COMBINATORIAL THEOREMS ON CLASSIFICATIONS OF SUBSETS OF A GIVEN SET

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Introduction

LET S be a set, and denote by $\Omega_n(S)$ the set of all subsets of S which contain exactly n elements. F. P. Ramsey† proved the following theorem:

Given any positive integers k, n, N, there is a positive integer M which has the following property. If $S = \{1, 2,..., M\}$, and Δ is any distribution of $\Omega_n(S)$ into k classes, there is always an element S' of $\Omega_N(S)$ such that the $\binom{N}{n}$ elements of $\Omega_n(S')$ belong to the same class of Δ.

We denote by $R(k, n, N)$ the least number M possessing this property, and we call R *Ramsey's function*. Clearly

$$R(1, n, N) = N; \qquad R(k, 1, N) = k(N-1)+1,$$

so that only the case $k \geqslant 2$; $N \geqslant n \geqslant 2$ is of interest.

All known proofs (**1**), (**2**), (**3**), (**4**) of Ramsey's theorem give upper estimates for R which are so large that they are hardly expressible explicitly in terms of the fundamental algebraic operations. A modification of the known methods of proof leads to a new upper estimate for R (Theorem 1) which is in general much better than the known estimates and which is, moreover, easily expressible in terms of $(n-1)$-fold exponentiation.

In (**4**), Ramsey's theorem, or rather its companion theorem‡ in which both S and S' are denumerably infinite, was generalized so as to cover the case of arbitrary distributions into infinitely many classes. The main step in the proof of this generalized Ramsey theorem consisted in showing that every 'invariant' distribution of $\Omega_n(S)$ is 'canonical'. A distribution Δ of $\Omega_n(S)$, where S is a set of real numbers, is called *invariant* if the following condition holds. Suppose that $A = \{a_1,..., a_n\}$; $B = \{b_1,..., b_n\}$ are any elements of $\Omega_n(S)$ belonging to the same class of Δ, and that $f(x)$ is any function defined and increasing in the union $A+B$ and having its values in S. Then always the sets $\{f(a_1),...,f(a_n)\}$ and $\{f(b_1),...,f(b_n)\}$ belong to the same class of Δ. A distribution Δ of $\Omega_n(S)$ is called *canonical* if integers k, ν_κ can be found satisfying

$$0 \leqslant k \leqslant n; \qquad 1 \leqslant \nu_1 < \nu_2 < ... < \nu_k \leqslant n,$$

such that the above sets A and B, under the assumption

$$a_1 < a_2 < ... < a_n; \qquad b_1 < ... < b_n,$$

† (**1**), Theorem B. ‡ (**1**), Theorem A.

belong to the same class of Δ if and only if $a_{\nu_1} = b_{\nu_1}; a_{\nu_2} = b_{\nu_2}; \ldots; a_{\nu_k} = b_{\nu_k}$. In this case we write $\Delta = \Delta^{(k)}_{\nu_1 \nu_2 \ldots \nu_k}$.

Clearly every canonical distribution is invariant. In (**4**), Theorem III, it was proved that if $S = \{1, 2, \ldots\}$, then every invariant distribution is canonical, so that the two classes of distributions coincide. It turns out that for *finite* sets S this is no longer true. In Theorem 2 of the present note necessary and sufficient conditions on n and N are established in order that every invariant distribution of $\Omega_n(\{1, 2, \ldots, N\})$ should be canonical.

By means of Theorem 1 and Theorem 2 we obtain a finitist version (Theorem 3) of the generalized Ramsey theorem of (**4**).

The next section of the paper is largely devoted to transfinite extensions of Ramsey's Theorem A in the case $k = n = 2$. The general type of problem arising in this field, of which we have only partial solutions, can be characterized as follows. Suppose that all sets of two real numbers are distributed into two classes K_1 and K_2. Let us say that an order type ϕ is *realizable in the class* K_λ if there exists a set X of real numbers, of order type ϕ under the natural order according to magnitude, such that $\Omega_2(X) \subset K_\lambda$. Ramsey's Theorem A ensures that the first infinite ordinal number $\omega = \overline{\{1, 2, \ldots\}}$ is always realizable in some class. A very interesting example due to Sierpiński (last section, Example 4 A) shows that it can happen that the only order types realizable in K_1 are denumerable ordinals, and at the same time the only types realizable in K_2 are the converses of denumerable ordinals, i.e. order types obtained from ordinal numbers by replacing every relation $x < y$ by the corresponding relation $x > y$. The most concrete result obtained in this note (Theorem 7) implies that *every order type* $\omega + m$ *is realizable in some class, for any finite* m. In Theorems 4–8 various results are established pointing in the direction of a fairly plausible conjecture: *in every distribution every denumerable ordinal is realized in some class.* The section concludes with two results (Theorems 9 and 10) concerning distributions of $\Omega_n(S)$ for arbitrary finite n and having any number of classes, finite or infinite.

In the final section of the paper a number of examples are given which show that in certain directions Ramsey's theorem cannot be generalized. Some of these examples give rise to unsolved problems, the most interesting of which seems to be the following one, discussed in §§ 2 and 3 of the last section. *Given an infinite set* S, *is it possible to divide all finite subsets of* S *into two classes in such a way that every infinite subset of* S *contains two finite subsets of the same number of elements but belonging to different classes?* The answer is in the affirmative if the cardinal of S is at most that of the continuum, and there are, in fact, many different methods of effecting the required classification. Nothing, however, is known for sets whose cardinal exceeds that of the continuum.

