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ON THE LAW OF THE ITERATED LOGARITHM

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Introduction

Let t be a real number $(0 \leqq t \leqq 1)$, and let $t = 0.\epsilon_1(t)\epsilon_2(t) \cdots$ be its dyadic expansion, or equivalently,

$$t = \frac{\epsilon_1(t)}{2} + \frac{\epsilon_2(t)}{2^2} + \cdots + \frac{\epsilon_n(t)}{2^n} + \cdots, \tag{0.1}$$

where $\epsilon_n(t) = 0$ or 1 according as the integral part of $2^n t$ is even or odd. It is well known that $\{\epsilon_n(t)\}$ $(n = 1, 2, \cdots)$ is an independent system in the sense of probability,[1] and that

$$\int_0^1 \epsilon_n(t)\, dt = \frac{1}{2}, \quad \int_0^1 \left(\epsilon_n(t) - \frac{1}{2}\right)^2 dt = \frac{1}{4}. \tag{0.2}$$

Let us further put

$$f_n(t) = \sum_{k=1}^{n} \epsilon_k(t) - \frac{n}{2}. \tag{0.3}$$

It was proved by A. Khintchine[2] and A. Kolmogoroff[3] that

$$\limsup_{n\to\infty} \frac{f_n(t)}{\left(\frac{n}{2}\log\log n\right)^{\frac{1}{2}}} = 1 \tag{0.4}$$

for almost all t.

Let $\varphi(n)$ be a monotone increasing non-negative function defined for all sufficiently large integers. Following P. Lévy we say that $\varphi(n)$ belongs to the *upper class* if, for almost all t, there exist only finitely many n such that

$$f_n(t) > \varphi(n); \tag{0.5}$$

and $\varphi(n)$ belongs to the *lower class* if, for almost all t, there exist infinitely many n such that (0.5) is true. According to the well-known law of 0 or 1, each $\varphi(n)$ must belong to one of these classes. Then the result of A. Khintchine and A. Kolmogoroff stated above means that $\varphi(n) = (1 + \epsilon)(\frac{1}{2}n \text{ loglog } n)^{\frac{1}{2}}$ belongs to the upper class if $\epsilon > 0$, and to the lower class if $\epsilon < 0$.

The purpose of the present paper is to give a sharpening of this result. The

[1] Cf. M. Kac and H. Steinhaus, *Sur les fonctions indépendentes*, Studia Math. 6 (1936), 46–58, 59–66, 89–97.

[2] A. Khintchine, *Asymptotische Gesetz der Wahrscheinlichkeitsrechnung*, Berlin, 1933.

[3] A. Kolmogoroff, *Über das Gesetz der iterierten Logarithmus*, Math. Annalen, 101 (1929), 126–135.

main results are stated in Theorems 1, 2, 3, 4, and 5 below. Among other results, it follows from Theorem 3 that, for $k > 3$,

$$\begin{aligned}\varphi(n) = \left(\frac{n}{2\log\log n}\right)^{\frac{1}{2}}\Big(\log\log n + \frac{3}{4}\log_3 n + \frac{1}{2}\log_4 n \\ + \cdots + \frac{1}{2}\log_{k-1} n + \left(\frac{1}{2}+\epsilon\right)\log_k n\Big)\end{aligned} \tag{0.6}$$

belongs to the upper class if $\epsilon > 0$ and to the lower class if $\epsilon \leqq 0$.

Our proof is direct and elementary. We do not assume the result of A. Khintchine and A. Kolmogoroff, and the paper can be read without knowledge of any particular results concerning the law of the iterated logarithm. The only facts we need are the notion of independence, and the well known inequality

$$c_1\frac{n}{x}e^{-2x^2/n} < Pr(A_n(x)) < c_2\frac{n}{x}e^{-2x^2/n}, \tag{0.7}$$

where

$$A_n(x) = E[t: f_n(t) > x] \tag{0.8}$$

means the set of all real numbers t $(0 \leqq t \leqq 1)$ satisfying $f_n(t) > x$, and $Pr(A)$ means the ordinary Lebesque measure of a measurable set A in the interval $0 \leqq t \leqq 1$. c_i $(i = 1, 2, \cdots)$ will denote positive constants.

Throughout the present paper, the sequence $\{m_n\}$ $(n = 1, 2, \cdots)$ defined by $m_1 = 1$ and

$$m_n = [e^{n/\log n}], \qquad n = 2, 3, \cdots, \tag{0.9}$$

will play a fundamental rôle. The fact that we adopt the sequence $\{m_n\}$ $(n = 1, 2, \cdots)$ instead of $\{a^n\}$ $(n = 1, 2, \cdots)$, which was used by A. Khintchine and A. Kolmogoroff, is essential in our proof, and will enable us to obtain our sharper results. The following inequalities, which are easy to prove, will be used very often:

$$m_n < m_{n+1} < c_3 m_n, \tag{0.10}$$

$$c_4\frac{m_n}{\log\log m_n} < m_{n+1} - m_n < c_5\frac{m_n}{\log\log m_n} \tag{0.11}$$

$$\begin{aligned}c_6\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}} < (m_{n+1}\log\log m_{n+1})^{\frac{1}{2}} \\ - (m_n \log\log m_n)^{\frac{1}{2}} < c_7\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}.\end{aligned} \tag{0.12}$$

It is not difficult to extend our results to the case in which the parameter n is continuous, i.e. the case of Brownian motion.[4] We can define the upper and

[4] Cf. A. Khintchine, loc. cit. 2. Cf. also N. Wiener, *Differential space*, Journal of Math. and Phys. 2 (1923), 131–174, and the book of P. Lévy quoted in footnote 5.

the lower classes in this case, and can obtain the corresponding results. It was stated by P. Lévy[3] that A. Kolmogoroff has proved the following result: Let $\psi(\lambda) = \varphi(\lambda)/\lambda^{\frac{1}{2}}$ be monotone increasing. Then a necessary and sufficient condition that $\varphi(\lambda)$ belong to the lower class is given by the divergence of the integral

$$(0.13) \qquad \int_0^\infty \psi(\lambda)e^{-2(\psi(\lambda))^2}\frac{d\lambda}{\lambda}.$$

It is easy to see that this is equivalent to Theorem 4. As far as I know, the proof of A. Kolmogoroff has not been published. Recently, J. Ville[5] proved that the divergence of (0.13) is necessary. This corresponds to a special case of Theorem 1, but his proof is entirely different from ours.

1

THEOREM 1. *$\varphi(n)$ belongs to the upper class if it is monotone increasing and if*

$$(1.1) \qquad \sum_{n=1}^{\infty} Pr(A_{m_n}(\varphi(m_n))) < \infty.$$

PROOF. First we remark that we may assume that

$$(1.2) \qquad \varphi(n) \leqq (n \operatorname{loglog} n)^{\frac{1}{2}}$$

for sufficiently large n. Indeed, otherwise we may consider $\varphi_1(n) = \min(\varphi(n), (n \operatorname{loglog} n)^{\frac{1}{2}})$ instead of $\varphi(n)$. It is clear that $\varphi_1(n)$ is monotone increasing, that $\varphi_1(n)$ satisfies (1.1) if $\varphi(n)$ does (because, by (0.7), $\varphi_0(n) = (n \log\log n)^{\frac{1}{2}}$ satisfies (1.1)); and that if $\varphi_1(n)$ belongs to the upper class so does $\varphi(n)$ too.

