ON THE ROOTS OF A POLYNOMIAL AND ITS DERIVATIVE

PAUL ERDÖS AND IVAN NIVEN

Let $r_1, r_2, \cdots, r_n$ be the roots of a polynomial $f(z)$ with complex coefficients, and let $R_1, R_2, \cdots, R_{n-1}$ be the roots of its derivative. N. G. de Bruijn[1] has proved that

$$(1) \qquad \frac{1}{n}\sum_{j=1}^{n} \left| I(r_j) \right| \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} \left| I(R_j) \right|,$$

when $f(z)$ has real coefficients; he raises the question whether this holds in general. We prove that this inequality holds when $f(z)$ has complex coefficients; also that (1) is an equality only when the roots of $f(z)$ are not both above and below the real axis. An immediate consequence of this is that if $D_l(z)$ represents the (positive) distance from z to any straight line l in the complex plane, then

$$(2) \qquad \frac{1}{n}\sum_{j=1}^{n} D_l(r_j) \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} D_l(R_j),$$

with the equality holding only when the r_j are not located on both sides of l. Further, if for any point A in the complex plane, $D_A(z)$ represents the distance from z to A, then

$$(3) \qquad \frac{1}{n}\sum_{j=1}^{n} D_A(r_j) \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} D_A(R_j),$$

with the equality holding only when all the r_j lie on a half line emanating from A.

If A is taken as the origin, we have

$$\frac{1}{n}\sum_{j=1}^{n} \left| r_j \right|^m \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} \left| R_j \right|^m$$

with $m=1$. This inequality, with $m=1, 2, 3, \cdots$, has been established by H. E. Bray[2] for the special case in which $f(z)$ is a real

Presented to the Society, April 26, 1947; received by the editors May 27, 1947.

[1] *On the zeros of a polynomial and of its derivative*, K. Akademie van Wetenschappen, Proceedings vol. 49 (1946) pp. 1037–1044. *Added in proof*: In a second paper by de Bruijn and T. A. Springer, *On the zeros of a polynomial and of its derivative* II, Ibid. vol. 50 (1947) pp. 264–270, the results of the present paper are obtained, and the inequality following (3) is obtained for any $m \geqq 1$.

[2] *On the zeros of a polynomial and its derivative*, Amer. J. Math. vol. 53 (1931) pp. 864–872.

polynomial with non-negative real roots.

If (3) is applied with A located successively at the roots of $f(z)$, and if these inequalities are summed, and if this process is repeated with A located successively at the roots of $f'(z)$, it is seen that the two resulting inequalities imply that

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|r_i - r_j| \geqq \frac{1}{(n-1)^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}|R_i - R_j|,$$

with equality only if all roots of $f(z)$ are equal. It seems likely that this inequality holds with the factors $1/n^2$ and $1/(n-1)^2$ replaced by $1/C_{n,2}$ and $1/C_{n-1,2}$, but we have not been able to prove this.

We now begin the proof of (1).

LEMMA 1. *In case all the roots of $f(z)$ lie on one side of or on the real axis, relation* (1) *holds with the equality sign. In case the roots lie both above and below the real axis, but the roots of $f'(z)$ lie on one side of or on the real axis, relation* (1) *holds with the inequality sign.*

PROOF. We recall Gauss' theorem that the roots of $f'(z)$ lie inside or on the convex polygon determined by the roots of $f(z)$; and they are on the polygon only when $f(z)$ has a multiple root or when all the roots of $f(z)$ lie on a line. Hence in the first statement in the lemma, the roots of $f'(z)$ lie on one side of or on the real axis. By relating the sum of the roots of a polynomial to the coefficients of the two highest powers, we see that

$$\frac{1}{n}\sum r_j = \frac{1}{n-1}\sum R_j,$$

whence

$$\frac{1}{n}\sum_{j=1}^{n} I(r_j) = \frac{1}{n-1}\sum_{j=1}^{n-1} I(R_j), \tag{4}$$

which proves the first statement in the lemma. The second statement is also a consequence of (4), because in that case all terms on the right side of (4) have like signs, whereas there are mixed signs in the sum on the left.

LEMMA 2. *Inequality* (1) *is equivalent to*

$$\frac{1}{n}\sum{}'\,|I(r_j)| \geqq \frac{1}{n-1}\sum{}'\,|I(R_j)|, \tag{5}$$

where $\sum'$ designates the sum over the roots lying below the real axis.

Equality in (1) *implies equality in* (5), *and conversely.*

PROOF. Subtract (4) from (1). We note in passing that the addition of (4) to (1) implies a result (5) with $\sum'$ representing the sum over the roots above the real axis.

We now proceed by induction, and assume that a result corresponding to (5) holds for all polynomials of degree less than n, with equality only when the roots are not on both sides of the real axis. (Note that this is true for quadratic polynomials.) Then we suppose, using Lemmas 1 and 2, that for some polynomial $f(z)$ of degree n with roots r_j on both sides of the real axis and derivative roots also on both sides

$$F = \frac{1}{n} \sum' \left| I(r_j) \right| - \frac{1}{n-1} \sum_{j=1}^{q} \left| I(R_j) \right| \leqq 0, \tag{6}$$

where $R_1, R_2, \cdots, R_q$ are the roots of $f'(z)$ which lie below the real axis. We show that these assumptions lead to a contradiction.

