

ON HIGHLY COMPOSITE NUMBERS

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Ramanujan† defines a number n to be highly composite if $d(m) < d(n)$ for all $m < n$. He proves that the limit of the quotient of two successive highly composite numbers is unity, and that the number of them not exceeding x is greater than‡

$$c \log x (\log \log x)^{\frac{1}{2}} (\log \log \log x)^{-\frac{3}{2}}.$$

In the present note I shall prove that the number of highly composite numbers not exceeding x is greater than

$$(\log x)^{1+c}$$

for a certain c . In fact I shall prove that if n is highly composite, then the next highly composite number is less than $n + n(\log n)^{-c}$; and the result just stated follows immediately from this. At present I cannot decide whether the number of highly composite numbers not exceeding x is greater than $(\log x)^k$ for every k .

The principal tool in the proof will be Ingham's improvement§ of Hoheisel's theorem. This asserts that if x is sufficiently large, then the number of primes in the interval $(x, x+x^{\frac{1}{2}})$ is asymptotic to $cx^{\frac{1}{2}}(\log x)^{-1}$.

First we state three lemmas, which will be proved at the end of the paper. They are contained substantially in the paper of Ramanujan, but we prove them here for completeness. Let $n = 2^{\kappa_2} 3^{\kappa_3} \dots p^{\kappa_p}$ be a sufficiently large highly composite number. Plainly

$$\kappa_2 \geq \kappa_3 \geq \dots \geq \kappa_p,$$

since otherwise, by rearranging the exponents, we could get a smaller number with the same $d(n)$.

* Received 14 February, 1944; read 15 June, 1944.

† Srinivasa Ramanujan, *Proc. London Math. Soc.* (2), 14 (1915), 347-409; or *Collected Papers* (Cambridge, 1927), 78-128.

‡ Throughout this paper, c will denote a positive absolute constant, not always the same.

§ A. E. Ingham, *Quart. J. of Math.* (Oxford series), 8 (1937), 255-266. In fact, Hoheisel's original theorem (with an unspecified constant less than 1 in the place of $\frac{1}{2}$) would suffice for our main result.

LEMMA 1. $c_1 \log n < p < c_2 \log n$.

LEMMA 2. If q is a prime satisfying $\frac{1}{2}p < q \leq p$, then $\kappa_q = 1$.

LEMMA 3. If q is a prime satisfying $2\sqrt{p} < q < 4\sqrt{p}$, then $\kappa_q = 2$.

THEOREM. There is a positive constant c such that, if n is highly composite, then there is a highly composite number n_1 satisfying

$$n < n_1 < n + n(\log n)^{-c}.$$

Proof. Let q be the largest prime with $\kappa_q \geq 2$. By Lemmas 2 and 3,

$$4\sqrt{p} < q < \frac{1}{2}p.$$

Put $q = p^\delta$, so that $\frac{1}{2} < \delta < 1$. By a well-known theorem* there exist positive integers s and t such that

$$(1) \quad s < p^{\frac{1}{3}}, \quad |s\delta - t| < p^{-\frac{1}{3}}.$$

Let the consecutive primes immediately greater than and less than p and q be denoted as follows:

$$\dots < p_2 < p_1 < p < P_1 < P_2 < \dots, \quad \dots < q_2 < q_1 < q < Q_1 < Q_2 < \dots$$

Put

$$u = q_1 q_2 \dots q_s, \quad U = Q_1 Q_2 \dots Q_s, \quad v = p_1 p_2 \dots p_t, \quad V = P_1 P_2 \dots P_t.$$

By the lemmas and the definition of q ,

$$(2) \quad \kappa_{q_i} \geq 2, \quad \kappa_{Q_i} = 1 \quad (i = 1, 2, \dots, s); \quad \kappa_{p_i} = 1 \quad (i = 1, \dots, t).$$

Put $n_1 = nV/u$, $n_2 = nU/v$. Clearly, by (2),

$$d(n_1) = d(n) 2^t \prod_{i=1}^s \frac{\kappa_{q_i}}{\kappa_{q_i} + 1} \geq d(n) 2^{t(\frac{2}{3})^s},$$

$$d(n_2) = d(n) 2^{-t(\frac{3}{2})^s}.$$

Hence either $d(n_1)$ or $d(n_2)$ is greater than or equal to $d(n)$. The argument is the same in either case; suppose $d(n_1) \geq d(n)$. Since n is highly composite, $n_1 > n$, and we have only to show that

$$(3) \quad n_1 < n + n(\log n)^{-c}.$$

* The result is due to Dirichlet; see, e.g., Hardy and Wright, *Introduction to the theory of numbers* (Oxford, 1938), 155–156.

