A THEOREM ON THE DISTRIBUTION OF THE VALUES OF L-FUNCTIONS

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1. Let (d/n) [where $d \equiv 0,\ 1 \pmod 4$, $d \neq u^2$, u integral] be Kronecker's symbol. Define for $s > 0$

$$L_d(s) = \sum_1^\infty \left(\frac{d}{n}\right) n^{-s}.$$

Denote by $g(a, x)$ the number of positive integers $d \leqslant x$ such that

$$d \equiv 0,\ 1\ (4);\ d > 0;\ d \neq u^2;$$
$$L_d(s) < a.$$

We prove the following

Theorem. *If $s > 3/4$ we have*

$$\lim_{x \to \infty} \frac{g(a, x)}{x/2} = g(a) \text{ exists};$$

furthermore $g(0) = 0$, $g(\infty) = 1$ and $g(a)$, the distribution function, is a continuous and strictly increasing function of a.

It is implicit in our theorem that for almost all d [i.e. with the exception of $o(x)$ integers $d \leqslant x$] $L_d(s) > 0$ provided that $s > 3/4$. This result seems to be new.

If the extended Riemann hypothesis holds, then of course $L_d(s) > 0$ for all d.

We can also prove our theorem when d runs over negative integral values whose absolute values do not exceed x. (Similar questions on the distribution functions of number theoretic functions were considered in several papers of Wintner, e.g. *Amer. Jour. of maths.* 63, (1941), 223-248; see also Jessen-Wintner, *Trans. Amer. Math. Soc.* 38 (1935), 48-88.)

2. Write for $s > 0$,

$$L_d(s, y) = \sum_{n \leqslant y} \left(\frac{d}{n}\right) n^{-s},$$
$$L_d^{(t)}(s, y) = \sum_{\substack{n \\ n \leqslant y}} \left(\frac{d}{n}\right) n^{-s},$$

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$$L_d^{(t)}(s) = \sum_n \left(\frac{d}{n}\right) n^{-s},$$

where, in the last two summations, n runs only over positive integers whose greatest prime factor does not exceed t. Clearly

$$L_d(s, x^{2/3}) - L_d^{(t)}(s, x^{2/3}) = \sum_{\substack{n \\ n \leqslant x^{2/3},\ P(n) > t}} \left(\frac{d}{n}\right) n^{-s},$$

where $P(n)$ denotes the greatest prime factor of n. Hence

$$| L_d(s, x^{2/3}) - L_d^{(t)}(s, x^{2/3}) |^2$$

$$= \sum \sum_{\substack{m,\, n \leqslant x^{2/3} \\ P(m),\, P(n) > t}} \left(\frac{d}{m}\right) \left(\frac{d}{n}\right) (mn)^{-s}$$

summing for all $d \equiv 0, 1 \ (4)$ which are in the range $2 \leqslant d \leqslant x$ and are not perfect squares:

$$\sum_d | L_d(s, x^{2/3}) - L_d^{(t)}(s, x^{2/3}) |^2$$

$$= \sum_d \sum_{\substack{m,\, n \leqslant x^{2/3} \\ P(m),\, P(n) > t}} \left(\frac{d}{m}\right) \left(\frac{d}{n}\right) (mn)^{-s} = \Sigma_1 + \Sigma_2,$$

where in Σ_1 the product mn is not a perfect square, while in Σ_2 the product mn is a square.

To estimate Σ_1 we use

LEMMA 1. *If k is not a perfect square we have*

$$\sum_d \left(\frac{d}{k}\right) = O(k^{\frac{1}{2}} \log k) + O(x^{\frac{1}{2}}).$$

The summation is for all d with

$$2 \leqslant d \leqslant x;\ d \equiv 0, 1 \ (4);\ d \neq u^2.$$

PROOF. Polya (*Gottinger Nachrichten*, 1918) proved that

$$\sum_a^b \chi(n) = O(k^{\frac{1}{2}} \log k)$$

if χ is a primitive character (mod k), $k > 1$. From this several writers deduced that this result is true for any non-principal character (mod k). We easily deduce

$$\sum_{\substack{a\\ n\equiv 0(4)}}^{b} \chi(n) = O(\sqrt{k}\log k), \sum_{\substack{a\\ n\equiv 1(4)}}^{b} \chi(n) = O(\sqrt{k}\log k)$$

which proves Lemma 1; the term $O(\sqrt{x})$ in the lemma is accounted for by the fact that the summation in our lemma excludes the d which are perfects quares.