The last example of the paper is not concerned with Ramsey's theorem but with the following theorem due to van der Waerden (**5**). *Given positive integers k and l, there is a positive integer m such that, if the set $\{1, 2, \ldots, m\}$ is divided into k classes, at least one class contains $l+1$ numbers which form an arithmetic progression.* The least number m possessing this property is denoted by $W(k, l)$ (*van der Waerden's function*). Our final example yields what seems to be the first non-trivial, no doubt extremely weak, lower estimate of W, namely $W(k, l) > ck^{\frac{1}{2}l}l^{\frac{1}{2}}$. An upper estimate of W, at any rate one which is easily expressible explicitly in terms of the fundamental algebraic operations, seems to be beyond the reach of methods available at present.

Notation and definitions

Brackets $\{\ldots\}$ are used exclusively in order to define sets by means of a list of the elements they contain. Thus order and multiplicities are irrelevant, so that $\{1, 1, 2\} = \{2, 1\}$. If A and B are sets, then $A+B$, AB, and $A-AB$ denote their union, intersection, and difference respectively, and $|A|$ denotes the cardinal of A. Set inclusion, in the wide sense, is denoted by $A \subset B$. The symbol $\{a_1, a_2, \ldots, a_n\}_{\neq}$ denotes the set $\{a_1, \ldots, a_n\}$ and also expresses the fact that $a_r \neq a_s$ $(1 \leqslant r < s \leqslant n)$, and similarly $\{b_1, \ldots, b_n\}_{<}$ denotes the set $\{b_1, \ldots, b_n\}$ and at the same time expresses the fact that $b_1 < \ldots < b_n$. Throughout, even in arguments involving transfinite ordinals, the symbol $x_1, x_2, \ldots$ denotes a sequence of type ω, where ω stands for the first infinite ordinal.

For typographical reasons we frequently replace symbols like

$$\sum_{\substack{1 \leqslant r \leqslant m \\ 1 \leqslant s \leqslant n}} a_{rs} \quad \text{by} \quad \sum {}^{1 \leqslant r \leqslant m}_{1 \leqslant s \leqslant n} \square\, a_{rs},$$

and similarly when in place of $\sum$ we have $\prod$, min, or max. As stated in the introduction, we put $\Omega_n(S) = \sum S' \subset S;\ |S'| = n \square \{S'\}$.

The letter Δ is used to denote partitions or distributions of sets S into classes, i.e. equivalence relations on S. The fact that Δ is a distribution of S is also expressed by saying that $\Delta(x)$ is *defined for* $x \in S$. We use the notation and the calculus of partitions developed in (**6**), which will now be briefly described.

A relation $x \equiv y \; (.\Delta)$ expresses the fact that $x, y \in S$, and that x and y belong to the same class of Δ. The number of non-empty classes of Δ is denoted by $|\Delta|$. The following two methods are employed for generating new distributions from given ones.

(i) If $\Delta_1, \ldots, \Delta_m$ are defined in S, then the equation

$$\Delta^*(x) = \prod 1 \leqslant \mu \leqslant m \square \Delta_\mu(x) \quad (x \in S)$$

defines that distribution Δ^* of S for which the relation

$$x \equiv y \; (.\Delta^*)$$

is equivalent to the system of relations

$$x \equiv y \; (.\Delta_\mu) \quad (1 \leqslant \mu \leqslant m).$$

(ii) If $\Delta(y)$ is defined for $y \in T$, and if $f(x)$ is a function on S into T, then the equation

$$\Delta'(x) = \Delta(f(x)) \quad (x \in S)$$

defines that distribution Δ' of S for which the relation

$$x_1 \equiv x_2 \; (.\Delta')$$

is equivalent to the relation

$$f(x_1) \equiv f(x_2) \; (.\Delta).$$

Estimate of Ramsey's function $R(k, n, N)$

We define a binary operation $*$ by putting, for positive numbers a and b,

$$a * b = a^b.$$

Furthermore, we put, for $n \geqslant 3$,

$$a_1 * a_2 * a_3 * \ldots * a_n = a_1 * (a_2 * (a_3 * (\ldots * (a_{n-1} * a_n) \ldots))).$$

Then, if $1 \leqslant m < n$,

$$a_1 * a_2 * \ldots * a_m * (a_{m+1} * \ldots * a_n) = a_1 * a_2 * \ldots * a_n,$$

where the symbol $a_1 * a_2 * \ldots * a_m$ has the value a_1 if $m = 1$.

THEOREM 1. *Ramsey's function R, defined in the introduction, satisfies the inequality*

$$R(k,n,N) \leqslant k * (k^{n-1}) * (k^{n-2}) * \ldots * (k^2) * [k(N-n)+1] \tag{1}$$

$$(k \geqslant 2;\; N \geqslant n \geqslant 2).$$

In particular, $R(k, 2, N) \leqslant k^{k(N-2)+1}$.

Ramsey, in his paper (**1**), states that his method yields the estimate

$$R(k, 2, N) \leqslant (\ldots((N!)!)! \ldots)! \quad (k-1 \text{ symbols '!'}),$$

but he emphasizes that he believes the right-hand side to be far too large.

The result obtained in (**2**) is

$$R(k, 2, N) \leqslant (k^{k(N-1)+2})/(k-1).$$

For $k = n = 2$, the result of (**3**) is slightly better than (1). We find from (**3**): $\quad R(2,2,N) \leqslant \binom{2N-2}{N-1} \sim \pi^{-\frac{1}{2}} 2^{2N-2} N^{-\frac{1}{2}} \quad (N \to \infty)$;

while Theorem 1 gives $\quad R(2, 2, N) \leqslant 2^{2N-3}$.

But for $k = 2; n = 3$ the comparison is heavily in favour of the new value. We find

from (**3**): an estimate considerably weaker than

$$R(2,3,N) \leqslant 2*2*\ldots*2 \quad (2N-2 \text{ 'factors'})$$

and from Theorem 1: $\qquad R(2,3,N) \leqslant 2^{4^{2N-5}}.$

In the special case $k = n = 2$ Theorem 1 asserts that *every graph of order* 2^{2N-3} *contains either N independent nodes or a complete subgraph of order N.* A transfinite analogue of this result is proved in (**8**).