Next we notice that, under the assumption (1.2), we have

$$(1.3) \qquad Pr(A_{m_{n+1}}(\varphi(m_n))) < c_8 Pr(A_{m_n}(\varphi(m_n))).$$

This is an easy consequence of the relations (0.7), (0.10) and (0.11). We omit the proof.

Now assume that Theorem 1 is not true. Then there exists a constant $c_9 > 0$ such that, for any $M_0 = m_{n_0}$, there exists an $N_0 = m_{n_0'}$ $(n_0' > n_0)$ such that

$$(1.4) \qquad Pr\Big(\sum_{M_0 < u \leqq N_0} A_u(\varphi(u))\Big) > c_9 > 0.$$

Let us put

$$(1.5) \qquad \begin{aligned} B(u) &= A_u(\varphi(u)) - A_u(\varphi(u)) \sum_{M_0 < v < u} A_v(\varphi(v)) \\ &= E[t: f_u(t) > \varphi(u); f_v(t) \leqq \varphi(v), M_0 < v < u]. \end{aligned}$$

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[4] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris, 1937.

[5] J. Ville, *Étude critique de la notion de collectif*, Paris, 1937.

Then $\{B(u)\}$ $(M_0 < u \leqq N_0)$ are mutually disjoint, and

$$(1.6)\qquad \sum_{M_0<u\leqq N_0} B(u) = \sum_{M_0<u\leqq N_0} A_u(\varphi(u)).$$

For each u $(M_0 < u \leqq N_0)$ take an n $(n_0 \leqq n < n_0')$ such that $m_n < u \leqq m_{n+1}$, and put

$$(1.7)\qquad \Delta^+_{u.m_{n+1}} = E[t: f_{m_{n+1}}(t) - f_u(t) \geqq 0].$$

Then it is clear that $B(u)$ and $\Delta^+_{u.m_{n+1}}$ are independent, and hence

$$(1.8)\qquad Pr(B(u)\cdot\Delta^+_{u.m_{n+1}}) = Pr(B(u))Pr(\Delta^+_{u.m_{n+1}}) \geqq \tfrac{1}{3}\cdot Pr(B(u)).$$

On the other hand, since $t \epsilon B(u)\cdot\Delta^+_{u,m_{n+1}}$ implies $f_{m_{n+1}}(t) \geqq f_u(t) > \varphi(u) \geqq \varphi(m_n)$, we have

$$(1.9)\qquad B(u)\cdot\Delta^+_{u,m_{n+1}} \subset A_{m_{n+1}}(\varphi(m_n))$$

for $m_n < u \leqq m_{n+1}$. Hence, since $\{B(u)\cdot\Delta^+_{u,m_{n+1}}\}$ $(M_0 < u \leqq N_0)$ are mutually disjoint, we have, by (1.3), (1.8), (1.6) and (1.4),

$$(1.10)\qquad \begin{aligned} c_8 \sum_{M_0<u\leqq N_0} Pr(A_{m_n}(\varphi(m_n))) &\geqq \sum_{M_0<u\leqq N_0} Pr(A_{m_{n+1}}(\varphi(m_n))) \\ &\geqq \sum_{M_0<u\leqq N_0} Pr(B(u)\cdot\Delta^+_{u,m_{n+1}}) \geqq \tfrac{1}{3} \sum_{M_0<u\leqq N_0} Pr(B(u)) \\ &\geqq \tfrac{1}{3}Pr\Big(\sum_{M_0<u\leqq N_0} B(u)\Big) = \tfrac{1}{3}Pr\Big(\sum_{M_0<u\leqq N_0} A_u(\varphi(u))\Big) > \frac{c_9}{2} > 0. \end{aligned}$$

Since c_8 and c_9 are positive constants, and since $M_0 = m_{n_0}$ can be arbitrarily large, this contradicts to the assumption (1.1). This proves Theorem 1.

COROLLARY 1. $\varphi(n) = (1/(2)^{\frac{1}{2}} + \epsilon)\,(n \log\log n)^{\frac{1}{2}}$ *belongs to the upper class for* $\epsilon > 0$.

COROLLARY 2. *The expression* (0.6) *belongs to the upper class for* $\epsilon > 0$.

PROOF. Follows immediately from Theorem 1 and (0.7).

2

THEOREM 2. *If* $\varphi(n)$ *is monotone increasing, then a necessary and sufficient condition that* $\varphi(n)$ *belong to the lower class is that, for almost all* t, *there exist infinitely many* n *such that*

$$(2.1)\qquad f_{m_n}(t) > \varphi(m_n).$$

PROOF. The sufficiency is obvious. In order to prove the necessity, let us assume that $\varphi(n)$ belongs to the lower class. First we remark that we may assume

$$(2.2)\qquad \varphi(n) \leqq (n \log\log n)^{\frac{1}{2}}$$

for sufficiently large n. Indeed, by Corollary 1 to Theorem 1, $\varphi_0(n) = (n \log\log n)^{\frac{1}{2}}$ belongs to the upper class. Hence, if we put $\varphi_1(n) = \min\,(\varphi(n), \varphi_0(n))$, then

$\varphi_1(n)$ belongs to the lower class if $\varphi(n)$ does; and if the necessity of the condition is proved for $\varphi_1(n)$, then it is obviously true for $\varphi(n)$ too.

By assumption, there exists a constant $c_{10} > 0$ such that for any $M_0 = m_{n_0}$ there exists an $N_0 = m_{n_0'}$ $(n_0' > n_0)$ such that

$$Pr\Big(\sum_{M_0 < u \leq N_0} A_u(\varphi(u))\Big) > c_{10}. \tag{2.3}$$

Let us put

$$\begin{aligned} C(u) &= A_u(\varphi(u)) - A_u(\varphi(u)) \cdot \sum_{u < v \leq N_0} A_v(\varphi(v)) \\ &= E[t: f_u(t) > \varphi(u); f_v(t) \leq \varphi(v),\ u < v \leq N_0]. \end{aligned} \tag{2.4}$$

Then $\{C(u)\}$ $(M_0 < n \leq N_0)$ are mutually disjoint, and

$$\sum_{M_0 < u \leq N_0} C(u) = \sum_{M_0 < u \leq N_0} A_u(\varphi(u)). \tag{2.5}$$

Fo r each u $(M_0 < u \leq N_0)$ take an n $(n_0 \leq n < n_0')$ such that $m_n < u \leq m_{n+}$ and put

$$\Delta^-_{m_n, u} = E[t: f_u(t) - f_{m_n}(t) \leq 0]. \tag{2.6}$$

It is to be noticed that $C(u)$ and $\Delta^-_{m_n, u}$ are not independent, but it can be shown by computations[7] that there exists a constant $c_{11} > 0$ such that

$$Pr(C(u) \cdot \Delta^-_{m_n, u}) > c_{11} Pr(C(u)). \tag{2.7}$$

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[7] We sketch the proof of (2.7): Let us put

$$C(u, k) = E[t: f_u(t) = k; \quad f_v(t) \leq \varphi(v), \quad u < v \leq N_0],$$

where $k > \varphi(u)$ is an integer or integer $+ \frac{1}{2}$ according as u is even or odd. Then a simple calculation with binomial coefficients shows that

$$Pr\Big(\sum_{\varphi(u) < k \leq \varphi(u) + u/\varphi(u)} C(u, k)\Big) > c_{12} Pr(C(u)).$$