In addition to the hypotheses just stated, we shall use induction on the number of roots of $f(z)$ lying above the real axis, and assume first that $f(z)$ has exactly one such root, say r_1: thus the first sum in F ranges over $j=2, 3, \cdots, n$. We consider what happens to F in (6) when r_1 is moved slightly, the other roots of $f(z)$ remaining fixed.

The first possibility is that $\sum_{j=1}^{q} \left| I(R_j) \right|$ for exactly these R_j does not change no matter in what direction the root r_1 is given a slight motion from its original position. Consider the roots R_j of $f'(z)$, each R_j being an analytic function[3] of r_1. More precisely, if r_1 moves to a value r in the neighborhood, then the R_j move to positions S_j given by

$$r = r_1 + t^\alpha, \qquad S_j = R_j + b_{1j}t + b_{2j}t^2 + \cdots, \tag{7}$$

α being a positive integer. For sufficiently small values of t, our assumption that

$$\sum_{j=1}^{q} \left| I(R_j) \right| = \sum_{j=1}^{q} \left| I(S_j) \right| \tag{8}$$

is the same as $\sum_{j=1}^{q} I(R_j) = \sum_{j=1}^{q} I(S_j)$, because the R_j and S_j are below the real axis. Thus for small values of the complex number t,

$$I\left(\sum_{j=1}^{q} b_{1j}t + \sum_{j=1}^{q} b_{2j}t^2 + \cdots \right) = 0$$

[3] Cf. G. A. Bliss, *Algebraic functions*, Amer. Math. Soc. Colloquium Publications, vol. 16, Theorem 13.1, p. 32.

and consequently $\sum b_{1j}=0$, $\sum b_{2j}=0$, and so on. By analytic continuation (8) holds when r_1 is moved to a position r on the real axis. But in this case those roots of $f'(z)$ which were on or above the real axis, namely $R_{q+1}, \cdots, R_{n-1}$, have moved to positions $S_{q+1}, \cdots, S_{n-1}$ below the real axis, so that we have

$$\frac{1}{n}\sum_{j=2}^{n}|I(r_j)| - \frac{1}{n-1}\sum_{j=1}^{n-1}|I(S_j)| < 0,$$

which contradicts Lemma 1.

Next suppose that $\sum_{j=1}^{q}|I(R_j)|$ changes when r_1 is moved to a neighboring position. Then (7) implies that r_1 can be moved in such a direction as to decrease $\sum_{j=1}^{q} I(R_j)$ and thus decrease F in (6), because

$$I\left(\sum_{j=1}^{q} S_j\right) = I\left(\sum_{j=1}^{q} R_j\right) + I\left(\sum_{j=1}^{q} b_{1j}t\right) + I\left(\sum_{j=1}^{q} b_{2j}t^2\right) + \cdots.$$

Let us move r_1 along a path so as to decrease F. It is clear that such a path will not lead to the real axis, because of Lemma 1, so we need treat only the possibilities of r_1 moving to infinity along a path in the upper half plane.

Consider, then, the polynomial $f(z)$ with roots $r, r_2, r_3, \cdots, r_n$, where $|r|$ is very large relative to $|r_j|$, $j=2, 3, \cdots, n$. If $g(z)$ is the polynomial with roots $r_2, r_3, \cdots, r_n$ then

$$f'(z) = g(z) + (z-r)g'(z). \tag{9}$$

Consider any fixed circle which has center at a root of $g'(z)$, but which does not pass through any root of $g'(z)$. On the circumference of such a circle, $|(z-r)g'(z)| > |g(z)|$ for sufficiently large r. By Rouché's theorem each such circle contains as many roots of $f'(z)$ as of $g'(z)$. Thus if $t_1, t_2, \cdots, t_{n-2}$ are the roots of $g'(z)$, all roots except one of $f'(z)$ can be written as

$$t_1 + \epsilon_{1r},\ t_2 + \epsilon_{2r},\ \cdots,\ t_{n-2} + \epsilon_{n-2r}$$

where every ϵ_{jr} tends to 0 as r tends to infinity. If the other root of $f'(z)$ is denoted by R, then since

$$\frac{1}{n}(r + r_2 + r_3 + \cdots + r_n) = \frac{1}{n-1}\left\{R + \sum_{j=1}^{n-2}(t_j + \epsilon_{jr})\right\},$$

we see that R is approximately $(n-1)r/n$, with a bounded error as r tends to infinity.

We first discuss the case where R is above the real axis. By our induction hypothesis applied to the polynomial $g(z)$,

$$\frac{1}{n-1}\sum_{j=2}^{n}|I(r_j)| \geqq \frac{1}{n-2}\sum_{j=1}^{n-2}|I(t_j)|. \tag{10}$$

Turning to $f(z)$, we see that the function F in (6) has the value

$$\frac{1}{n}\sum_{j=2}^{n}|I(r_j)| - \frac{1}{n-1}\sum_{j=1}^{n-2}|I(t_j+\epsilon_{jr})|, \tag{11}$$

since we are assuming that R has positive imaginary part. But the ϵ_{jr} are arbitrarily small, so (10) implies that (11) is positive, and this contradicts our assumption that F is negative.