Proof of Theorem 1. Let $k \geqslant 2; N \geqslant n \geqslant 2$. We first dispose of a trivial case. If $N = n$, then $R(k,n,N) = n$, and the theorem asserts that

$$n \leqslant k*(k^{n-1})*(k^{n-2})*\ldots*(k^2)*[k.0+1]. \tag{2}$$

If $n = 2$, then (2) asserts that $2 \leqslant k$, which is true. Now suppose that (2) holds for all k and some $n = n_0 \geqslant 2$. Then, when $n = n_0$ is replaced by n_0+1, the right-hand side of (2) increases by at least a unit, so that (2) holds for $n = n_0+1$. Hence the conclusion holds for $N = n$, and we may assume that $N > n$.

Let A be a finite set. The construction we shall describe will be possible provided that $|A|$ is sufficiently large. A sufficient condition on $|A|$ will be determined after the construction has been defined. Throughout this proof the letter B denotes subsets of A such that $|B| = n-2$.

We are given a distribution Δ of $\Omega_n(A)$, such that $|\Delta| \leqslant k$. We choose $\{a_1, a_2,\ldots, a_{n-1}\}_{\neq} \subset A$ and put

$$\Delta_{n-1}(x) = \Delta(\{a_1,\ldots,a_{n-1},x\}) \quad (x \in A-\{a_1,\ldots,a_{n-1}\}).$$

Then $|\Delta_{n-1}| \leqslant k$, and there is $A_n \subset A-\{a_1,\ldots,a_{n-1}\}$ such that $|\Delta_{n-1}| = 1$ in A_n,†

$$|A_n| \geqslant (|A|-n+1)k^{-1}.$$

We choose $a_n \in A_n$ and put

$$\Delta_n(x) = \prod B \subset \{a_1,\ldots,a_{n-1}\} \square \Delta(B+\{a_n,x\}) \quad (x \in A_n-\{a_n\}).$$

Then $|\Delta_n| \leqslant k*\binom{n-1}{n-2}$, and there is $A_{n+1} \subset A_n-\{a_n\}$ such that $|\Delta_n| = 1$ in A_{n+1},

$$|A_{n+1}| \geqslant (|A_n|-1)k^{-\binom{n-1}{n-2}}.$$

Generally, let $m \geqslant n$, and suppose that elements $a_1,\ldots, a_{m-1}$ and sets $A_n, A_{n+1},\ldots, A_m$ have been defined, and that $A_m \neq 0$. Then we choose $a_m \in A_m$ and put

$$\Delta_m(x) = \prod B \subset \{a_1,\ldots,a_{m-1}\} \square \Delta(B+\{a_m,x\}) \quad (x \in A_m-\{a_m\}).$$

Then $|\Delta_m| \leqslant k*\binom{m-1}{n-2}$, and there is $A_{m+1} \subset A_m-\{a_m\}$ such that $|\Delta_m| = 1$ in A_{m+1},

$$|A_{m+1}| \geqslant (|A_m|-1)k^{-\binom{m-1}{n-2}}.$$

Now put $\qquad l = 1+R(k,n-1,N-1).$

† i.e. all elements of A_n belong to the same class of Δ_{n-1}.

Then $l \geqslant N > n$. If $|A|$ is sufficiently large, then $A_m \neq 0$ $(n \leqslant m \leqslant l)$, so that $a_1,..., a_l$ exist. Let

$$\Delta'(\{\rho_1,..., \rho_{n-1}\}) = \Delta(\{a_{\rho_1},..., a_{\rho_{n-1}}, a_l\}) \quad (1 \leqslant \rho_1 < ... < \rho_{n-1} < l).$$

Then $|\Delta'| \leqslant k$, and by definition of l there is $D \subset \{1,..., l-1\}$ such that $|\Delta'| = 1$ in $\Omega_{n-1}(D)$, $|D| = N-1$. Finally, we put

$$A' = \{a_l\} + \sum \rho \in D \,\square\, \{a_\rho\} = \{a_{\lambda_1},..., a_{\lambda_N}\}_{\neq},$$

say. Then
$$|\Delta| = 1 \quad \text{in} \quad \Omega_n(A'). \tag{3}$$

For let $C = \{a_{\rho_1},..., a_{\rho_n}\} \subset A'$; $1 \leqslant \rho_1 < ... < \rho_n \leqslant l$. Then, since $a_{\rho_n}, a_l \in A_{\rho_n}$, we have $a_{\rho_n} \equiv a_l \ (.\Delta_{\rho_{n-1}})$, and hence

$$\{a_{\rho_1},..., a_{\rho_{n-1}}, a_{\rho_n}\} \equiv \{a_{\rho_1},..., a_{\rho_{n-1}}, a_l\} \quad (.\Delta). \tag{4}$$

Also, since $\{a_{\rho_1},..., a_{\rho_{n-1}}\} \subset A'$, we have

$$\{a_{\rho_1},..., a_{\rho_{n-1}}\} \equiv \{a_{\lambda_1},..., a_{\lambda_{n-1}}\} \quad (.\Delta'),$$

i.e.
$$\{a_{\rho_1},..., a_{\rho_{n-1}}, a_l\} \equiv \{a_{\lambda_1},..., a_{\lambda_{n-1}}, a_l\} \quad (.\Delta). \tag{5}$$

By (4) and (5), $C \equiv C_0 \ (.\Delta)$, where $C_0 = \{a_{\lambda_1},..., a_{\lambda_{n-1}}, a_l\}$ is independent of the choice of C. Hence (3) follows.