Thus it suffices to show that, for $\varphi(u) < k \leq \varphi(u) + u/\varphi(u)$,

$$Pr(C(u, k) \cdot \Delta^-_{m_n, u}) > c_{13} Pr(C(u, k)).$$

Now, it is easy to see that

$$\frac{Pr(C(u, k))}{Pr(C(u, k) \cdot \Delta^-_{m_n, u})} < c_{14} \binom{u}{\frac{u}{2} + k} \Big/ \binom{m_n}{\frac{u}{2} + k};$$

and a simple calculation shows that

$$\binom{u}{\frac{u}{2} + k} > c_{15} \binom{m_n}{\frac{u}{2} + k},$$

which completes the proof of (2.7).

On the other hand, since $t \in C(u)\cdot\Delta^-_{m_n,u}$ implies $f_{m_n}(t) \geqq f_u(t) > \varphi(u) \geqq \varphi(m_n)$, we have

$$C(u)\cdot\Delta^-_{m_n,u} \subset A_{m_n}(\varphi(m_n)) \tag{2.8}$$

for $m_n < u \leqq m_{n+1}$. Hence, since $\{C(u)\cdot\Delta^-_{m_n,u}\}$ $(M_0 < u \leqq N_0)$ are mutually disjoint, we have, by (2.7) and (2.5),

$$\begin{aligned}\sum_{M_0<m_n\leqq N_0} Pr(A_{m_n}(\varphi(m_n))) &\geqq \sum_{M_0<u\leqq N_0} Pr(C(u)\cdot\Delta^-_{m_n,u}) \\ &\geqq c_{11}\sum_{M_0<u\leqq N_0} Pr(C(u)) = c_{11}Pr(\sum_{M_0<u\leqq N_0} C(u)) \\ &= c_{11}Pr(\sum_{M_0<u\leqq N_0} A_u(\varphi(u))) > c_{10}\cdot c_{11} > 0.\end{aligned} \tag{2.9}$$

Since c_{10} and c_{11} are absolute positive constants, and since $M_0 = m_{n_0}$ can be taken arbitrarily large, this means that the set of all t for which the inequality (2.1) holds for infinitely many n, has positive measure. By the law of 0 or 1, this set must have measure 1, and thus Theorem 2 is proved.

3

THEOREM 3. *Let $\varphi(n)$ be monotone increasing and let us assume that*

$$\varphi(m_{n+1}) - \varphi(m_n) > c_{12}\,(m_n/\log\log m_n)^{\frac{1}{2}}. \tag{3.1}$$

Then a necessary and sufficient condition that $\varphi(n)$ belong to the lower class is that

$$\sum_{n=1}^{\infty} Pr(A_{m_n}(\varphi(m_n))) = \infty. \tag{3.2}$$

PROOF. The necessity follows from Theorem 1, without assuming (3.1). In order to prove that the condition (3.1) is sufficient, let us assume that $\varphi(n)$ is monotone increasing and satisfies (3.1) and (3.2). We first notice that (3.1) and (0.12) imply

$$\varphi(m_{n+1}) - \varphi(m_n) > c_{13}((m_{n+1}\log\log m_{n+1})^{\frac{1}{2}} - (m_n\log\log m_n)^{\frac{1}{2}}), \tag{3.3}$$

and hence

$$\varphi(m_n) > c_{14}(m_n\log\log m_n)^{\frac{1}{2}}. \tag{3.4}$$

From (3.4) and (0.7) it follows easily that

$$\lim_{n\to\infty} Pr(A_{m_n}(\varphi(m_n))) = 0. \tag{3.5}$$

Next we notice that we may assume

$$\varphi(n) \leqq (n\log\log n)^{\frac{1}{2}}, \tag{3.6}$$

for sufficiently large n. Indeed, otherwise we may consider $\varphi_1(n) = \min(\varphi(n), (n\log\log n)^{\frac{1}{2}})$ instead of $\varphi(n)$. Since $\varphi_0(n) = (n\log\log n)^{\frac{1}{2}}$ clearly satisfies (3.1), $\varphi_1(n)$ satisfies it too. Further, it is obvious that (3.2) is satisfied by $\varphi_1(n)$ whenever it is satisfied by $\varphi(n)$. Moreover, since $\varphi_0(n)$ belongs to the upper class, by the corollary to Theorem 1, $\varphi_1(n)$ belongs to the lower class at the same time as $\varphi(n)$.

Because of the law of 0 or 1, and because of Theorem 2, it is sufficient to prove that there exists a constant $c_{15} > 0$ such that there exists, for any $M_0 = m_{n_0}$, an $N_0 = m_{n_0'}$ $(n_0' > n_0)$ such that

$$Pr\Big(\sum_{M_0 < m_n \leqq N_0} A_{m_n}(\varphi(m_n))\Big) > c_{15}. \tag{3.7}$$

Let $\delta > 0$ be a small positive number, which we shall determine later. Then, by (3.2) and (3.4), there exists an N such that, for any $M_0 = m_{n_0} > N$, an $N_0 = m_{n_0'}$ $(n_0' > n_0)$ exists such that

$$\delta < \sum_{M_0 < m_n \leqq N_0} Pr(A_{m_n}(\varphi(m_n))) < 2\delta. \tag{3.8}$$

We shall prove that if δ is chosen sufficiently small (but fixed), then (3.7) is satisfied, with the same integers M_0 and N_0 as in (3.8), by a suitable positive constant $c_{15} > 0$.

In order to prove this, let us first put

$$\begin{aligned} D(m_n) &= A_{m_n}(\varphi(m_n)) - A_{m_n}(\varphi(m_n))\cdot \sum_{m_n < m_{n+r} \leqq N_0} A_{m_{n+r}}(\varphi(m_{n+r})) \\ &= E[t: f_{m_n}(t) > \varphi(m_n);\, f_{m_{n+r}}(t) \leqq \varphi(m_{n+r}),\ m_n < m_{n+r} \leqq N_0]. \end{aligned} \tag{3.9}$$

Then $\{D(m_n)\}$ $(M_0 < m_n \leqq N_0)$ are mutually disjoint, and

$$\sum_{M_0 < m_n \leqq N_0} D(m_n) = \sum_{M_0 < m_n \leqq N_0} A_{m_n}(\varphi(m_n)). \tag{3.10}$$

Let us further put

$$\begin{aligned} D_1(m_n) &= A_{m_n}(\varphi(m_n)) - A_{m_n}\left(\varphi(m_n) + \frac{c_{12}}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}\right) \\ &= E\left[t: \varphi(m_n) < f_{m_n}(t) \leqq \varphi(m_n) + \frac{c_{12}}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{3.11}$$

