A NOTE ON FAREY SERIES

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[This note was received in the form of a letter addressed, through the *Quarterly Journal*, to the late Dr. Mayer. It has been put into its present form by the kindness of Professor Davenport.]

In extension of Dr. Mayer's theorems on the ordering of Farey series,* the following theorem can be proved:

THEOREM: *There exists an absolute constant* c *such that, if* $n > ck$, *and if*

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots$$

are the Farey fractions of order n, *then* $\frac{a_x}{b_x}$ *and* $\frac{a_{x+k}}{b_{x+k}}$ *are similarly ordered.*

Proof. As in Dr. Mayer's paper, we observe first that, if a_x/b_x and a_y/b_y (the latter being the greater) are not similarly ordered, then $a_y \geqslant a_x+1$, $b_y \leqslant b_x-1$, and therefore it suffices to prove that there are at least k Farey fractions between

$$\frac{a_x}{b_x} \quad \text{and} \quad \frac{a_x+1}{b_x-1}.$$

Case I. Suppose that $a_x/b_x < \frac{1}{6}$. In this case, we note that

$$\frac{a_x+1}{b_x-1}-\frac{a_x}{b_x} = \frac{a_x+b_x}{(b_x-1)b_x} > \frac{1}{b_x} \geqslant \frac{1}{n},$$

and we shall prove that there are at least k Farey fractions in the interval $\left(\frac{a_x}{b_x}, \frac{a_x}{b_x}+\frac{1}{n}\right)$. Let

$$\frac{a_x}{b_x}, \quad \frac{a_{x+1}}{b_{x+1}}, \quad \dots, \quad \frac{a_y}{b_y}$$

be the Farey fractions in this interval. Since the difference between two consecutive fractions is less than $\frac{1}{n}$, we have

$$\frac{1}{n} < \frac{a_{y+1}}{b_{y+1}}-\frac{a_x}{b_x} < \frac{2}{n}.$$

* A. E. Mayer, *Quart. J. of Math.* (Oxford), 13 (1942), 186–7, Theorems 1, 2.

If $n > 60$, it follows that $a_{y+1}/b_{y+1} < \frac{1}{6}+\frac{1}{30} = \frac{1}{5}$, so that $b_j \geqslant 6$ for $x \leqslant j \leqslant y+1$.

Now

$$\frac{a_{y+1}}{b_{y+1}}-\frac{a_x}{b_x} = \sum_{j=x}^{y}\left(\frac{a_{j+1}}{b_{j+1}}-\frac{a_j}{b_j}\right) = \sum_{j=x}^{y}\frac{1}{b_j b_{j+1}} < \sum_{j=x}^{y}\frac{2}{n\min(b_j, b_{j+1})},$$

since $b_j+b_{j+1} > n$. Thus

$$\sum \equiv \sum_{j=x}^{y}\frac{1}{\min(b_j, b_{j+1})} > \frac{1}{2}. \tag{1}$$

We write

$$\sum = \sum\nolimits_1 + \sum\nolimits_2, \tag{2}$$

where $\sum_1$ is extended over those values of j for which

$$\min(b_j, b_{j+1}) < 8k,$$

and $\sum_2$ over the others. Plainly

$$\sum\nolimits_2 < \frac{y-x+1}{8k}.$$

If there is only one value of j (with $x \leqslant j \leqslant y+1$) for which $b_j < 8k$, then there are at most two terms in $\sum_1$, and, since $b_j \geqslant 6$, we have $\sum_1 \leqslant \frac{1}{3}$. If there are several such values of j, let them be $r_1, r_2,\ldots, r_t$. We have

$$\frac{2}{n} > \frac{a_{r_t}}{b_{r_t}}-\frac{a_{r_1}}{b_{r_1}} = \sum_{l=1}^{t-1}\left(\frac{a_{r_{l+1}}}{b_{r_{l+1}}}-\frac{a_{r_l}}{b_{r_l}}\right) \geqslant \sum_{l=1}^{t-1}\frac{1}{b_{r_l}b_{r_{l+1}}} > \frac{1}{8k}\sum_{l=1}^{t-1}\frac{1}{b_{r_l}}.$$

Hence

$$\sum_{l=1}^{t-1}\frac{1}{b_{r_l}} < \frac{16k}{n},$$

and the same holds for the sum from 2 to t. Thus

$$\sum_{l=1}^{t}\frac{1}{b_{r_l}} < \frac{32k}{n},$$

and, since each b_{r_l} occurs in at most two terms in $\sum_1$, it follows that

$$\sum\nolimits_1 < \frac{64k}{n} < \frac{1}{3},$$

provided that $n > 192k$.