We shall now obtain a value for $|A|$ which will ensure that the construction can be carried through. For such a value we shall then have

$$R(k, n, N) \leqslant |A|.$$

Put
$$t_n = k^{-1}(|A|-n+1),$$
$$t_{m+1} = k^{-\binom{m-1}{n-2}}(-1+t_m) \quad (n \leqslant m < l).$$

All we require is that $t_l > 0$. Now, if $k^{-\binom{m}{n-2}} = k_m$,

$$t_l = k_{l-2}(-1+k_{l-3}(...(-1+k_{n-1}(-1+t_n))...))$$
$$= -k_{l-2} - k_{l-2}k_{l-3} - ... - k_{l-2}k_{l-3}...k_{n-1} + k_{l-2}...k_{n-1}t_n.$$

Hence a sufficient condition on $|A|$ is

$$k_{l-2}...k_{n-2}(|A|-n+1) > k_{l-2} + k_{l-2}k_{l-3} + ... + k_{l-2}...k_{n-1},$$

i.e.
$$|A|-n+1 > k^{\binom{l-3}{n-2}+...+\binom{n-2}{n-2}} + k^{\binom{l-4}{n-2}+...+\binom{n-2}{n-2}} + ... + k^{\binom{n-2}{n-2}}.$$

A possible value is

$$|A| = n + k^{\binom{l-2}{n-1}} + k^{\binom{l-3}{n-1}} + ... + k^{\binom{n-1}{n-1}},$$

so that

$$R(k, n, N) \leqslant n + \sum_{\lambda=n-1}^{l-2} k * \binom{\lambda}{n-1}$$
$$\leqslant n + \sum_{\lambda=n-1}^{l-2} k * \lambda * (n-1) \leqslant n + \sum_{\lambda=n-1}^{l-2} [k*(\lambda+1)*(n-1) - k*\lambda*(n-1)]$$
$$= n + k*(l-1)*(n-1) - k*(n-1)*(n-1) \leqslant k*(l-1)*(n-1)$$
$$= k*R(k, n-1, N-1)*(n-1),$$
$$R(k, n, N)*n \leqslant (k^n)*R(k, n-1, N-1)*(n-1).$$

After $n-1$ applications of this last inequality we get

$$R(k,n,N)*n \leqslant (k^n)*(k^{n-1})*\ldots*(k^2)*R(k,1,N-n+1)*1$$
$$= (k^n)*(k^{n-1})*\ldots*(k^2)*[k(N-n)+1],$$

which is the desired result.

Invariant and canonical distributions

The terms *invariant distribution* and *canonical distribution* are defined in the introduction.

THEOREM 2. *Let n and N be integers, $1 \leqslant n \leqslant N$, and let Δ be an invariant distribution of $\Omega_n(S)$, where $S = \{1,2,\ldots,N\}$. Then Δ is canonical, provided that at least one of the three conditions* (i) $n = 1$, (ii) $n < \frac{1}{2}N$, (iii) $n = N$ *holds. If, on the other hand, the numbers n and N do not satisfy any of the conditions* (i), (ii), (iii), *then there exists an invariant distribution of $\Omega_n(S)$ which is not canonical.*

Proof. We recall the following definitions used in (4). If A_μ, B_μ are sets of positive integers $(1 \leqslant \mu \leqslant m)$, then the relation

$$A_1:A_2:\ldots:A_m = B_1:B_2:\ldots:B_m$$

means that there exists a function $f(x)$, defined and increasing on $A_1+A_2+\ldots+A_m$, which maps, for all μ, A_μ on B_μ. The relation

$$A:A' = B:B' = C:C'$$

means that simultaneously

$$A:A' = B:B'; \qquad B:B' = C:C'.$$

It is worth noting that the relations

$$A:A' = B:B'; \qquad A':A'' = B':B''$$

do not imply that $A:A'' = B:B''$. This is shown by the example in which A, A', A''; B, B', B'' are

$$\{1,3\}, \quad \{2,4\}, \quad \{3,5\}; \qquad \{1,4\}, \quad \{2,5\}, \quad \{3,6\}$$

respectively.

We begin by proving the last part of Theorem 2. If n and N do not satisfy any of the conditions (i)–(iii) then

$$2 \leqslant n < N \leqslant 2n. \tag{6}$$

Put $S = \{1,\ldots,N\}$; $A_0 = \{1,2,\ldots,n\}$; $B_0 = \{N-n+1, N-n+2,\ldots,N\}$, and define a distribution Δ' of $\Omega_n(S)$ as follows. If $A, B \in \Omega_n(S)$, then $A \equiv B\ (.\Delta')$ if and only if either $A = B$ or $A = A_0$; $B = B_0$ or $A = B_0$; $B = A_0$. Then Δ' is invariant. For let $A, B, C, D \in \Omega_n(S)$; $A \equiv B\ (.\Delta')$,

$$A:B = C:D. \tag{7}$$

We have to deduce that $C \equiv D\ (.\Delta')$. If $A = B$, then, by (7), $C = D$, and hence $C \equiv D\ (.\Delta')$. If $A \neq B$, then we may assume that $A = A_0$; $B = B_0$. Then, in view of (6), the relation (7) implies that $C = A_0$; $D = B_0$. For, clearly, the order relations between the elements of A_0 and B_0 are such that there exists only one pair of elements of $\Omega_n(S)$ for which they all hold. Hence in any case $C \equiv D\ (.\Delta')$.

We now show that Δ' is not canonical. We have $1 < N-n+1$,

$$\{1, 2,..., n\} \equiv \{N-n+1, N-n+2,..., N\} \quad (.\Delta').$$

Hence, if Δ' were canonical, say $\Delta' = \Delta^{(k)}_{\nu_1\nu_2...\nu_k}$, then the only possibility would be $k = 0$, i.e. $|\Delta'| = 1$. But this is a contradiction against

$$\{1, 2,..., n\} \not\equiv \{1, 2,..., n-1, n+1\} \quad (.\Delta').$$

Thus the last part of Theorem 2 is proved.