Then a simple computation will show that[8]

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[8] We have clearly

$$\frac{Pr(D_1(m_n))}{Pr(A_{m_n}(\varphi(m_n)))} = \frac{\sum' \binom{m_n}{u}}{\sum_{u > \varphi(m_n)} \binom{m_n}{u}}$$

where the dash indicates that u runs only over the interval

$$\varphi(m_n), \qquad \varphi(m_n) + \frac{c_{12}}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}.$$

A simple calculation shows that

$$\sum{}' \binom{m_n}{u} \Big/ \sum_{u > \varphi(m_n)} \binom{m_n}{u} > c_{16}$$

which proves (3.12).

$$Pr(D_1(m_n)) > c_{16}Pr(A_{m_n}(\varphi(m_n))). \tag{3.12}$$

Let us put

$$D_2(m_n) = D_1(m_n)\cdot E[t: f_{m_{n+r-1}}(t) - f_{m_{n+r}}(t) \leqq 0, r = 1, 2, \cdots, h], \tag{3.13}$$

where h is a positive integer which we shall determine later. Then it is easy to see that

$$Pr(D_2(m_n)) \geqq 2^{-h}Pr(D_1(m_n)), \tag{3.14}$$

and that $t \in D_2(m_n)$ implies

$$f_{m_{n+r}}(t) \leqq f_{m_n}(t) < \varphi(m_{n+r}), \tag{3.15}$$

for $r = 1, 2, \cdots, h$. Let us further put

$$D_3(m_n) = D_2(m_n)\cdot E[t: f_{m_{n+r}}(t) \leqq \varphi(m_{n+r}), m_{n+h} < m_{n+r} \leqq N_0]. \tag{3.16}$$

Then it is clear that $D_3(m_n) \subset D(m_n) \subset A_{m_n}(\varphi(m_n))$. In order to complete the proof of Theorem 3, it is sufficient to prove that, if δ is chosen sufficiently small and if h is chosen sufficiently large (but both fixed), then there exists a constant $c_{17} > 0$ such that

$$Pr(D_3(m_n)) > c_{17}Pr(A_{m_n}(\varphi(m_n))). \tag{3.17}$$

Indeed, (3.17) will imply

$$\begin{aligned} Pr\Big(\sum_{M_0 < m_n \leqq N_0} A_{m_n}(\varphi(m_n))\Big) &= Pr\Big(\sum_{M_0 < m_n \leqq N_0} D(m_n)\Big) \\ &= \sum_{M_0 < m_n \leqq N_0} Pr(D(m_n)) \geqq \sum_{M_0 < m_n \leqq N_0} Pr(D_3(m_n)) \\ &> c_{17} \sum_{M_0 < m_n \leqq N_0} Pr(A_{m_n}(\varphi(m_n))) > c_{17}\cdot\delta, \end{aligned} \tag{3.18}$$

which means that (3.7) is satisfied by $c_{15} = c_{17}\cdot\delta > 0$, thus completing the proof of Theorem 3.

The rest of the proof of Theorem 3 is devoted to establishing the relation (3.17). For this purpose, put

$$D_{3,r}(m_n) = D_2(m_n)\cdot A_{m_{n+r}}(\varphi(m_{n+r})), \tag{3.19}$$

for all integers r such that $m_{n+h} < m_{n+r} \leqq N_0$. It is easy to see that

$$D_2(m_n) \subset D_3(m_n) + \sum_{m_{n+h} < m_{n+r} \leqq N_0} D_{3,r}(m_n). \tag{3.20}$$

We shall evaluate $Pr(\sum_{m_{n+h} < m_{n+r} \leqq N_0} D_{3,r}(m_n))$ by decomposing the sum into three parts: $\sum_{m_{n+h} < m_{n+r} \leqq 2m_n}$, $\sum_{2m_n < m_{n+r} \leqq m_n \log m_n}$, and $\sum_{m_n \log m_n < m_{n+r} \leqq N_0}$.

In the first place, $t \in D_{3,r}(m_n)$ implies

$$
\begin{aligned}
f_{m_{n+r}}(t) - f_{m_{n+h}}(t) &\geqq f_{m_{n+r}}(t) - f_{m_n}(t) \\
&> \varphi(m_{n+r}) - \varphi(m_n) - \frac{c_{12}}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}} \\
&> \sum_{k=0}^{r-1} c_{12}\left(\frac{m_{n+k}}{\log\log m_{n+k}}\right)^{\frac{1}{2}} - \frac{c_{12}}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}} \\
&> \frac{c_{12}\, r}{2}\left(\frac{m_n}{\log\log m_n}\right).
\end{aligned} \tag{3.21}
$$

Hence

$$
Pr(D_{3,r}(m_n)) \leqq \alpha_r \cdot Pr(D_2(m_n)), \tag{3.22}
$$

where

$$
\begin{aligned}
\alpha_r &= Pr\left(E\left[t : f_{m_{n+r}}(t) - f_{m_{n+h}}(t) > \frac{c_{12}\, r}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}\right]\right) \\
&= Pr\left(A_{m_{n+r}-m_{n+h}}\left(\frac{c_{12}\, r}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}\right)\right) \\
&< c_2 \frac{(m_{n+r} - m_{n+h})^{\frac{1}{2}}}{\frac{c_{12}\, r}{2}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}} \exp\left[-\frac{2\left(\frac{c_{12}\, r}{2}\right)^2 \frac{m_n}{\log\log m_n}}{m_{n+r} - m_{n+h}}\right].
\end{aligned} \tag{3.23}
$$

Since, on the other hand, $m_{n+h} < m_{n+r} \leqq 2m_n$ implies

$$
\begin{aligned}
m_{n+r} - m_{n+h} &\leqq \sum_{k=0}^{r-1} (m_{n+k+1} - m_{n+k}) \\
&< c_5 r \frac{m_{n+r}}{\log\log m_{n+r}} \leqq 2c_5 r \frac{m_n}{\log\log m_n},
\end{aligned} \tag{3.24}
$$

we have, by (0.7),

$$
\alpha_r < c_{18} e^{-c_{19} r} \tag{3.25}
$$

for $m_{n+h} < m_{n+r} \leqq 2m_n$. Consequently,

$$
Pr\Big(\sum_{m_{n+h} < m_{n+r} \leqq 2m_n} D_{3,r}(m_n)\Big) < c_{18} \cdot Pr(D_2(m_n)) \cdot \sum_{r=h+1}^{\infty} e^{-c_{19} r}. \tag{3.26}
$$

Secondly, $t \in D_{3,r}(m_n)$ and $2m_n < m_{n+r} \leqq m_n \log m_n$ imply

$$
\begin{aligned}
f_{m_{n+r}}(t) - f_{m_{n+h}}(t) &> \frac{c_{12}}{2} \sum_{k=0}^{r-1} \left(\frac{m_{n+k}}{\log\log m_{n+k}}\right)^{\frac{1}{2}} \\
&> \frac{c_{12}}{2c_7} \sum_{k=0}^{r-1} \left((m_{n+k+1} \log\log m_{n+k+1})^{\frac{1}{2}} - (m_{n+k} \log\log m_{n+k})^{\frac{1}{2}}\right) \\
&> c_{20}(m_{n+r} \log\log m_{n+r})^{\frac{1}{2}}.
\end{aligned} \tag{3.27}
$$

