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CORRECTIONS TO TWO OF MY PAPERS

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In my paper *"On the divergence properties of the Lagrange interpolation polynomials,"* (Annals of Math. Vol. 42, (1941), p. 309–315) I stated that, if $x_0 = \cos \frac{p}{q}\pi$ (p and q odd), and the fundamental points of the interpolation are the roots of the Tchebicheff polynomial $T_n(x)$, then there exists a continuous function $f(x)$ such that $\lim L_n(f(x_0)) = \infty$.

Dr. Schönberg has pointed out that the proof there given is not correct. There is a trivial error in lemma 1; namely, it is possible that $x_i^{(m)} = x_j^{(n)}$. Nevertheless it is possible to save almost everything, practically without modifying the proof. We prove the following slightly weaker.

THEOREM. *There exists a continuous function $f(x)$ such that if $x_0 = \cos \frac{p}{q}\pi$, where p and q are odd, then* $\lim | L_n(f(x_0)) | = \infty$.

Proof. We need

LEMMA 1. *If $x_i^{(m)} \neq x_j^{(n)}$ then $| x_i^{(m)} - x_j^{(n)} | > \frac{1}{m^3}$ for $m \geqq n$.*

Proof. As in the paper.

Everything is now unchanged until the bottom of page 311. We have there

$$f(x) = \sum_{n=n_0}^{\infty} \epsilon_n \frac{f_n(x)}{\sqrt{\log n}}.$$

where $\epsilon_n = \pm 1$ and will be determined later; the definition of $f_n(x)$ is the same as in the paper.

$L_n(\varphi_2(x_0)) = 0$ still holds (p. 313 top). It suffices to show that, for $r > n$, $f_r(x_k^{(n)}) = 0$. And this is true, for otherwise either

$$x_l^{(r)} = x_k^{(n)},$$

which is impossible since $(2l - 1, r) = 1$, or we have

$$x_l^{(r)} \neq x_k^{(n)} \quad \text{and} \quad | x_l^{(r)} - x_k^{(n)} | < \frac{1}{2^{2r}},$$

which does not hold by lemma 1.

Define now $\epsilon_n = \text{signum } L_n(\varphi_1(x_0))$; then clearly

$$| L_n(f(x_0)) | \leqq \left| L_n\left(\frac{\epsilon_n f_n(x_0)}{\sqrt{\log n}}\right) \right|$$

and the rest of the proof is unchanged.

At present I cannot decide whether a continuous function $f(x)$ exists such that $\lim L_n(f(x_0)) = \infty$, or whether a continuous $f(x)$ exists with $\lim L_n(f(x_0)) = a$, where $a \neq f(x_0)$.

Added in proof. By a more careful analysis, I can now show the following theorem: *Let E be any closed set, then there exists a continuous $f(x)$ such that the limit points of L_n $(f(x_0))$ is precisely the set E. The set E can consist of the point $+\infty$ alone.* This of course is a generalization of the result mentioned before.

In my paper *"On some asymptotic formulas in the theory of factorisatio numerorum"* (Annals of Math. Vol. 42, (1941) p. 989–993) the main theorem is stated incorrectly. The correct statement is as follows:

Let $1 < a_1 < a_2 < \cdots$ be a sequence of integers such that for some ρ, $\sum_{i=1}^{\infty} \frac{1}{a_i^\rho} = 1$ and $\sum \frac{\log a_i}{a_i^\rho}$ converges and not all the a_i's are powers of a_1. Denote by $f(n)$ the number of factorisations of n into the a_i's. We consider order in other words a_1a_2 and $a_2 \cdot a_1$ are different factorisations. Also $f(1) = 1$. Denote $F(n) = \sum_{k=1}^{n} f(k)$. Then we have

$$F(n) = cn^\rho(1 + 0(1)).$$

The proof remains entirely unchanged: in fact this theorem is the one really proved in the paper.

It might be of some interest to investigate what happens if the conditions of our theorem are not satisfied. There are three cases: I. $\sum_{i=1}^{\infty} \frac{1}{a_i^k}$ diverges for all k. Then it is easy to see that $\lim \frac{F(n)}{n^k} = \infty$ for all k.

