}, m_i), $i = 1, 2 \cdots r$ is not less than

$$\left(\frac{\beta}{2} - \epsilon\right)m_r \geqq \left(\frac{\beta}{2} - \epsilon\right)n.$$

But for all these a_i's with the exception of $a_{k_1}, a_{k_2}, \cdots, a_{k_r}$, $a + 1$ is not in A.

Moreover, the intervals (a_{k_i}, m_i) do not contain only a's; else, whenever $p > a_{k_i}$ is such that (a_{k_i}, p) does contain integers not in A, we have $p > m_i$. Therefore $f(a_{k_i}, p) \geqq (\alpha - \frac{1}{2}\beta)(p - a_{k_i} + 1)$ (by definition of m_i); so that the modified density of the positive terms of $\{a_j - a_{k_i}\}$ $(j = 1, 2 \cdots)$ is $\geqq \alpha - \frac{1}{2}\beta$, and we are in Case 1. Thus each of the intervals (a_{k_i}, m_i) has to contain an x which is in A, such that $x + 1$ is not in A. Hence, finally, the number of integers $\leqq n$ of the form $a_i + 1$ which are not in A is $\geqq (\frac{1}{2}\beta - \epsilon)(n - 1)$. Hence the number of integers $\leqq n$ of the form $\{a_i, a_i + 1\}$ is not less than $(\alpha + \frac{1}{2}\beta - \epsilon)n - 1$, if n is large enough, which completes the proof of our theorem.

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