We now suppose that n and N satisfy at least one of the conditions (i)–(iii), and that Δ is an invariant distribution of $\Omega_n(S)$, where $S = \{1,..., N\}$. We shall prove that Δ is canonical. If in what follows we assume $N = \infty$ so that (ii) holds, then we obtain a new proof of Theorem I of (4) which is simpler than the original proof.

All congruences are understood $(.\Delta)$. First of all, let $n = 1$. If there is a set $\{x_0, y_0\}_< \subset S$ such that $\{x_0\} \not\equiv \{y_0\}$, then we have, for $\{x, y\}_< \subset S$,

$$\{x_0\}:\{y_0\} = \{x\}:\{y\}$$

and hence, by the invariance of Δ, $\{x\} \not\equiv \{y\}$. Hence, if $n = 1$, then either $\Delta = \Delta^{(0)}$ or $\Delta = \Delta^{(1)}_1$, so that Δ is canonical.

Next, let $n = N$. Then Δ is canonical for the trivial reason that

$$|\Omega_n(S)| = 1.$$

Thus we need only to consider the case $2 \leqslant n < \frac{1}{2}N$. Put

$$H_\nu = \{1, 2,..., \nu-1, \nu+1,..., n+1\} \quad (1 \leqslant \nu \leqslant n+1).$$

Let $I = \{\nu_1, \nu_2,..., \nu_k\}_<$ be the set of all ν such that

$$1 \leqslant \nu \leqslant n; \qquad H_\nu \not\equiv H_{\nu+1}.$$

Then $0 \leqslant k \leqslant n$. We shall prove that $\Delta = \Delta^{(k)}_{\nu_1...\nu_k}$.

Define n operators T_ν by putting, for $1 \leqslant \nu \leqslant n$,

$$T_\nu(A, B) = (\{a_1,..., a_{\nu-1}, \min(a_\nu, b_\nu), a_{\nu+1},..., a_n\}, \{b_1,..., \min(a_\nu, b_\nu),..., b_n\}),$$

whenever $\quad A = \{a_1,..., a_n\}_< \subset S; \qquad B = \{b_1,..., b_n\}_< \subset S.$ (8)

Thus, in particular,

$$T_\nu(H_\nu, H_{\nu+1}) = (H_{\nu+1}, H_{\nu+1}) \quad (1 \leqslant \nu \leqslant n).$$

(α) Suppose that (8) holds, and $a_\nu = b_\nu$ $(\nu \in I)$.

We have to prove that $A \equiv B$. Put

$$\rho(A, B) = \sum a_\nu \neq b_\nu \,\square\, 1.$$

We may assume that $\rho(A, B) > 0$, and we may use induction with respect to $\rho(A, B)$. Let $\nu_0 = \min a_\nu \neq b_\nu \square \nu$. Then $\nu_0 \bar{\in} I$. We may assume that $a_{\nu_0} < b_{\nu_0}$. Then

$$T_{\nu_0}(A, B) = (A, B'),$$

where

$$B : B' = H_{\nu_0} : H_{\nu_0+1}.$$

Also, since $\nu_0 \bar{\in} I$, $H_{\nu_0} \equiv H_{\nu_0+1}$. Hence, since Δ is invariant, $B \equiv B'$. But $\rho(A, B') < \rho(A, B)$, and therefore, using the induction hypothesis, $A \equiv B' \equiv B$.

(β) Suppose that (8) holds, and $A \equiv B$.

We have to prove that $a_\nu = b_\nu$ $(\nu \in I)$.

Suppose that, on the contrary, there exists $\nu' \in I$ such that $a_{\nu'} \neq b_{\nu'}$. Let ν_0 be the smallest of such indices ν'. Put

$$T_{\nu_0-1} T_{\nu_0-2} \dots T_1(A, B) = (C, D).$$

This means that, in particular, $A = C$, $B = D$, if $\nu_0 = 1$. Then, using the definition of I, we deduce that $C \equiv A \equiv B \equiv D$. Let

$$C = \{c_1, \dots, c_n\}_<; \qquad D = \{d_1, \dots, d_n\}_<.$$

Then

$$c_\nu = d_\nu \quad (\nu < \nu_0); \qquad c_{\nu_0} \neq d_{\nu_0},$$

and we may assume that $c_{\nu_0} < d_{\nu_0}$. Since $N > 2n$, it is possible to find sets E and F such that

$$E = \{e_1, \dots, e_n\}_< \subset S - \{N\}; \qquad F = \{f_1, \dots, f_n\}_< \subset S - \{N\},$$
$$C : D = E : F.$$

Then

$$e_\nu = f_\nu \quad (\nu < \nu_0); \qquad e_{\nu_0} < f_{\nu_0}.$$

Also, by the invariance of Δ, $E \equiv F$. Put

$$E' = \{e_1, \dots, e_{\nu_0}, e_{\nu_0+1} + 1, e_{\nu_0+2} + 1, \dots, e_n + 1\},$$
$$F' = \{f_1, \dots, f_{\nu_0-1}, f_{\nu_0} + 1, f_{\nu_0+1} + 1, \dots, f_n + 1\}.$$

Then E' and F' are obtained from E and F respectively by an application of the mapping

$$x \to x \quad (x \leqslant e_{\nu_0}); \qquad x \to x + 1 \quad (x > e_{\nu_0}),$$

and therefore we have $E : F = E' : F'$. Hence, by the invariance of Δ, $E' \equiv F'$.