Hence

$$Pr(D_{3,r}(m_n)) < \beta_r \cdot Pr(D_2(m_n)) \tag{3.28}$$

for $2m_n < m_{n+r} \leqq m_n \log m_n$, where

$$\begin{aligned} \beta_r &= Pr(E[t: f_{m_{n+r}}(t) - f_{m_{n+h}}(t) > c_{20}(m_{n+r} \log \log m_{n+r})^{\frac{1}{2}}]) \\ &= Pr(A_{m_{n+r}-m_{n+h}}(c_{20}(m_{n+r} \log \log m_{n+r})^{\frac{1}{2}})) \\ &< c_2 \frac{(m_{n+r} - m_{n+h})^{\frac{1}{2}}}{c_{20}(m_{n+r} \log \log m_{n+r})^{\frac{1}{2}}} \exp\left[-\frac{2c_{20}^2 m_{n+r} \log \log m_{n+r}}{m_{n+r} - m_{n+h}}\right] \\ &< c_{21} e^{-c_{22} \log \log m_{n+r}} < \frac{c_{21}}{(\log m_n)^{c_{22}}}. \end{aligned} \tag{3.29}$$

On the other hand, the number of m_{n+r}'s satisfying $2m_n < m_{n+r} \leqq m_n \log m_n$ does not exceed $c_{23}(\log \log m_n)^2$.[9] Hence we have

$$Pr\Big(\sum_{2m_n < m_{n+r} \leqq m_n \log m_n} D_{3,r}(m_n)\Big) < Pr(D_2(m_n)) \cdot c_{24} \frac{(\log \log m_n)^2}{(\log m_n)^{c_{21}}}. \tag{3.30}$$

Lastly, $t \in D_{3,r}(m_n)$ and $m_n \log m_n < m_{n+r} \leqq N_0$ imply

$$\begin{aligned} f_{m_{n+r}}(t) - f_{m_{n+h}}(t) &> \varphi(m_{n+r}) - \varphi(m_n) - \frac{c_{12}}{2}\left(\frac{m_n}{\log \log m_n}\right)^{\frac{1}{2}} \\ &> \varphi(m_{n+r}) - 2(m_n \log \log m_n)^{\frac{1}{2}}. \end{aligned} \tag{3.31}$$

Hence

$$Pr(D_{3,r}(m_n)) < \gamma_r \cdot Pr(D_2(m_n)) \tag{3.32}$$

for $m_n \log m_n < m_{n+r} \leqq N_0$, where

$$\begin{aligned} \gamma_r &= Pr(E[t: f_{m_{n+r}}(t) - f_{m_{n+h}}(t) > \varphi(m_{n+r}) - 2(m_n \log \log m_n)^{\frac{1}{2}}]) \\ &= Pr(A_{m_{n+r}-m_{n+h}}(\varphi(m_{n+r}) - 2(m_n \log \log m_n)^{\frac{1}{2}})) \\ &< c_2 \frac{(m_{n+r} - m_{n+h})^{\frac{1}{2}}}{\varphi(m_{n+r}) - 2(m_n \log \log m_n)^{\frac{1}{2}}} \\ &\qquad \cdot \exp\left[-\frac{2(\varphi(m_{n+r}) - 2(m_n \log \log m_n)^{\frac{1}{2}})^2}{m_{n+r} - m_{n+h}}\right] \\ &< c_2 \frac{(m_{n+r})^{\frac{1}{2}}}{\varphi(m_{n+r}) - 2(m_n \log \log m_n)^{\frac{1}{2}}} \\ &\qquad \cdot \exp\left[-\frac{2(\varphi(m_{n+r}))^2}{m_{n+r}} + \frac{8\varphi(m_{n+r})(m_n \log \log m_n)^{\frac{1}{2}}}{m_{n+r}}\right]. \end{aligned} \tag{3.33}$$

[9] It follows from (0.11) that the number of m_n's in the interval $(x, 2x)$ does not exceed $c_{10} \log \log x$. Thus the number of m_n's in the interval $(x, x \log x)$ does not exceed

$$c_{10} \log \log x \frac{\log \log x}{\log 2} < c_{13}(\log \log x)^2.$$

On the other hand, for sufficiently large n, $m_{n+r} > m_n \log m_n$ implies

$$\varphi(m_{n+r}) - 2(m_n \log\log m_n)^{\frac{1}{2}} > \tfrac{1}{2}\varphi(m_{n+r}), \tag{3.34}$$

$$\varphi(m_{n+r})(m_n \log\log m_n)^{\frac{1}{2}} < m_{n+r}. \tag{3.35}$$

Hence

$$\begin{aligned}\gamma_r &< c_{25}\cdot\frac{(m_{n+r})^{\frac{1}{2}}}{\varphi(m_{n+r})}\exp\left[-\frac{2(\varphi(m_{n+r}))^2}{m_{n+r}}\right]\\ &< c_{26}\, Pr(A_{m_{n+r}}(\varphi(m_{n+r}))).\end{aligned} \tag{3.36}$$

Consequently,

$$\begin{aligned}Pr\Big(&\sum_{m_n \log m_n < m_{n+r} \leqq N_0} D_{3,r}(m_n)\Big)\\ &< Pr(D_2(m_n))\cdot c_{25} \sum_{M_0 < m_{n+r} \leqq N_0} Pr(A_{m_{n+r}}(\varphi(m_{n+r})))\\ &< c_{26}\cdot 2\delta\cdot Pr(D_2(m_n)).\end{aligned} \tag{3.37}$$

Combining (3.26), (3.30) and (3.37), we have finally

$$\begin{aligned}Pr\Big(&\sum_{m_{n+h} < m_{n+r} \leqq N_0} D_{3,r}(m_n)\Big)\\ &< Pr(D_2(m_n))\left\{c_{18}\cdot\sum_{r=h+1}^{\infty} e^{-c_{19}r} + c_{24}\frac{(\log\log m_n)^2}{(\log m_n)^{c_{22}}} + c_{26}\cdot 2\delta\right\}.\end{aligned} \tag{3.38}$$

Hence, if we take h sufficiently large and δ sufficiently small, then we have

$$Pr\Big(\sum_{m_{n+h} < m_{n+r} \leqq N_0} D_{3,r}(m_n)\Big) < \theta\cdot Pr(D_2(m_n)), \tag{3.39}$$

where θ is a constant with $0 < \theta < 1$. Consequently, by (3.20),

$$\begin{aligned}Pr(D_3(m_n)) &> (1-\theta)\cdot Pr(D_2(m_n))\\ &> 2^{-h}(1-\theta)\cdot Pr(D_1(m_n)) > (1-\theta)\cdot c_{27} Pr(A_{m_n}(\varphi(m_n) \quad (\text{by } 3.12)\\ &> c_{17} Pr(A_{m_n}(\varphi(m_n))),\end{aligned} \tag{3.40}$$

which proves (3.17). The proof of Theorem 3 is completed.

