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A CONJECTURE IN ELEMENTARY NUMBER THEORY

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A well known conjecture of Catalan states that *if $f(n)$ is the sum of all divisors of n except n, then the sequence of iterates of $f(n)$ is either eventually periodic or ends at* 1. It not only seems impossible to prove this, but it is also very difficult to verify.[1]

Another conjecture of Poulet,[2] which appears equally difficult to prove, has the doubtful merit that it is easy to verify. Let $\sigma(n)$ be the sum of all divisors of n, and let $\phi(n)$ be Euler's function. Then *for any integer n the sequence*

$$f_0(n) = n, \qquad f_{2k+1}(n) = \sigma(f_{2k}(n)), \qquad f_{2k}(n) = \phi(f_{2k-1}(n))$$

is eventually periodic.

We have verified this conjecture to $n=10000$ (extending Poulet's verification) by using Glaisher's tables.[3] The checking was facilitated by the following observation: if the conjecture is to be checked for all $n<x$, it is enough to find a member of the sequence other than the first which is less than x.

The longest cycle found was in the sequence $f_k(9216)$. It starts with $f_6(9216)$, and is: 34560, 122640, 27648, 81800, 30976, 67963, 54432, 183456, 48384, 163520, 55296, 163800, 34560. However our method of checking does not show that this is the largest cycle up to 10000, and in fact Poulet found that $f_k(1800)$ has the same length 12.

As a rule $\phi(\sigma(n))$ is less than n. In fact, it can be shown that *for every $\epsilon>0$, $\phi(\sigma(n))<\epsilon n$, except for a set of density* 0. The proof follows from the following two observations:

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[1] L. E. Dickson, *Theorems and tables on the sum of the divisors of a number*, Quart. J. Math. vol. 44 (1913) pp. 264–296, and P. Poulet, *La Chasse aux Nombres*, vol. 1, pp. 68–72, and vol. 2, p. 188.

[2] P. Poulet, *Nouvelles suites arithmétiques*, Sphinx vol. 2 (1932) pp. 53–54.

[3] J. W. L. Glaisher, *Number-divisor tables*, British Association for the Advancement of Science, Mathematical Tables, vol. 8.

(1) *For a given prime p, the set of all n such that $\sigma(n) \equiv 0 \pmod{p}$ is of density* 1.

The set of all integers not divisible by any prime q of the form $px-1$ is of density zero, since $\sum_q 1/q$ diverges. Hence the set of all integers divisible by a prime $q>N$ of this type is of density 1. But the set of all integers divisible by q^2, $q>N$, is of density less than $\sum_{q>N} 1/q^2 = o(1)$. Therefore, if x is large, the number of n less than x such that $\sigma(n) \equiv 0 \pmod{p}$ exceeds $(1-\epsilon)x$.

(2) *Except for ϵx integers n less than x, $\sigma(n) < c(\epsilon)n$.*

This follows from the fact that $\sum_{n<x} \sigma(n) \sim \pi^2 n^2/12$.

Choose p so that $\prod_{q \leq p} (1-1/q) < \delta/c(\epsilon)$. Then, if x is sufficiently large, all except $\eta x + \epsilon x$ integers n less than x have $\sigma(n) < c(\epsilon)n$, $\sigma(n) \equiv 0 \pmod{q}$ for all $q \leq p$. But, with these exceptions, $\phi[\sigma(n)] < \delta n$, which completes the proof, since η and ϵ are arbitrary.

In much the same way it can be shown that *for every $c>0$, $\sigma[\phi(n)] > cn$ except for a set of density zero.*

Actually, much more can be shown. *Except for a set of density zero, $e^\gamma \phi[\sigma(n)]$ log log log $n \sim \sigma(n)$, and $e^{-\gamma}\sigma[\phi(n)]/$log log log $n \sim \phi(n)$,* where γ is Euler's constant. The proof is suppressed, but it might be noted that the reason for this result is that, for almost all n, $\phi(n)$ and $\sigma(n)$ are both divisible by all primes less than $(\log \log n)^{1-\epsilon}$, and by relatively few primes greater than $(\log \log n)^{1+\epsilon}$.

There exist numbers for which $\phi(\sigma(n)) = n$. Up to 2500 these numbers are 1, 2, 8, 12, 128, 240, 720; while two further solutions are 2^{15} and 2^{31}. Poulet gives many others; we do not know whether there are infinitely many solutions.

We state two further conjectures:

(a) Form the sequence $\sigma(n)$, $\sigma(\sigma(n))$, $\phi(\sigma(\sigma(n)))$, $\sigma(\phi(\sigma(\sigma(n))))$ in which the functions are successively applied in the order $\sigma, \sigma, \phi, \sigma, \sigma, \phi, \sigma, \sigma, \phi, \cdots$. This sequence seems to tend to infinity if n is large enough.

(b) On the other hand, the sequence $\phi(n), \phi(\phi(n)), \sigma(\phi(\phi(n))), \cdots$, in which the order is $\phi, \phi, \sigma, \phi, \phi, \sigma, \phi, \phi, \sigma, \cdots$, seems to converge to 1, for all n.

Obviously many more such conjectures can be formulated.

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