If, on the other hand, R is below the real axis, we argue as follows. Since the roots of $g(z)$ are on or below (with at least one below) the real axis, by Gauss' theorem a particular root t_j of $g'(z)$ is on the real axis only if it is a multiple root of $g(z)$, in which case it is a multiple root of $f(z)$ and the corresponding $\epsilon_{jr}=0$. Thus we can take r sufficiently large so that the roots $t_j+\epsilon_{jr}$ of $f'(z)$ are all on or below the real axis. Hence all the roots of $f'(z)$ lie on or below the real axis, and we use the second part of Lemma 1 to complete the proof.

Having completed the proof of (5) in case there is exactly one root of $f(z)$ with positive imaginary part, let us now use induction and assume that (5) holds for all polynomials of degree n with $k-1$ roots above the real axis. Then consider any polynomial of degree n with k roots, $r_1, r_2, \cdots, r_k$, above the real axis. We proceed as in the case where there is exactly one root with positive imaginary part. We move r_1 to the real axis if (a) such motion does not alter $\sum_{j=1}^{q} I(R_j)$; otherwise (b) we move r_1 along a path which decreases this sum. In case (a) the problem is reduced to that of a polynomial with $k-1$ roots having positive imaginary part: likewise in case (b) when r_1 moves along a path crossing or touching the real axis. (Note that if by such motion of r_1, either one or more of the R_j for $j\leqq q$ move to positions above the real axis or one or more of the R_j for $j>q$ move to positions below the real axis, such alterations work in our favor in decreasing F.) Finally, if in case (b) r_1 moves to infinity in the upper half plane, we handle the problem in the following way.

As before we use the notation r for the new location of r_1, and $t_j+\epsilon_{jr}$ $(j=1, 2, \cdots, n-2)$ and R for the roots of $f'(z)$. In case R is above the real axis, we argue as above, with (10) and (11) suitably altered.

In case R is below the real axis, the argument for the case of exactly one root with positive imaginary part is inadequate. If $\sum''$ designates the sum over those of the indicated roots which lie above the real axis, then

$$0 < \frac{1}{n-1}\sum_{j=2}^{n}{}'' \left| I(r_j) \right| - \frac{1}{n-2}\sum_{j=1}^{n-2}{}'' \left| I(t_j) \right|, \tag{12}$$

by the induction hypothesis applied to $g(z)$ and the remark following the proof of Lemma 2. Define a by

$$a = \sum_{j=2}^{n}{}'' \left| I(r_j) \right| - \sum_{j=1}^{n-2}{}'' \left| I(t_j) \right|.$$

Clearly $a>0$, by (12), and a is independent of r, R, and the ϵ_{jr}. Choose r sufficiently large so that $a>n\sum_{j=1}^{n-2}\left|\epsilon_{jr}\right|$. Then (12) implies that

$$\begin{aligned} 0 &< (n-1)\sum_{j=2}^{n}{}'' \left| I(r_j) \right| - n\sum_{j=1}^{n-2}{}'' \left| I(t_j) \right| - a \\ &< (n-1)\sum_{j=2}^{n}{}'' \left| I(r_j) \right| - n\sum_{j=1}^{n-2}{}'' \left| I(t_j) \right| - n\sum_{j=1}^{n-2} \left| \epsilon_{jr} \right| \\ &< (n-1)\sum_{j=2}^{n}{}'' \left| I(r_j) \right| - n\sum_{j=1}^{n-2}{}'' \left| I(t_j + \epsilon_{jr}) \right|. \end{aligned}$$

This is the inequality we want for the roots of $f(z)$ and $f'(z)$ provided we add $\left|I(r)\right|$ to the first sum, which works in our favor. This completes the proof of (5) and (1).

To prove (2), we observe that if the roots of $f(z)$ are translated or rotated about the origin, the roots of $f'(z)$ undergo an identical translation or rotation. This remark not only proves (2) but also shows that we need prove (3) only when A is the origin. Next we note that if $D(\theta, r)$ denotes the distance from r to a line through the origin with direction angle θ, then

$$\int_0^{2\pi} D(\theta, r)d\theta = 4\left| r \right|.$$

Thus (3) is obtained from (2) by integration over the direction angle. Since there is equality in (2) only if all the r_j are on the same side of l, we see that there is equality in (3) only if the r_j lie on a half line out of A.

REMARK. Let $f(z)$ have exactly one root, say r_1, below the real axis; let r_2 be the root, apart from r_1, with least imaginary part. Then there is no root of $f'(z)$ below the line

$$y = I(r_1) + \frac{1}{n} I(r_2 - r_1).$$

This can be seen by translating all roots a distance $I(r_2)$ in the direction of the negative imaginary axis, and by application of (5). It can also be obtained directly from the well known relation $\sum_j (R - r_j)^{-1} = 0$.

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