Similarly, if

$$E'' = \{e_1, \dots, e_{\nu_0-1}, e_{\nu_0} + 1, e_{\nu_0+1} + 1, \dots, e_n + 1\},$$

then $E : F = E'' : F'$, since E'' and F' are the images of E and F respectively under the mapping

$$x \to x \quad (x < e_{\nu_0}); \qquad x \to x + 1 \quad (x \geqslant e_{\nu_0}).$$

Hence $E'' \equiv F'$, and so, finally, $E' \equiv F' \equiv E''$. But

$$E' : E'' = H_{\nu_0+1} : H_{\nu_0},$$

and therefore, by the invariance of Δ, $H_{\nu_0+1} \equiv H_{\nu_0}$, i.e. $\nu_0 \bar{\in} I$, which is the desired contradiction. This proves Theorem 2.

An estimate of the generalized Ramsey function $R^*(n, N)$

THEOREM 3. *Given any integers n and N such that $1 \leqslant n \leqslant N$, there exists a positive integer M^* having the following property. If $S = \{1, 2,..., M^*\}$, and if Δ is any distribution of $\Omega_n(S)$ into any number of classes, then there is $S^* \subset S$, where $|S^*| = N$, such that Δ is invariant in $\Omega_n(S^*)$. If, in addition $N > 2n$, then Δ is canonical in $\Omega_n(S^*)$. If $R^*(n, N)$ is the least possible value of M^*, then*

$$R^*(n, N) \leqslant R(2^{\frac{1}{2}\binom{2n}{n}[\binom{2n}{n}-1]}, 2n, N+n-1), \tag{9}$$

where R is Ramsey's function.

Proof. Let M^* be the number on the right-hand side of (9), and

$$S = \{1,..., M^*\}.$$

Consider any distribution Δ of $\Omega_n(S)$. We define $\Delta'(X)$ for $X \in \Omega_{2n}(S)$ as follows. We put $E \equiv F\ (.\Delta')$ if and only if $E, F \in \Omega_{2n}(S)$, and the following condition holds. Whenever

$$\left.\begin{array}{c} A,\, B \in \Omega_n(E); \qquad C,\, D \in \Omega_n(F) \\ A:B:E = C:D:F \end{array}\right\}, \tag{10}$$

then the two relations

$$A \equiv B\ (.\Delta); \qquad C \equiv D\ (.\Delta) \tag{11}$$

are either both true or both false.

For fixed E the number of unordered pairs A, B is $n' = \frac{1}{2}\binom{2n}{n}\left[\binom{2n}{n}-1\right]$. Hence $|\Delta'| \leqslant 2^{n'}$, and therefore, by definition of M^*, there is

$$S' = \{a_1, a_2,..., a_{N+n-1}\}_< \subset S$$

such that $|\Delta'| = 1$ in $\Omega_{2n}(S')$. Then the set $S^* = \{a_1,..., a_N\}$ has the required property. For suppose that A, B, C, $D \in \Omega_n(S^*)$,

$$A \equiv B\ (.\Delta); \qquad A:B = C:D. \tag{12}$$

Then $|A+B| = |C+D| = m$, say, where $n \leqslant m \leqslant 2n$. If $m = n$, then $C = D$, and so $C \equiv D\ (.\Delta)$. If $m > n$, then we put

$$E = A+B+\{a_{N+1}, a_{N+2},..., a_{N+2n-m}\},$$
$$F = C+D+\{a_{N+1}, a_{N+2},..., a_{N+2n-m}\}.$$

Then E, $F \in \Omega_{2n}(S')$, and hence, by choice of S', $E \equiv F\ (.\Delta')$. Now, (10) holds and the first relation (11) is true. Therefore, by definition of Δ', both relations (11) are true.

Hence, in any case, (12) implies (11), and so Δ is invariant. The rest of Theorem 3 follows from Theorem 2.

Some more definitions. Transfinite extensions of Ramsey's theorem

We consider a fixed, non-empty, set S of real numbers. The letters A, B, X, Y denote subsets of S, and always $|A|, |B| > \aleph_0$. The *order type* of X under the natural order by magnitude is denoted by $\bar{X}$. The letters α, β, μ, ν, denote ordinal numbers, and always

$$0 \leqslant |\alpha|, |\beta| \leqslant \aleph_0; \qquad 0 \leqslant |\mu|, |\nu| \leqslant 2^{\aleph_0},$$

and the letters m, n denote positive finite ordinal numbers. By ϕ is denoted any order type of cardinal $|\phi| \leqslant \aleph_0$, and ϕ^* denotes the *converse* of ϕ, i.e. the order type obtained from ϕ by replacing every relation $x < y$ by the corresponding relation $x > y$. We recall that, according to the definition of multiplication of order types, $(\phi_1 \phi_2)^* = \phi_1^* \phi_2^*$. As usual, ω denotes the least denumerable ordinal number.

We shall be concerned with a fixed distribution Δ of $\Omega_n(S)$ into non-empty classes K_λ. In the special case $n = |\Delta| = 2$ we denote the classes of Δ, in some fixed order, by K_1, K_2, and we define certain sets and numbers as follows.

$$F_\lambda(A) = \sum X \subset A;\ \Omega_2(X) \subset K_\lambda \,\square\, \{\bar{X}\},$$
$$F'_\lambda(A) = \prod B \subset A \,\square\, F_\lambda(B),$$
$$f_\lambda(A) = \min \mu \mathrel{\bar{\in}} F_\lambda(A) \,\square\, \mu,$$
$$f'_\lambda(A) = \min B \subset A \,\square\, f_\lambda(B).$$

Thus $F_\lambda(A)$ is the set of all order types ϕ which are 'realizable' by some suitable subset of A all of whose subsets of two elements belong to K_λ, and $F'_\lambda(A)$ is the set of all those ϕ which are even realizable in every non-denumerable subset of A. Also, $f_\lambda(A)$ is the least ordinal number which is not realizable in A, and $f'_\lambda(A)$ is the least ordinal number which is not realizable in every non-denumerable subset of A.†

It follows from the definitions that, if $A \subset B$, then

$$1 \in F'_\lambda(B) \subset F'_\lambda(A) \subset F_\lambda(A) \subset F_\lambda(B),$$
$$2 \leqslant f'_\lambda(B) \leqslant f'_\lambda(A) \leqslant f_\lambda(A) \leqslant f_\lambda(B).$$

It is well known that corresponding to every ϕ there is a set X such that $\bar{X} = \phi$. Furthermore, if ϕ_0 denotes the order type of the set of all rational numbers, then $\phi_0 \in F_\lambda(A)$ implies that every $\phi \in F_\lambda(A)$.