COROLLARY 1. *$\varphi(n) = (1/\sqrt{2} + \epsilon)(n \log\log n)^{\frac{1}{2}}$ belongs to the lower class for $\epsilon \leqq 0$.*

COROLLARY 2. *The expression (0.6) belongs to the lower class for $\epsilon \leqq 0$.*

PROOF. Follows immediately from Theorem 3 and (0.7).

4

THEOREM 4. *Let $\varphi(n)/n^{\frac{1}{2}}$ be monotone increasing. Then a necessary and sufficient condition that $\varphi(n)$ belong to the lower class is that*

$$\sum_{n=1}^{\infty} Pr(A_{m_n}(\varphi(m_n))) = \infty. \tag{4.1}$$

We need the following

LEMMA 1. *Let $M_1 < N_1 < M_2 < N_2 < \cdots < M_i < N_i < \cdots$ be a sequence of positive integers tending to infinity, and let $\varphi(n)$ be such that*

$$\varphi(m_{n+1}) - \varphi(m_n) > c_{28}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}} \quad M_i < m_n \quad m_{n+1} \leqq N_i, \tag{4.2}$$

$$\varphi(m_n) > c_{29}(m_n \log\log m_n)^{\frac{1}{2}}, \qquad M_i < m_n < m_{n+1} \leqq N_i, \tag{4.3}$$

$$\sum_{M_i < m_n \leqq N_i} Pr(A_{m_n}(\varphi(m_n))) > c_{30}. \tag{4.4}$$

Then $\varphi(n)$ belongs to the lower class.

We do not give the proof of Lemma 1, since it can be carried out in the same way as in Theorem 3.

PROOF OF THEOREM 4. The necessity of the condition is clear by Theorem 1. In order to prove that it is sufficient, let us assume that $\varphi(n)/n^{\frac{1}{2}}$ is monotone increasing and that (4.1) is satisfied. We shall prove that there exists a sequence of integers $M_1 < N_1 < M_2 < N_2 \quad \cdots < M_i < N_i < \cdots$ tending to infinity, which satisfies the conditions of Lemma 1.

If we have

$$\varphi(m_n) < \tfrac{1}{10}(m_n \log\log m_n)^{\frac{1}{2}} \tag{4.5}$$

for all sufficiently large n, then the fact that $\varphi_0(n) = \frac{1}{10}(n \log\log n)^{\frac{1}{2}}$ belongs to the lower class (see Corollary 1 to Theorem 3), together with Theorem 2, will imply that $\varphi(n)$ belongs to the lower class. On the other hand, if

$$\varphi(m_n) > \tfrac{1}{20}(m_n \log\log m_n)^{\frac{1}{2}} \tag{4.6}$$

for sufficiently large n, then

$$\varphi(m_{n+1}) \geqq \left(\frac{m_{n+1}}{m_n}\right)^{\frac{1}{2}} \varphi(m_n) > \left(1 + \frac{c_{31}}{\log\log m_n}\right)\varphi(m_n) \tag{4.7}$$

by (0.11), and hence

$$\varphi(m_{n+1}) - \varphi(m_n) > c_{31}\frac{\varphi(m_n)}{\log\log m_n} > \frac{c_{31}}{20}\left(\frac{m_n}{\log\log m_n}\right)^{\frac{1}{2}}. \tag{4.8}$$

Consequently, by Theorem 3, $\varphi(n)$ must belong to the lower class again.

Thus, in order to prove Theorem 4, we have only to consider the case when there exist two sequences of integers tending to infinity $\{M_i\} = \{m_{n_i}\}$ $(i = 1, 2, \cdots)$ and $\{N_i\} = \{m_{n_i'}\}$ $(i = 1, 2, \cdots)$ such that $M_1 < N_1 < M_2 < N_2 < \cdots < M_i < N_i < \cdots$, and

$$\varphi(M_i) = \varphi(m_{n_i}) \geqq \tfrac{1}{10}(M_i \log\log M_i)^{\frac{1}{2}}, \tag{4.9}$$

$$\varphi(N_i) = \varphi(m_{n_i'}) \leqq \tfrac{1}{20}(N_i \log\log N_i)^{\frac{1}{2}}. \tag{4.10}$$

We may assume that

$$\varphi(m_n) < \tfrac{1}{10}(m_n \log\log m_n)^{\frac{1}{2}} \tag{4.11}$$

for $M_i < m_n \leqq N_i$ (i.e. for $n_i < n \leqq n_i'$) $(i = 1, 2, \cdots)$.

We shall prove that the conditions of Lemma 1 are all satisfied by these $\{M_i\}$ $(i = 1, 2, \cdots)$ and $\{N_i\}$ $(i = 1, 2, \cdots)$. Since $\varphi(M_i)/(M_i)^{\frac{1}{2}} \leqq \varphi(N_i)/N_i^{\frac{1}{2}}$ by assumption, we have $\frac{1}{10}(\log\log M_i)^{\frac{1}{2}} \leqq \frac{1}{20}(\log\log N_i)^{\frac{1}{2}}$ for $i = 1, 2, \cdots$. Since $M_i, N_i \to \infty$ as $i \to \infty$, it follows that we have $M_i^4 \leqq N_i$ for sufficiently large i.

Let now $M_i < m_n < N_i$. Then

$$Pr(A_{m_n}(\varphi(m_n))) > c_1 \frac{(m_n)^{\frac{1}{2}}}{\varphi(m_n)} e^{-2(\varphi(m_n))^2/m_n}$$

$$> c_1 \frac{10}{(\log\log m_n)^{\frac{1}{2}}} e^{-(\log\log m_n)/50} = \frac{10\, c_1}{(\log\log m_n)^{\frac{1}{2}}(\log m_n)^{1/50}} \tag{4.12}$$

$$> \frac{1}{(\log m_n)^{1/49}}$$

for sufficiently large i. Since $2\cdot\log M_i\cdot\log\log M_i < n < 3\cdot\log M_i\cdot\log\log M_i$ implies $\log M_i < n/\log n < 4\log M_i$, or equivalently $M_i < e^{n/\log n} < M_i^4$, for sufficiently large i, we have

$$\sum_{M_i < m_n \leqq N_i} Pr(A_{m_n}(\varphi(m_n))) > \sum_{M_i < m_n \leqq N_i} \frac{1}{(\log m_n)^{1/49}}$$

$$> \sum_{M_i < m_n \leqq M_i^4} \frac{1}{(\log m_n)^{1/49}} > \sum_{2p_i < n \leqq 3p_i} \frac{1}{n^{1/49}}, \tag{4.13}$$

where $p_i = \log N_i\cdot\log\log N_i$. Thus (4.4) is satisfied. (4.3) is clearly satisfied with $c_{29} = \frac{1}{20}$; (4.8) shows that (4.2) is also satisfied. This completes the proof of Theorem 4.

5

THEOREM 5. *Let $\varphi(n)$ satisfy*

$$\varphi(n) > c_{32}(n\log\log n)^{\frac{1}{2}}, \tag{5.1}$$

$$\sum_{n=1}^{\infty} Pr(A_{m_n}(\varphi(m_n))) = \infty. \tag{5.2}$$

Then $\varphi(n)$ belongs to the lower class.