From (1) and (2), we have $\sum_2 > \frac{1}{6}$, that is

$$\frac{y-x+1}{8k} > \frac{1}{6}, \qquad y-x+1 > \tfrac{4}{3}k > k+1$$

for $k \geqslant 3$. This proves the result in Case I.

Case II. Suppose now that $a_x/b_x \geqslant \frac{1}{6}$. In this case,

$$\frac{a_x+1}{b_x-1}-\frac{a_x}{b_x}=\frac{a_x+b_x}{(b_x-1)b_x}>\frac{7}{6n}.$$

We shall prove that the interval

$$\left(\frac{a_x}{b_x},\frac{a_x}{b_x}+\frac{7}{6n}\right)$$

contains at least k Farey fractions. For this interval we have, in place of (1),

$$\sum_{j=x}^{y}\frac{1}{\min(b_j,b_{j+1})}>\frac{7}{12}.$$

If $b_j \geqslant 6$ for $x \leqslant j \leqslant y+1$, the proof of case (I) remains valid. Hence we can suppose that one of the b_j does not exceed 5. But, if $b_r \leqslant 5$, then

$$\frac{2}{n}>\left|\frac{a_j}{b_j}-\frac{a_r}{b_r}\right|\geqslant\frac{1}{5b_j}$$

for $j \neq r$, whence $b_j > \frac{1}{10}n > 40k$, provided that $n > 400k$. So every b_j except b_r satisfies $b_j > 40k$.

Since the difference between two consecutive Farey fractions is at most $1/2(n-1)$, we have (omitting in the summations $j = r$ and $j+1 = r$)

$$\sum_{j=x}^{y}{}'\left(\frac{a_{j+1}}{b_{j+1}}-\frac{a_j}{b_j}\right)>\frac{7}{6n}-\frac{2}{2(n-1)}>\frac{1}{10n}.$$

Hence
$$\frac{1}{10n}<\sum_{j=x}^{y}{}'\frac{1}{b_jb_{j+1}}<\frac{2}{n}\sum_{j=x}^{y}{}'\frac{1}{\min(b_j,b_{j+1})},$$

whence
$$\sum_{j=x}^{y}{}'\frac{1}{\min(b_j,b_{j+1})}>\frac{1}{20}.$$

Since $\min(b_j,b_{j+1}) > 40k$ in this sum, we have

$$\frac{y-x+1}{40k}>\frac{1}{20},\qquad y-x+1>2k\geqslant k+1.$$

This completes the proof.

I have not been able to find the best possible value for the constant c in the above result. It is easy to prove the following results, which are closely connected with that proved above:

(i) To every $\epsilon > 0$ there exists a $c = c(\epsilon)$ such that any interval of length $(1+\epsilon)/n$ contains at least cn Farey fractions of order n.

(ii) If $f(n) \to \infty$ as $n \to \infty$, any interval of length $n^{-1}f(n)$ contains

$$\frac{3}{\pi^2} nf(n) + o(nf(n))$$

Farey fractions of order n.

It may be of interest to remark that Lemma 1 of Dr. Mayer's paper can be strengthened as follows: There exists a constant c_1 such that any interval of length $L = k^{c_1}$ contains a set of at least k mutually prime integers. This can be proved by Brun's method. It would be interesting to have a good estimate for the best possible value $L(k)$ of L from below. It follows from a result of Rankin* that

$$L(k) > c_2 \frac{k \log k \log \log \log k}{(\log \log k)^2}.$$

* *J. of London Math. Soc.* 13 (1938), 242.