THEOREM 4. *If S is the set of all rational numbers, and $n = |\Delta| = 2$, then either* (i) $\omega \in F_1(S)$, *or* (ii) $\omega^* \in F_1(S)$, *or* (iii) *every* $\phi \in F_2(S)$.

† Ordinal numbers realizable in a given ordered set were studied by J. C. Shepherdson, *Proc. London Math. Soc.* (3), 1 (1951), 291–307.

A common hypothesis of Theorems 5–8 is $|S| > \aleph_0$; $n = |\Delta| = 2$.

THEOREM 5. *Either* (i) $\omega \in F_1(S)$, *or* (ii) *every* $\alpha \in F_2(S)$.

THEOREM 6. *If $f'_1(S)$ is not a limit number, then there is A and $\mu > 0$ such that*

$$f_1(A) = f'_1(A) = f'_1(S),$$
$$f_2(A) = f'_2(A) = \omega^\mu.$$

If $\phi_\nu \in F'_2(A)$ $(\nu < \alpha)$, *then* $\left(\sum_\nu \phi^*_\nu\right)^* \in F'_2(A)$. *If* $\phi \in F'_2(A)$, *then every* $\phi\alpha^* \in F'_2(A)$. *Finally, every* $\alpha^* \in F'_2(A)$.

THEOREM 7. *Either* (i) *every* $\omega + m \in F_1(S)$, *or* (ii) *every* $\omega . m \in F_2(S)$.

THEOREM 8. *Either* (i) $\omega + \omega^* \in F_1(S)$, *or* (ii) *every* $\alpha \in F_2(S)$, *or* (iii) *every* $\alpha^* \in F_2(S)$.

THEOREM 9. *If $|S| > \aleph_0$, and n and Δ are arbitrary, but such that, for every $a \in S$,*

$$|\sum \{a, x_2, \ldots, x_n\}_< \in K_\lambda \square \{\lambda\}| < \aleph_0,\dagger$$
$$|\sum \{x_1, \ldots, x_{n-1}, a\}_< \in K_\lambda \square \{\lambda\}| < \aleph_0,$$

then there is $S' \subset S$ such that $|S'| = \aleph_0$, and $|\Delta| = 1$ in $\Omega_n(S')$.

THEOREM 10. *If $|S| > \aleph_0$, and n and Δ are arbitrary, but such that, for every $\{a_1, \ldots, a_{n-1}\}_< \subset S$,*

$$|\sum \{a_1, \ldots, a_{n-1}, x\}_< \in K_\lambda \square \{\lambda\}| \leqslant \aleph_0,$$

then there is a set $S' = \{x_1, x_2, \ldots\}_< \subset S$ such that in $\Omega_n(S')$ Δ is canonical and, in fact, $\Delta = \Delta^{(k)}_{\nu_1 \ldots \nu_k}$, where $0 \leqslant k \leqslant n-1$; $1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k \leqslant n-1$.

The following theorem of Dushnik and Miller (**7**) belongs to the group of Theorems 4–10.

If $|S| > \aleph_0$, *then either* (i) $\omega \in F_1(S)$, *or* (ii) $\omega^* \in F_1(S)$, *or* (iii) $F_2(S)$ *contains some non-denumerable order type.*

Proofs of Theorems 4–10

Proof of Theorem 4. The letter I denotes open intervals. A set X is called *i-dense* if there exists some I such that X is dense in I.

LEMMA 1. *If neither X nor Y is i-dense then $X+Y$ is not i-dense.*

For, given any I, there is $I_1 \subset I$ such that $XI_1 = 0$, and then there is $I_2 \subset I_1$ such that $YI_2 = 0$. Then $(X+Y)I_2 = 0$.

We now prove the theorem. We introduce the notation

$$L_\lambda(a) = \sum \{x, a\}_< \in K_\lambda \square \{x\}; \qquad R_\lambda(a) = \sum \{a, x\}_< \in K_\lambda \square \{x\}, \quad (13)$$
$$U_\lambda(a) = L_\lambda(a) + R_\lambda(a).$$

† Here, as in similar cases later on, the summation indices or symbols are all symbols which are not stated to have fixed values. Thus, in the present case, summation extends over varying $x_2, \ldots, x_n, \lambda$.

A set $S' \subset S$ is said to be *of type* 1 if there is $a_0 \in S'$ such that $S'U_1(a_0)$ is i-dense, and *of type* 2 if there is no such a_0.

Case 1. Suppose that every i-dense set $S' \subset S$ is of type 1. Then there is $a_0 \in S_0 = S$ such that $S_1 = S_0 U_1(a_0)$ is i-dense. Generally, if S_m has already been defined for some m, and if S_m is i-dense, then there is $a_m \in S_m$ such that $S_{m+1} = S_m U_1(a_m)$ is i-dense. Then $S_0 \supset S_1 \supset \dots$, and if $0 \leqslant r < s$, then

$$a_s \in S_s \subset S_{r+1} = S_r U_1(a_r); \qquad \{a_r, a_s\} \in K_1.$$

Since every sequence contains a monotonic subsequence—this is, in fact, a simple case of Ramsey's theorem—there is a set $\{b_1, b_2, \dots\} \subset \{a_1, a_2, \dots\}$ such that either $b_1 < b_2 < \dots$ or $b_1 > b_2 > \dots$. Hence either (i) or (ii) of Theorem 4 holds.