Thus using Lemma 1 we have (since $s > \frac{3}{4}$)

$$|\Sigma_1| < c \sum_{m,\, n \leqslant x^{2/3}} \frac{\sqrt{(mn)}\log(mn) + \sqrt{x}}{(mn)^s}$$
$$< c\left\{ x^{\frac{2}{3}(3-2s)+\varepsilon} + x^{\frac{2}{3}(2-2s)+\frac{1}{2}} \right\} = o(x). \qquad (1)$$

In Σ_2, $mn = w^2$, hence

$$|\Sigma_2| \leqslant \sum_{\substack{w=1\\ P(w)>t}}^{x^{2/3}} \frac{xd(w^2)}{w^{2s}} < \frac{cx}{\sqrt{t}}, \qquad (2)$$

where $d(n)$ denotes the number of divisors of n. We used here the well-known fact that $d(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$ and also that $s > \frac{3}{4}$.

Thus we obtain from (1) and (2)

LEMMA 2. *Let $s > \frac{3}{4}$. Then there exists an absolute constant c so that*

$$\sum_d |L_d(s, x^{\frac{2}{3}}) - L_d^{(t)}(s, x^{\frac{2}{3}})|^2 < \frac{cx}{\sqrt{t}}.$$

3. We next estimate

$$L_d(s) - L_d(s, x^{2/3}) = \sum_{n > x^{2/3}} \left(\frac{d}{n}\right) n^{-s}$$
$$= \frac{O(\sqrt{d}\log d)}{x^{2/3s}} = O(x^{\frac{1}{2}-2/3s+\varepsilon}) = o(1). \qquad (3)$$

Further we have

$$L_d^{(t)}(s) - L_d^{(t)}(s, x^{\frac{2}{3}}) = \sum_{\substack{n \geqslant x^{2/3}\\ P(n) \leqslant t}} \left(\frac{d}{n}\right) n^{-s}$$
$$= O\Big(\sum_{\substack{n \geqslant x^{2/3}\\ P(t) \leqslant t}} n^{-s}\Big), \qquad (4)$$

since the number of integers $n \leqslant x$ for which $P(n) \leqslant t$ is less than

$$\left(\frac{\log x}{\log 2}\right)^{\pi(t)} < \left(\frac{\log x}{\log 2}\right)^t,$$

where $\pi(y)$ denotes the number of primes $\leqslant y$. We obtain by partial summation from (4) that

$$|L_d^{(t)}(s)-L_d^{(t)}(s, x^{\frac{2}{3}})| < c\left(\frac{\log x}{\log 2}\right)^t x^{-\frac{2}{3}s} = o(1), \qquad (5)$$

as x tends to ∞, for any fixed t.

Combining Lemma 2 with (3) and (5) we obtain

LEMMA 3. *For every positive* δ *there exist an* ε *and* t_0 *so that for the number of integers* d *with* $1 < d \leqslant x$ $[d \equiv 0, 1\ (4),$ $d \neq u^2]$ *satisfying*

$$|L_d^{(t)}(s) - L_d(s)| > \varepsilon$$

is $\leqslant \delta x$ *whenever* $t > t_0$ *and* x *is large enough* $[x > x_0(\delta)]$.

5. We define the primes

$$p_1 < p_2 < p_3 \ldots . < p_k$$

and all less than t as follows:

$p_1 = 5$; p_{i+1} is the least prime satisfying

$$p_{i+1}^s > 10\, p_i^s \quad (1 \leqslant i \leqslant k-1).$$

We define the signature of d with respect to a set of primes as the set of values of (d/p), where p runs through the given set of primes.

Denote by $q_1, q_2, \cdots, q_m$ the primes $\leqslant t$ which are distinct from the p's. We have $k+m = \pi(t)$.

Let d_1 and d_2 be two values of d which have different signatures with respect to the p's but the same signature with respect to the q's. Let p_j $(j \leqslant k)$ be the first prime p for which the signatures of d_1 and d_2 disagree. Then we clearly have

$$\text{Max}\left(\frac{L_{d_1}^{(t)}(s)}{L_{d_2}^{(t)}(s)}, \frac{L_{d_2}^{(t)}(s)}{L_{d_1}^{(t)}(s)}\right) \geqslant (1+p_j^{-s}) \prod_{t=j+1}^{k}\left(\frac{1-p_t^{-s}}{1+p_t^{-s}}\right)$$

$$> (1+p_j^{-s}) \prod_{t=j+1}^{k}(1-2p_t^{-s}) > (1+p_j^{-s})\left\{1-2\sum_{t=j+1}^{k} p_t^{-s}\right\}$$

$$> (1+p_j^{-s})\left\{1-\frac{2}{p_j^s}\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)\right\}$$

$$= (1+p_j^{-s})\left(1-\frac{2}{9}p_j^{-s}\right)$$

$$= 1+\tfrac{7}{9}p_j^{-s}-\tfrac{2}{9}p_j^{-2s} > 1+\tfrac{5}{9}p_j^{-s} > 1+\tfrac{1}{2}p_j^{-s}$$
$$\geqslant 1+\tfrac{1}{2}p_k^{-s}. \qquad (6)$$