Since $s < p^{3/2} < q^{3/2}$, and $t < p^{3/2} + 1$, it follows at once from Ingham's theorem that

$$q_s > q - q^{3/2}, \quad Q_s < q + q^{3/2}, \quad p_t > p - p^{3/2}, \quad P_t < p + p^{3/2}.$$

Thus

$$\begin{aligned} n_1 &< n(p + p^{3/2})^t (q - q^{3/2})^{-s} \\ &= np^t q^{-s} \exp \{t \log (1 + p^{-3/2})\} \exp \{-s \log (1 - q^{-3/2})\} \\ &< np^{t-s} \exp (tp^{-3/2} + 2sq^{-3/2}) < n \exp \{p^{-3/2} \log p + 2p^{-3/2} + 2q^{-3/2}\} \\ &< n(1 + p^{-\alpha}) \end{aligned}$$

for any absolute constant $\alpha < \frac{3}{2}$. Since $p > c \log n$, this proves (3).

We conclude by proving the lemmas.

Proof of Lemma 1. We first show that $q^{\kappa_q} < p^6$ for any $q < p$. Suppose this false for a particular q . Determine a so that $p^2 < q^a < p^3$, which implies $2a < \kappa_q$. Put $n_1 = nP_1/q^a$. Then

$$d(n_1) = d(n) 2^{\left(\frac{\kappa_{q_1} + 1 - a}{\kappa_{q_1} + 1}\right)} > d(n).$$

But on the other hand, by Bertrand's postulate*, $P_1 < p^2 < q^a$, and hence $n_1 < n$. This is a contradiction.

Thus $q^{\kappa_q} < p^6$, and so

$$n < p^{6\pi(p)},$$

and by the prime number theorem this implies $p > c \log n$.

The other result is now immediate from the prime-number theorem, since

$$n > 2 \cdot 3 \dots p = \exp (\log 2 + \dots + \log p) > e^{p(1-\epsilon)}$$

for any fixed $\epsilon > 0$.

Proof of Lemma 2. Suppose that $\frac{1}{2}p < q \leq p$, and $\kappa_q \geq 2$. Put

$$n_1 = \frac{nP_1P_2}{q_1q_2q_3}.$$

Then

$$d(n_1) = d(n) \cdot 4 \prod_{i=1}^3 \frac{\kappa_{q_i}}{\kappa_{q_i} + 1} \geq d(n) \cdot 4 \cdot \left(\frac{2}{3}\right)^3 > d(n).$$

But by the prime-number theorem, $n_1 < n$, and so we have a contradiction.

* For a proof see Landau, *Handbuch*, 89, or Ramanujan, *Collected Papers*, 208.

Proof of Lemma 3. Suppose first that $q > 2\sqrt{p}$ and $\kappa_q \geq 3$. Put $n_1 = nP_1P_2/q_1q_2q_3q_4$. Then $n_1 < n$ and

$$d(n_1) = d(n) \cdot 4 \prod_{i=1}^4 \frac{\kappa_{q_i}}{\kappa_{q_i} + 1} \geq d(n) \cdot 4 \cdot \left(\frac{3}{4}\right)^4 > d(n),$$

a contradiction. Hence $\kappa_q \leq 2$ for $q > 2\sqrt{p}$

Suppose, secondly, that q is the least prime for which $\kappa_q = 1$, and that $q < 4\sqrt{p}$. Put $n_1 = nQ_1Q_2 \dots Q_7/p_1p_2p_3p_4$. Then $n_1 < n$ and

$$d(n_1) = d(n) \left(\frac{3}{2}\right)^7 \frac{1}{2^4} > d(n),$$

a contradiction. This completes the proof of Lemma 3.

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