II. For all values of k for which $\sum_{i=1}^{\infty} \frac{1}{a_i^k}$ converges $\sum_{i=1}^{\infty} \frac{1}{a_i^k} < 1$. Clearly, there exists a ρ such that for every ϵ, $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho+\epsilon}}$ converges but $\sum \frac{1}{a_i^{\rho-\epsilon}}$ diverges. We can easily see that $\sum_{i=1}^{\infty} \frac{1}{a_i^\rho}$ converges and is <1. For if $\sum_{i=1}^{\infty} \frac{1}{a_i^\rho}$ diverged we would have, for sufficiently small ϵ, $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho+\epsilon}} > 1$; and since, for large k, $\sum_{i=1}^{\infty} \frac{1}{a_i^k} < 1$, there would exist a k_0 such that $\sum_{i=1}^{\infty} \frac{1}{a_i^{k_0}} = 1$—which contradicts the hypothesis.

Now we show that

$$\lim \frac{F(n)}{n^\rho} = 0 \tag{1}$$

and

$$\lim \frac{F(n)}{n^{\rho-\epsilon}} = \infty. \tag{2}$$

Suppose (1) does not hold. Write $\sum \frac{1}{a_i^\rho} = A < 1$ and $c = \limsup \frac{F(u)}{u^\rho}$.

We have

$$F(u) = \sum_{i=1}^{\infty} F\left(\frac{u}{a_i}\right) + 1,$$

so that

$$\frac{F(u)}{u^\rho} \leqq c \sum_{i=1}^{\infty} \frac{1}{a_i^\rho} + o(1) < Ac + o(1)$$

for sufficiently large u. This is possible only if $c = 0$. (2) can be shown by similar arguments.

III. There exists a ρ such that $\sum_{i=1}^{\infty} \frac{1}{a_i^\rho} = 1$, but $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^\rho}$ diverges. It seems likely that in this case

$$\lim \frac{F(n)}{n^\rho} = 0.$$

But I am only able to prove that

$$\underline{\lim} \frac{F(n)}{n^\rho} = 0. \tag{3}$$

Suppose that (3) is not satisfied. Let the greatest lower bound of $\frac{F(n)}{(n+1)^\rho}$ be c $(c > 0)$. Choose k so large that

$$\sum_{i=1}^{k} \frac{\log a_i}{a_i^\rho} > \frac{2}{\rho c}.$$

Denote by $g(n)$ the number of the factorisations of n as the product of the a_i for $i \leqq k$, and let $G(n) = \sum_{u=1}^{n} g(u)$. Clearly for $m \leqq a_k$, $G(m) = F(m)$. Thus for $m \leqq a_k$

$$\frac{G(m)}{(m+1)^\rho} \geqq c.$$

Next we prove that for all m

$$\frac{G(n)}{(n+1)^{\rho'}} \geqq c \tag{4}$$

where $\sum_{i=1}^{k} \frac{1}{a_i^{\rho'}} = 1$, $(\rho' \leqq \rho)$.

Clearly (4) holds for all $n \leqq a_k$. We prove (4) by induction. Assume it for n: we shall prove it for $n+1$. We have

$$G(n+1) = \sum_{i=1}^{k} G\left[\frac{n+1}{a_i}\right] + 1 \geqq c \sum_{i=1}^{k} \left(\left[\frac{n+1}{a_i}\right] + 1\right)^{\rho'} + 1.$$

Therefore

$$\frac{G(n+1)}{(n+2)^{\rho'}} \geqq c \sum_{i=1}^{k} \frac{1}{a_i^{\rho'}} = c,$$

which proves (4). Thus $\lim \frac{G(n)}{n^{\rho'}} \geqq c$. (We know from my paper that the limit exists.)

Put $h(s) = \sum_{i=1}^{k} \frac{1}{a_i^s}$; then clearly

$$\frac{1}{2 - h(s)} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Therefore a simple calculation shows that if $\lim \frac{G(n)}{n^{\rho'}}$ exists the limit equals

$$\left(\rho' \sum_{i=1}^{k} \frac{\log a_i}{a_i^{\rho'}}\right)^{-1} < \frac{\rho c}{2\rho'} \leqq \frac{c}{2},$$

which proves (3). It is easy to construct sequences a_i, with $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}}, = 1$, $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^{\rho}} = \infty$ and

$$\lim \frac{F(n)}{n^{\rho}} = 0. \tag{5}$$

But I can not prove that (5) holds for all such sequences a_i.