In the above we used that $2p_j^{-s} < 1$ which follows from $s > \frac{3}{4}$, $p_1 = 5$. Also we used $p_{i+1}^s > 10\, p_i^s$. We next prove

LEMMA 4. *If $a > 0$ and $0 < \varepsilon < \dfrac{a}{4\,p_k^s+1}$ the inequality*

$$a-\varepsilon < L_d^{(t)}(s) < a+\varepsilon$$

cannot be satisfied by $d = d_1$ and $d = d_2$ if d_1 and d_2 are values of d such that their signature with respect to the primes p is different, while their signature with respect to the primes q is the same.

From (6) it follows that Lemma 4 is true if ε is so small that

$$\frac{a+\varepsilon}{a-\varepsilon} < 1+\frac{1}{2\,p_k^s},$$
$$\frac{\varepsilon}{a} < \frac{2^{-1}\,p_k^{-s}}{2+2^{-1}\,p_k^{-s}},$$
$$\varepsilon < \frac{a}{1+4\,p_k^s}.$$

This proves the lemma.

Let y_s denote the number of $d \leqslant x$ $[d > 0;\ d \equiv 0,\ 1\ (4);\ d \neq u^2]$ such that all the d's have a fixed signature with respect to the q's. Clearly s assumes

$$h = 3^m = 3^{\pi(t)-k}$$

values, and

$$y_1+y_2+\cdots+y_h = \frac{x}{2}+O(\sqrt{x}),$$

where the constant implied in the O is an absolute one.

Again the d's (which have a fixed signature with respect to the q's) fall into $3^k = g$ classes according to their signature with respect to the p's. Thus

$$y_s = z_{1s}+z_{2s}+\cdots+z_{gs} \quad (1 \leqslant s \leqslant h). \qquad (7)$$

Clearly

$$gh = 3^{\pi(t)}.$$

Next we prove

LEMMA 5. *For $x > x_0\,(k)$ we have*

$$\frac{z_{bs}}{y_s} < \frac{1}{2^k} \; [1 \leqslant b \leqslant g].$$

PROOF. The d's $\leqslant x$ which have a particular signature with respect to the q's are (by assumption) y_s in number ($s = 1, 2, 3, \ldots, h$). Let

$$q_{\alpha_1}, q_{\alpha_2}, \cdots, q_{\alpha_w}$$

be the primes q for which $(d/q) = 0$; let

$$q_{\beta_1}, q_{\beta_2}, \cdots, q_{\beta_{w'}}$$

be the primes for whih $(d/q) = +1$; finally let

$$q_{\gamma_1}, q_{\gamma_2}, \cdots, q_{\gamma_{w''}}$$

be the primes q for which $(d/q) = -1$. We have

$$w + w' + w'' = \pi(t) - k = m.$$

It is evident that

$$\begin{aligned} y_s &= \frac{x}{2} \prod_{n=1}^{w} q_{\alpha_n}^{-1} \prod_{n=1}^{w'} \left(\frac{q_{\beta_n} - 1}{2\, q_{\beta_n}}\right) \prod_{n=1}^{w''} \left(\frac{q_{\gamma_n} - 1}{2\, q_{\gamma_n}}\right) + O(\sqrt{x}) \\ &= \frac{Qx}{2} + O(\sqrt{x}), \end{aligned} \tag{8}$$

where the constant in the last O may also depend on t. Consider next the value of z_{bs}. This number is the number of $d \leqslant x$ which have the above signature with respect to the q's and also have a fixed signature with respect to the p's. Write

$$\left(\frac{d}{p}\right) = 0 \;\text{ for } p = p_{\alpha_n} \; (1 \leqslant n \leqslant v).$$

$$\left(\frac{d}{p}\right) = +1 \;\text{ for } p = p_{\beta_n} \; (1 \leqslant n \leqslant v').$$

$$\left(\frac{d}{p}\right) = -1 \;\text{ for } p = p_{\gamma_n} \; (1 \leqslant n \leqslant v'').$$

Then

$$v + v' + v'' = k.$$

Further it is evident that

$$\begin{aligned} z_{bs} &= \frac{Qx}{2} \prod_{n=1}^{v} p_{\alpha_n}^{-1} \prod_{n=1}^{v'} \left(\frac{p_{\beta_n} - 1}{2 p_{\beta_n}}\right) \prod_{n=1}^{v''} \left(\frac{p_{\gamma_n} - 1}{2 p_{\gamma_n}}\right) \\ &= PQ\frac{x}{2} + O(\sqrt{x}). \end{aligned} \tag{9}$$

From (8) and (9), we have

$$z_{bs}/y_s = P + O(x^{-\frac{1}{2}}). \qquad (10)$$

The lemma thus follows since $P < 2^{-k}$ [$k > 1$].

