ON ARITHMETICAL PROPERTIES OF LAMBERT SERIES

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Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \text{ and } g(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sin \frac{n\pi}{2}.$$

Chowla* has proved that if t is an integer $\geqslant 5$, then $g(1/t)$ is irrational. He also conjectures that for rational $|x| < 1$ both $f(x)$ and $g(x)$ are irrational.

In the present note we prove the following

Theorem. *Let $|t| > 1$ be any integer. Then both $f(1/t)$ and $g(1/t)$ are irrational.*

We only give the details for $f(1/t)$; the proof for $g(1/t)$ follows by the method of this note and that of Chowla.

Let us first assume that t is positive and that n is large. Put $k = [(\log n)^{1/10}]$ and let $p_1, p_2, \ldots,$ be the sequence of consecutive primes greater than $(\log n)^2$. Put

$$A = \left\{ \prod_{1 \leqslant i \leqslant \frac{k(k+1)}{2}} p_i \right\}^t.$$

From elementary results about the distribution of primes it follows that $p_i < 2\ (\log n)^2$ for $i \leqslant \dfrac{k(k+1)}{2}$. Thus by a simple computation we obtain

$$A < \left\{ 2\ (\log n)\ \right\}^{tk^2} < e^{(\log n)^{1/4}}. \tag{1}$$

Consider now the following congruences:

$$x \equiv p_1^{\,t-1} \pmod{p_1^t}$$
$$x+1 \equiv (p_2\, p_3)^{t-1} \left\{ \bmod\ (p_2\, p_3)^t \right\}$$
$$\cdots$$
$$x+k-1 \equiv (p_u p_{u+1} \ldots p_{u+k-1})^{t-1} \left\{ \bmod\ (p_u \ldots p_{u+k-1})^t \right\}, \tag{2}$$

* *Proc. Nat. Inst. of Sciences of India*, 13 (1947).

where $u = \frac{k(k-1)}{2} + 1$. The integers less than n satisfying the congruences (2) are clearly of the form

$$x + y.A,\ 0 < x < A,\ 0 \leqslant y < [n/A].$$

We evidently have from (2) that for $0 \leqslant j < k$

$$d(x+j+y.A) \equiv 0 \pmod{t^{j+1}},$$

where $d(m)$ denotes, as usual, the number of divisors of m Thus if we rewrite

$$\sum_{r < x+k+yA} \frac{d(r)}{t^r}$$

in the scale of t, then $t^{-x-yA+1}$ will be the lowest power of t which will occur.

Now if we proceed to determine y in such a way that

$$\sum_{r \geqslant x+k+yA} \frac{d(r)}{t^r} < \frac{1}{t^{x+k/2+yA}}, \tag{3}$$

then the representation of $\sum_{r=1}^{\infty} d(r)/t^r$ in the scale of t will contain at least $\frac{1}{2}k$ consecutive zeros. Thus since $k = [(\log n)^{1/10}]$ can be made arbitrarily large, our number is irrational. [It is clear that the representation of $\sum_{r=1}^{\infty} d(r)/t^r$ in the scale of t is not finite, since

$$\sum_{r \geqslant x+k+yA} d(r)/t^r > 0.]$$

To complete our proof we will determine a value $y_0 < [n/A]$ satisfying (3). First of all we show that

$$\sum_{r > x+k+10\log n+yA} \frac{d(r)}{t^r} < \frac{1}{t^{x+k+yA}}. \tag{4}$$

Now (4) follows by a simple computation by remarking that $d(r) < r$ and $k = (\log n)^{1/10}$. Thus it will suffice to find a $y_0 < [n/A]$ for which

$$\sum{}' \frac{d(r)}{t^r} < \tfrac{1}{2} \frac{1}{t^{x+k/2+y_0A}}, \tag{5}$$

where the dash indicates that

$$x+k+y_0A \leqslant r \leqslant x+k+y\,A+10 \log n;$$

clearly if y_0 satisfies (5) it also satisfies (3). Thus r lies in one of the $[10 \log n]$ arithmetic progressions

$$x+k+s+yA,\ y < [n/A],\ 0 \leqslant s < 10 \log n.$$

First we prove that there exists a $y_0 < [n/A]$ so that

$$d(x+k+s+y_0.A) < 2^{k/4}, \text{ for all } 0 \leqslant s < 10 \log n. \tag{6}$$

It is easy to see that

$$(x+k+s, A) = 1 \text{ for all } 0 \leqslant s < 10 \log n.$$

For, if not, then there exists an s so that

$$x+k+s \equiv 0 \pmod{p_j}, \text{ where } j \leqslant \frac{k(k+1)}{2}.$$

But from (2) we have

$$x+i \equiv 0 \pmod{p_j} \text{ for some } i < k.$$

Thus $k+s-i \equiv 0 \pmod{p_j}$, which is impossible since

$$0 < k+s-i < 11 \log n \text{ and } p_j > (\log n)^2.$$

This completes the proof.

Put $x+k+s = \vartheta$. We have from $(\vartheta, A) = 1$,

$$\sum_{y \leqslant [n/A]} d(\vartheta+yA) < 2 \sum_{l=1}^{\sqrt{n}} \left(\frac{n}{Al}+1\right) < c\frac{n \log n}{A},$$

since $A < n^{\varepsilon}$. Thus the number of y's for which

$$d(\vartheta+yA) > 2^{k/4} \text{ is less then } c\frac{n \log n}{A.2^{k/4}},$$

and the number of y's for which for some s

$$d(x+k+s+y.A) > 2^{k/4} \text{ is less than}$$

$$10c\frac{n\,(\log n)^2}{A.2^{k/4}} < \tfrac{1}{2}\frac{n}{A}.$$

Thus there clearly exists a $y_0 < [n/A]$ satisfying (6). Now clearly

$$\sum{}' \frac{d(r)}{t^r} < 2^{k/4} \sum{}' \frac{1}{t^r} < \tfrac{1}{2} \frac{1}{t^{x+k/2+y_0A}},$$

which proves (5) and completes the proof of the theorem for $t > 1$.

If t is negative the proof is similar to the one just given except that we have to make sure that the expansion of $\sum_{r=1}^{\infty} d(r)/t^r$ in the scale of t is not finite. This is certainly the case if we can prove the existence of a $y_0 < [n/A]$ satisfying (6), for which

$$\sum_{r > x+k+30A} d(r)/t^r \neq 0.$$

This can be done by methods similar to those used above. We do not give the details.

The analogous problems about

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{t^n}, \sum_{n=1}^{\infty} \frac{\phi'(n)}{t^n}, \sum_{n=1}^{\infty} \frac{\vartheta(n)}{t^n},$$

where $\phi(n)$ denotes Euler's ϕ-function, $\phi'(n)$ denotes the sum of the divisors of n, and $\vartheta(n)$ denotes the number of prime factors of n, seem to present difficulties.