SOME PROPERTIES OF PARTIAL SUMS OF THE HARMONIC SERIES

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It has been proved that $\sum_{k=m}^{n} k^{-1}$ cannot be an integer[1] for any pair of positive integers m and n. More generally, $\sum_{k=0}^{n}(m+kd)^{-1}$ cannot be an integer.[2] We prove two theorems of a similar nature.

THEOREM 1. *There is only a finite number of integers n for which one or more of the elementary symmetric functions of* $1, 1/2, 1/3, \cdots, 1/n$ *is an integer.*

PROOF. Let $\sum_{k,n}$ denote the kth symmetric function of $1, 1/2, 1/3, \cdots, 1/n$. Since each term of $\sum_{k,n}$ is contained $k!$ times in the expansion of $(1+1/2+ \cdots +1/n)^k$, we have, for $k>3 \log n$ and n sufficiently large,

$$\sum_{k,n} < \frac{(1+1/2+\cdots+1/n)^k}{k!} < \frac{(1+\log n)^k}{k!} < 1,$$

where the second inequality arises from the usual comparison of $\log n$ with the harmonic series, and the third inequality is implied by the hypothesis $k>3 \log n$.

Henceforth we take $k<3 \log n$. By a theorem of A. E. Ingham[3] there is a prime between x and $x+x^{5/8}$. This implies that there is a prime p between $1+n/(k+1)$ and n/k for $k<3 \log n$ and n sufficiently large. Hence $\sum_{k,n}$ contains the term

$$\frac{1}{p} \cdot \frac{1}{2p} \cdots \frac{1}{kp} = \frac{1}{k!p^k}.$$

Now $(k!, p)=1$ since $k<n/(k+1)$, and hence no other term in $\sum_{k,n}$ has a denominator divisible by p^k. So if $\sum_{k,n}=a/b$, we know that $p^k | b$ and $p \nmid a$, which proves the theorem.

By a similar but more complicated argument we can prove the same

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[1] Cf. Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925, chap. 8, p. 159, problem 250.

[2] Cf. T. Nagell, *Eine Eigenschaft gewissen Summen*, Skrifter Oslo, no. 13 (1923) pp. 10–15.

[3] *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) p. 256. This result is actually stronger than necessary for our use here. The classical estimates will suffice.

result for the elementary symmetric functions of $1/m$, $1/(m+1)$, $\cdots$, $1/n$, and of $1/m$, $1/(m+d)$, $1/(m+2d)$, $\cdots$, $1/(m+nd)$.

It should be noted that $\sum_{2,3}$ is an integer; we know of no other integral case. Theorem 1 can be proved without the use of the prime number theorem, and this proof could be used to determine the bound on n, above which the result of the theorem holds. For smaller values of n, $\sum_{k,n}$ could be checked, but the proof is complicated and the limits would be large.

THEOREM 2. *No two partial sums of the harmonic series can be equal; that is, it is not possible that*

$$1/m + 1/(m+1) + \cdots + 1/n = 1/x + 1/(x+1) + \cdots + 1/y. \tag{1}$$

PROOF. We assume that $n<x$. Clearly if (1) has a solution, then any prime divisor of one of the denominators must divide another. Hence by Bertrand's postulate we can be certain that $y<2x-1$, since otherwise a prime $p>n$ would be one of the denominators on the right side of (1).

LEMMA. *Any solution of* (1) *must satisfy* $y<x+x^{1/2}-1$.

To prove this we use a theorem of Sylvester and Schur[4] which states that if $n>k$, then in the set n, $n+1$, $\cdots$, $n+k-1$ there is an integer containing a prime divisor greater than k. In our case $x>y-x+1$, so that there is a prime $p>y-x+1$ which divides one and only one (say ap) of the integers x, $x+1$, $x+2$, $\cdots$, y. Also p must divide one (say bp) of the set m, $m+1$, $m+2$, $\cdots$, n, and certainly not more than one, since $n-m<y-x$. Then $1/ap$ and $1/bp$ are the only terms in equation (1) whose denominators are divisible by p, and since

$$1/bp - 1/ap = (a-b)/abp,$$

we conclude that p must divide $a-b$, whence $a-b \geqq p$ and $a \geqq p+1$. This implies that

$$y \geqq ap \geqq p^2 + p > (y-x+1)^2 + y - x + 1$$

or

$$x - 1 > (y-x+1)^2,$$

which proves the lemma.

Next we obtain estimates for the expressions in (1). First we note that

[4] Cf. Paul Erdös, J. London Math. Soc. vol. 9 (1934) p. 282.

$$\log \frac{2k+1}{2k-1} = \log\left(1+\frac{1}{2k}\right) - \log\left(1-\frac{1}{2k}\right)$$
$$= \frac{1}{k} + \sum_{j=1}^{\infty} \frac{2}{(2j+1)(2k)^{2j+1}}.$$

Solving for $1/k$, and summing the result for $k=m, m+1, \cdots, n$, we obtain

$$(2)\qquad \frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{n} = \log\frac{2n+1}{2m-1} - \sum_{k=m}^{n}\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}},$$

and similarly

$$(3)\qquad \frac{1}{x}+\frac{1}{x+1}+\cdots+\frac{1}{y} = \log\frac{2y+1}{2x-1} - \sum_{k=x}^{y}\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}}.$$

Now (1) and our assumption that $n<x$ imply that for any $j\geqq 1$,

$$\sum_{k} \frac{2}{(2j+1)(2k)^{2j+1}}$$

is greater when summed over $k=m, m+1, \cdots, n$ than over $k=x, x+1, \cdots, y$ and so, comparing the right sides of (2) and (3), we see that

$$(2n+1)/(2m-1) > (2y+1)/(2x-1).$$

Thus, ignoring the sum on the right side of (3), we may write

$$(4)\qquad \log\frac{(2n+1)(2x-1)}{(2m-1)(2y+1)} < \sum_{k=m}^{n}\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}}.$$

The infinite sum on the right can be replaced by $4/3$ times the first term, since each term is more than 4 times the next. The numerator of the fraction on the left exceeds the denominator by at least 2, since both are odd, and hence the left side exceeds

$$\log\left(1+\frac{2}{(2m-1)(2y+1)}\right) > \frac{1}{(2m-1)(2y+1)}.$$

Thus we have

$$(5) \qquad \frac{1}{(2m-1)(2y+1)} < \sum_{k=m}^{n} \frac{2\cdot 4/3}{3(2k)^3} < \frac{1}{9m^2} \sum_{k=m}^{n} \frac{1}{k} = \frac{1}{9m^2} \sum_{k=x}^{y} \frac{1}{k}\cdot$$

But the last sum has fewer than $x^{1/2}$ terms (by the lemma) and each term is not greater than $1/x$. And since $(2m-1)(2y+1) < 4my$, inequality (5) implies that

$$\frac{1}{4my} < \frac{1}{9m^2}\cdot\frac{x^{1/2}}{x}$$

or

$$(6) \qquad 9mx^{1/2} < 4y.$$

But also $1/m \leqq 1/m + \cdots + 1/n < x^{1/2}\cdot(1/x) = 1/x^{1/2}$, so that $x^{1/2} < m$, which together with (6) implies that $9x < 4y$, which contradicts the lemma. This completes the proof of Theorem 2.

In conclusion, we observe that $1/2+1/3+1/4 \equiv 1/12 \pmod 1$. Whether the sums in equation (1) are congruent (mod 1) for infinitely many values m, n, x, y is an unsolved problem.

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