Consider now the d's which have the same signature as the numbers of y_s ($1 \leqslant s \leqslant h$). By Lemma 4 at most max (z_{bs}) [$1 \leqslant b \leqslant 3^k$] of them satisfy the inequality

$$a - \varepsilon < L_d^{(t)}(s) < a + \varepsilon. \qquad (11)$$

Hence by Lemma 5 the total number of d's not exceeding x which satisfy (11) is at most

$$z_{\alpha 1} + z_{\beta 2} + z_{\gamma 3} + \ldots < 2^{-k}(y_1 + y_2 + y_3 + \ldots) = 2^{-k}\left\{\frac{x}{2} + O(\sqrt{x})\right\}.$$

Thus, we have (choose $2^{-k} \leqslant \delta$)

Lemma 6. *Given any positive* δ, *there exist* t_0, ε, x_0 *such that the number of positive* $d \leqslant x$ *with* $d \equiv 0, 1$ (4), $d \neq u^2$, *and*

$$a - \varepsilon < L_d^{(t)}(s) < a + \varepsilon \qquad (12)$$

is less than δx *for all* $t > t_0$, $x > x_0$.

The case $a = 0$ needs special discussion. Here (12) has to be replaced by

$$0 < L_d^{(t)}(s) < \varepsilon \qquad (13)$$

whence

$$S_d = \prod_{p \leqslant t} \left\{1 - \left(d/p\right) p^{-s}\right\} > \varepsilon^{-1}.$$

Now the sum

$$\sum_{1 \leqslant d \leqslant x} S_d = x/2 + O(\sqrt{x}),$$

where d runs over integers which are $\equiv 0, 1$ (4), not perfect squares, and $\leqslant x$.

It easily follows that Lemma 6 is true with $a = 0$ when we replace (12) by (13).

6. **Proof of the theorem.** Denote by $g_t(a, x)$ the number of integers $d \leqslant x$ [$d \equiv 0, 1$ (4), $d \neq u^2$, $d > 0$] for which

$$L_d^{(t)}(s) \leqslant a.$$

It is easy to see that

$$\lim_{x\to\infty} \frac{g_t(a, x)}{x/2} = g(a, t)$$

exists. This follows from the following simple observation. The expression

$$\prod_{p\leqslant t} \left\{ 1-\left(\frac{d}{p}\right) p^{-s} \right\}^{-1}$$

is periodic in d (mod $p_1 p_2 \ldots p_w$), where $p_1, \ldots, p_w$ are all the primes $\leqslant t$. As d goes from 1 to $p_1 p_2 \ldots p_w$ suppose that there are N_t values of d for which

$$0 < L_d^{(t)}(s) \leqslant a.$$

Then

$$\lim_{x\to\infty} \frac{g_t(a, x)}{x/2} = \frac{2N_t}{p_1 p_2 \ldots p_w}.$$

We next prove

$$\lim_{t\to\infty} g(a, t) = g(a),$$

where $g(a)$ was defined in § 1.

To do this it will suffice to show that given an arbitrary positive η we can find t_0, x_0 such that

$$|g_t(a, x) - g(a, x)| < \eta x$$

for $t > t_0$, $x > x_0(\eta)$.

We split the integers $d \leqslant x$ $[d \equiv 0, 1\ (4),\ d \neq u^2]$ which satisfy

$$L_d^{(t)}(s) \leqslant a,\ L_d(s) > a$$

or

$$L_n^{(t)}(s) > a,\ L_d(s) \leqslant a$$

in.o two classes:

I $$|L_d^{(t)}(s) - L_d(s)| > \varepsilon.$$

By Lemma 3 the number of these integers is $< \delta x$.

II $$a - \varepsilon < L_d^{(t)}(s) < a + \varepsilon.$$

By Lemma 6 the number of these integers is $< \delta x$.

This completes the proof of our theorem. The fact that $g(a)$ is a continuous and strictly increasing function of a follows easily by the arguments of Lemmas 3 and 6.

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