To prove Theorem 5 we need the following

LEMMA 2. *Let $\varphi(n)$ be monotone increasing, and let $\{m_{n_i}\}$ $(i = 1, 2, \cdots)$ be a subsequence of $\{m_n\}$ $(n = 1, 2, \cdots)$ such that*

$$\varphi(m_{n_{i+1}}) \geqq \varphi(m_{n_i}) + c_{33}((m_{n_{i+1}}\log\log m_{n_{i+1}})^{\frac{1}{2}} - (m_{n_i}\log\log m_{n_i})^{\frac{1}{2}}) \tag{5.3}$$

$$\sum_{i=1}^{\infty} Pr(A_{m_{n_i}}(\varphi(m_{n_i}))) = \infty. \tag{5.4}$$

Then $\varphi(n)$ belongs to the lower class.

Since the proof of Lemma 2 can be carried out exactly as in the proof of Theorem 3, we omit the proof.

PROOF OF THEOREM 5. As in the proof of Theorem 3, we may assume that

$$\varphi(n) \leqq (n \log \log n)^{\frac{1}{2}} \tag{5.5}$$

for sufficiently large n. We shall find a subsequence $\{m_{n_i}\}$ $(i = 1, 2, \cdots)$ of $\{m_n\}$ $(n = 1, 2, \cdots)$ which satisfies the conditions of Lemma 2. For this purpose we classify the integers m_n into two classes. The first class I consists of all integers m_p for which

$$\varphi(m_q) \geqq \varphi(m_p) + \epsilon((m_q \log \log m_q)^{\frac{1}{2}} - (m_p \log \log m_p)^{\frac{1}{2}}) \tag{5.6}$$

for all $q \geqq p$, where ϵ is a positive constant with $0 < \epsilon < c_{32}$ which we shall determine later. All other integers m_p will belong to the second class II. We shall prove that, if we denote by $\{m_{n_i}\}$ $(i = 1, 2, \cdots, m_{n_i} < m_{n_{i+1}})$ the integers of the class I, then this sequence satisfies the conditions of Lemma 2. Indeed, (5.3) is clear from (5.6). In order to prove (5.4) for the m_{n_i}'s of the class I, let us denote by II_i the set of all integers m_p of the class II such that $m_p < m_{n_i}$ and

$$\varphi(m_{n_i}) < \varphi(m_p) + \epsilon((m_{n_i} \log \log m_{n_i})^{\frac{1}{2}} - (m_p \log \log m_p)^{\frac{1}{2}}). \tag{5.7}$$

By definition, for each m_p of the class II, there exists an m_q $(m_q > m_p)$ such that

$$\varphi(m_q) < \varphi(m_p) + \epsilon((m_q \log \log m_q)^{\frac{1}{2}} - (m_p \log \log m_p)^{\frac{1}{2}}). \tag{5.8}$$

Because of (5.1) and the relation $\epsilon < c_{32}$, there exists, for each m_p of II, a largest integer m_q $(m_q > m_p)$ satisfying (5.8). This m_q clearly belongs to I. Hence we have $\sum_{i=1}^{\infty} \mathrm{II}_i = \mathrm{II}$ (II_i are not necessarily mutually disjoint).

Thus in order to prove (5.4), we need only prove that there exists a constant $c_{34} > 0$ such that

$$\sum_{m_p \in \mathrm{II}_i} Pr(A_{m_p}(\varphi(m_p))) < c_{34} Pr(A_{m_{n_i}}(\varphi(m_{n_i}))). \tag{5.9}$$

For this purpose we shall first show that

$$m_{n_i} < c_{35} m_p \tag{5.10}$$

for all $m_p \in II_i$, where c_{35} is independent of i and p. Indeed, if (5.10) is false, we have

$$\begin{aligned} \varphi(m_{n_i}) &< \varphi(m_p) + \epsilon((m_{n_i} \log \log m_{n_i})^{\frac{1}{2}} - (m_p \log \log m_p)^{\frac{1}{2}}) \\ &< (m_p \log \log m_p)^{\frac{1}{2}} + \epsilon(m_{n_i} \log \log m_{n_i})^{\frac{1}{2}} \qquad (5.11) \\ &< \left(\frac{1}{\sqrt{c_{35}}} + \epsilon\right)(m_{n_i} \log \log m_{n_i})^{\frac{1}{2}}, \end{aligned}$$

and this is a contradiction to (5.1) if c_{35} is sufficiently large.

By (5.5), (5.7) and (5.10), if $m_p = m_{n_i - k} \epsilon \mathrm{II}_i$, then we have

$$Pr(A_{n_p}(\varphi(m_p))) < c_2 \frac{m_p^{\frac{1}{2}}}{\varphi(m_p)} e^{-2(\varphi(m_p))^2/m_p}$$

$$< \frac{c_2}{(\log\log m_p)^{\frac{1}{2}}}$$

$$\cdot \exp\left[-\frac{2\{\varphi(m_{n_i}) - \epsilon((m_{n_i}\log\log m_{n_i})^{\frac{1}{2}} - (m_p \log\log m_p)^{\frac{1}{2}})\}^2}{m_{n_i}}\right]$$

(5.12)
$$< \frac{c_2}{(\log\log m_p)^{\frac{1}{2}}}$$

$$\cdot \exp\left[-\frac{2(\varphi(m_{n_i}))^2 - 4\epsilon\varphi(m_{n_i})((m_{n_i}\log\log m_{n_i})^{\frac{1}{2}} - (m_p\log\log m_p)^{\frac{1}{2}})}{m_{n_i} - k\cdot c_5 \dfrac{m_p}{\log\log m_p}}\right]$$

$$< \frac{c_{36}}{(\log\log m_n)^{\frac{1}{2}}} \exp\left[-\frac{2(\varphi(m_{n_i}))^2}{m_{n_i}}\right]\cdot \eta^2 < c_{37}\cdot Pr(A_{m_{n_i}}(\varphi(m_{n_i})))\cdot \eta^2$$

where

$$\eta = \exp\left[\frac{(\varphi(m_{n_i}))^2}{m_{n_i}} - \frac{(\varphi(m_{n_i}))^2 - 2\epsilon\varphi(m_{n_i})((m_{n_i}\log\log m_{n_i})^{\frac{1}{2}} - (m_p\log\log m_p)^{\frac{1}{2}})}{m_{n_i} - kc_{38}\dfrac{m_{n_i}}{\log\log m_{n_i}}}\right]$$

$$< \exp\left[\frac{(\varphi(m_{n_i}))^2}{m_{n_i}} - \frac{(\varphi(m_{n_i}))^2 - 2\epsilon\varphi(m_{n_i})((m_{n_i}\log\log m_{n_i})^{\frac{1}{2}} - (m_p\log\log m_p)^{\frac{1}{2}})}{m_{n_i}} \cdot\left(1 + k\frac{c_{39}}{\log\log m_{n_i}}\right)\right]$$ (5.13)

$$< \exp\left[\frac{2\epsilon\varphi(m_{n_i})}{m_{n_i}}((m_{n_i}\log\log m_{n_i})^{\frac{1}{2}} - (m_p\log\log m_p)^{\frac{1}{2}}) - \frac{kc_{40}(\varphi(m_{n_i}))^2}{m_{n_i}\log\log m_{n_i}}\right]$$

$$< \exp\left[\frac{2\epsilon\varphi(m_{n_i})}{m_{n_i}}\cdot k\cdot c_7\left(\frac{m_{n_i}}{\log\log m_{n_i}}\right)^{\frac{1}{2}} - k\frac{c_{40}(\varphi(m_{n_i}))^2}{m_{n_i}\log\log m_{n_i}}\right]$$

$$< \exp\left[(2\epsilon\cdot c_7 c_{32} - c_{40}\cdot c_{32}^2)\cdot k\right].$$