Professor Hille has given the following result. (Acta Arithmetica Vol. 2, p. 140): Let $p_1 < p_2 < \cdots$ be a sequence of primes, and let $a_1 < a_2 < \cdots$ be the integers composed of the p's. Denote by $f(n)$ the number of factorisations of n into the product of the a's, and let $F(n) = \sum_{m=1}^{n} f(m)$. If $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}} = 1$ then $\lim \frac{F(n)}{n^{\rho}} = \left(\rho \sum_{i=1}^{\infty} \frac{\log a_i}{a_i}\right)^{-1}$. His proof (which uses the theorem of Wiener Ikehara) seems to apply only if $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^{\rho}} < \infty$. If (5) is always true in case iii, Hille's result would follow even if $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^{\rho}} = \infty$.

Recently I found in the literature a few results, which I proved in my paper "Elementary proof of some asymptotic formulas in the theory of partitions" (Annals of Math. Vol. 43). On p. 447 I prove the following result: Denote by $P_r(n)$ the number of partitions of n into powers of r then $\lim \frac{\log P_r(n)}{(\log n)^2} = \frac{1}{2 \log r}$. This result was proved by Mahler (London Math. Soc. Journal, Vol. 15, p. 123.) Mahlers proof is completely different from mine. He also obtains

$$c_1 \frac{r^{-\frac{1}{2}n(n-1)}(zr)^n}{n!} < P_r(rz) < c_2 \frac{r^{-\frac{1}{2}n(n-1)}(rz)^n}{n!}, \quad \text{where} \quad r^{n-1} n \leqq z < r^n(n+1).$$

On p. 448 I prove the following two results:

I. Let $a_1 < a_2 < \cdots$ be a sequence of integers of positive density α, the a's have common factor 1. Denote by $p(n)$ the number of partition of n into the a's. Then $\log p(n) \sim \pi\sqrt{\frac{2}{3}\alpha n}$.

II. Let $a_1 < a_2 < \cdots$ be a sequence of integers such that every large integer is the sum of different a's. Denote by $P(n)$ the number of partitions of n into different a's. Then $\log P(n) \sim \pi\sqrt{\frac{1}{3}\alpha n}$. Similar results were proved by K. Knopp (Schriften der Königsberger Gel. Ges. Math. und Nat. Klasse, 2 Jahr. Heft. 3 1925). His proofs are quite different from mine and are more complicated.

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ADDENDUM

BY EINAR HILLE

The objection raised by Dr. Erdös to formula (4.3) of my paper "A Problem in 'Factorisatio Numerorum' " is well founded. However, the results on pp. 139–140 are entirely correct if the basis P contains only a finite number of primes p_{i_ν}. When the basis is infinite it is necessary to assume that $\lim_{s\to\sigma_0} \zeta(s; P) > 2$ where $\zeta(s; P) = \prod_{\nu=1}^{\infty} [1 - p_{i_\nu}^{-s}]^{-1}$ and $\sigma_0 = \sigma_0(P)$ is the abscissa of convergence of the infinite product. This assumption implies that the equation $\zeta(s; P) = 2$ has a root $\rho(P)$ which exceeds σ_0. If this assumption is satisfied, formulas (3.8), (3.9), (4.1), (4.3), and (5.1) remain valid. If, instead, $\zeta(\sigma_0; P) = 2$ so that $\rho(P) = \sigma_0$, the Ikehara-Wiener theorem does not apply; the analysis breaks down completely and cannot be saved by assuming that $\zeta'(\sigma_0; P)$ is finite. Though formula (4.3) still makes sense, it is at best unproved. If $\zeta(\sigma_0; P) < 2$, the formula becomes meaningless and it is not enough to replace $\rho(P)$ by σ_0 since Erdös has proved [formula (1) above] that in this case $F(n) = o(n^{\sigma_0})$ while it is not necessarily true that $\zeta'(\sigma_0 ; P)$ is infinite.

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