Hence, if we take ϵ sufficiently small, then

$$\eta < e^{-c_{42}k} \tag{5.14}$$

with a positive constant c_{42}. Hence

$$\begin{aligned} \sum_{m_p \in \Pi_i} Pr(A_{m_p}(\varphi(m_p))) &< c_{37} Pr(A_{m_{n_i}}(\varphi(m_{n_i}))) \cdot \sum_{k=1}^{\infty} e^{-2c_{42}k} \\ &= \frac{c_{37}}{1 - e^{-2c_{42}}} Pr(A_{m_{n_i}}(\varphi(m_{n_i}))) \end{aligned} \tag{5.15}$$

which proves (5.9). This completes the proof of Theorem 5.

Before concluding this chapter, let us add some more results without proof.

1). *If $\varphi(n)$ is monotone increasing and belongs to the lower class, then $\varphi(n) + c\,(n/\log\log n)^{\frac{1}{2}}$ belongs to the lower class for all c.*

This result is the best possible. For, if $\psi(n) \to \infty$, then we can find a monotone function $\varphi(n)$ belonging to the lower class such that $\varphi(n) + \psi(n)(n/\log\log n)^{\frac{1}{2}}$ belongs to the upper class.

2). *If $\varphi(n)$ is monotone increasing and belongs to the lower class, then $\varphi(n) + c(n/\varphi(n))$ belongs to the lower class for all c.* Since we can always assume that $\varphi(n) < (n \log\log n)^{\frac{1}{2}}$, 2) is slightly stronger than 1).

3). *Let $\varphi(n)$ be monotone increasing, and suppose that it belongs to the upper class. Then for almost all t, there exist only finitely many n such that for some $m < n$, $|f_n(t) - f_m(t)| > \varphi(n)$.*

4). *For almost all t, we have*

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} f_k(t)}{\frac{1}{2} n^{\frac{3}{2}} \left(\frac{\log\log n}{2}\right)^{\frac{1}{2}}} = 1. \tag{5.16}$$

Professor J. L. Doob suggested that if $n_1 < n_2 < \cdots$ is a sequence of integers with $n_{i+1}/n_i > c > 1$, then for almost all t,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} \frac{\sum_{k=1}^{n_i} f_k(t)}{n_i^{\frac{3}{2}}} = 0. \tag{5.17}$$

Indeed, it is not difficult to show that (5.17) holds. In fact, the condition $n_{i+1}/n_i > c > 1$ can be weakened, but it is necessary that n_i tends to infinity with a certain speed (quicker than i).

5). *There exists a continuous strictly decreasing function $\psi(x)$ defined for $0 \leqq x \leqq 1$, with $\psi(0) = 1$, $\psi(1) = 0$, such that, for almost all t, the upper density of the set of n's for which*

$$f_n(t) \geqq x\left(\frac{n \log\log n}{2}\right)^{\frac{1}{2}} \tag{5.18}$$

is exactly $\psi(x)$.[10]

[10] This problem was suggested by W. Ambrose.

6

In this final chapter, we shall construct an increasing function $\varphi(n)$ such that

$$\sum_{n=1}^{\infty} Pr(A_{m_n}(\varphi(m_n))) = \infty, \tag{6.1}$$

and nevertheless $\varphi(n)$ belongs to the upper class. This shows that the *converse of Theorem* 1 *is not true.*

We put

$$p_k = 2^{2^{2^k}}, \qquad k = 1, 2, \cdots, \tag{6.2}$$

and define $\varphi(n)$ as follows:

$$\varphi(n) = \log k \cdot \sqrt{p_k}, \qquad \text{for } p_{k-1} < n \leqq p_k. \tag{6.3}$$

It follows from (0.11) that the number of m_n's satisfying $\frac{1}{2}p_k < m_n \leqq p_k$ is $\geqq c_{43} \log\log p_k$ and hence $\geqq c_{44}2^k$. Consequently, from (0.7) we have

$$\sum_{\frac{1}{2}p_k < m_n \leqq p_k} Pr(A_{m_n}(\varphi(m_n))) > \frac{c_1}{\log k} e^{-4(\log k)^2} \cdot c_{44} 2^k \geqq c_{45} > 0. \tag{6.4}$$

Since this is true for each k, (6.1) is proved.

Denote now

$$M_k = E[t: \max_{1 \leqq n \leqq p_k} f_n(t) > \log k \cdot \sqrt{p_k}]. \tag{6.5}$$

In order to show that $\varphi(n)$ belongs to the upper class, it is clearly sufficient to prove that

$$\sum_{k=1}^{\infty} Pr(M_k) < \infty. \tag{6.6}$$

It is easy to see that[11]

$$Pr(M_k) \leqq 2Pr(E[t: f_{p_k}(t) > \log k\sqrt{p_k}]. \tag{6.7}$$

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[11] In general, we have

$$Pr(E[t: \max_{1 \leqq n \leqq p} f_n(t) > x]) \leqq 2\, Pr(E[t: f_p(t) > x]).$$

Indeed, we have

$$\begin{aligned} Pr(E[t: \max_{1 \leqq n \leqq p} f_n(t) > x]) &= Pr(E[t: f_p(t) > x]) \\ &\quad + \sum_{n=1}^{p-1} Pr(E[t: f_1(t) \leqq x, \cdots, f_{n-1}(t) \leqq x, f_n(t) > x, f_p(t) \leqq x]) \\ &= Pr(E[t: f_p(t) > x]) \\ &\quad + \sum_{n=1}^{p-1} Pr(E[t: f_1(t) \leqq x, \cdots, f_{n-1}(t) \leqq x, f_n(t) > x, f_p(t) \geqq 2f_n(t) - x]) \\ &\leqq Pr(E[t: f_p(t) > x]) \\ &\quad + \sum_{n=1}^{p-1} Pr(E[t: f_1(t) \leqq x, \cdots, f_{n-1}(t) \leqq x, f_n(t) > x, f_p(t) > x]) \\ &= 2\, Pr(E[t: f_p(t) > x]). \end{aligned}$$

Thus from (0.7) we have

$$\sum_{k=1}^{\infty} Pr(M_k) \leqq 2 \cdot c_2 \cdot \sum_{k=1}^{\infty} \frac{1}{\log k} e^{-2(\log k)^2} < \infty, \tag{6.8}$$

which proves (6.6).

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