Case 2. There is a set $S' \subset S$ which is i-dense and of type 2.

Then we can choose I such that S' is dense in I. We choose $I_\nu \subset I$ $(\nu = 1, 2, \dots)$ such that the I_ν are dense in I, which means that if $I' \subset I$ then $I_\nu \subset I'$ for some ν. Then we can find $x_1 \in I_1 S'$. Suppose that for some m the numbers $x_1, \dots, x_m$ have already been defined, and that, if $m > 1$,

$$x_{\nu+1} \in I_{\nu+1} S' U_2(x_1) U_2(x_2) \dots U_2(x_\nu) \quad (0 < \nu < m).$$

Then none of the sets $S'U_1(x_\nu)$ $(1 \leqslant \nu \leqslant m)$ is i-dense, and therefore, by Lemma 1, the set

$$\{x_1, \dots, x_m\} + \sum 1 \leqslant \nu \leqslant m \,\square\, S'U_1(x_\nu)$$

is not i-dense. Hence it is possible to choose

$$x_{m+1} \in I_{m+1} S' U_2(x_1) U_2(x_2) \dots U_2(x_m).$$

This process defines the sequence $x_1, x_2, \dots$ which is dense in I and satisfies

$$\{x_r, x_s\} \in K_2 \quad (1 \leqslant r < s).$$

Thus (iii) of Theorem 4 holds, and the proof is completed.

Proof of Theorem 5. If $a < b$, then we denote by (a, b) the open interval with ends a and b. A set A is called a *full set* if for $a < b < c$; $b \in A$ always

$$|(a, b)A| > \aleph_0; \qquad |(b, c)A| > \aleph_0.$$

LEMMA 2. *Given any A, there is a full set $B \subset A$.*

Proof of Lemma 2. Put, for $\delta > 0$,

$$A_-(\delta) = \sum a \in A;\ |(a-\delta, a)A| \leqslant \aleph_0 \,\square\, \{a\};$$

$$A_+(\delta) = \sum a \in A;\ |(a, a+\delta)A| \leqslant \aleph_0 \,\square\, \{a\}.$$

Let $a \in A_-(\delta)$. We shall prove that a is not a point of condensation of $A_-(\delta)$. If $(a, a+\delta)A_-(\delta) = 0$, then

$$|(a-\delta, a+\delta)A_-(\delta)| \leqslant |(a-\delta, a)A| \leqslant \aleph_0,$$

and our assertion is proved. If there is $b \in (a, a+\delta)A_-(\delta)$, then

$$a-\delta < b-\delta < a < b,$$

$$|(b-\delta, b)A_-(\delta)| \leqslant |(a-\delta, a)A| + |(b-\delta, b)A| \leqslant \aleph_0,$$

and the same conclusion follows. Therefore no point of $A_-(\delta)$ is a point of condensation of $A_-(\delta)$, and so $|A_-(\delta)| \leqslant \aleph_0$. Similarly, $|A_+(\delta)| \leqslant \aleph_0$. Then the set

$$B = A - \sum_m \left\{A_-\left(\frac{1}{m}\right) + A_+\left(\frac{1}{m}\right)\right\}$$

is non-denumerable and full, and the lemma follows.

We now prove Theorem 5. Let us suppose that $\omega \mathrel{\overline{\in}} F_1(S)$. Then there is A such that

$$|AR_1(a)| \leqslant \aleph_0 \quad (a \in A), \tag{14}$$

where $R_1(a)$ is defined by (13). For otherwise we could define sequences A_m, a_m as follows. Put $A_0 = S$. Then, since (14) is false for $A = A_0$, there is $a_0 \in A_0$ such that $|A_1| > \aleph_0$, where $A_1 = A_0 R_1(a_0)$. Then there is $a_1 \in A_1$ such that $|A_2| > \aleph_0$, where $A_2 = A_1 R_1(a_1)$. Generally,

$$a_m \in A_m = A_{m-1} R_1(a_{m-1}) \quad (m = 1, 2, \ldots).$$

If we put $X = \{a_0, a_1, \ldots\}$, then $\overline{X} = \omega; \Omega_2(X) \subset K_1$, which is a contradiction against $\omega \mathrel{\overline{\in}} F_1(S)$. Hence (14) holds for some suitable A. By Lemma 2 we can find a full set $B \subset A$.

Now consider any ordinal $\alpha \geqslant \omega$. We can choose a set $T = \{t_1, t_2, \ldots\}_{\neq}$ of real numbers such that $\overline{T} = \alpha$. Then we can successively find intervals $I_1, I_2, \ldots$ such that

$$\left.\begin{aligned} I_m = (b_m, c_m); \qquad & I_r I_s = 0 \quad (r \neq s) \\ \overline{\{b_1, b_2, \ldots, b_m\}} &= \overline{\{t_1, t_2, \ldots, t_m\}} \\ |I_m B| > \aleph_0 \quad & (m = 1, 2, \ldots) \end{aligned}\right\}. \; *) \tag{15}$$

For if $I_1, \ldots, I_{m-1}$ have already been chosen, then we can choose $d \in B$ such that

$$\overline{\{b_1, \ldots, b_{m-1}, d\}} = \overline{\{t_1, \ldots, t_{m-1}, t_m\}} \quad **)$$

and then put $I_m = (d-\epsilon, d+\epsilon)$ for a sufficiently small ϵ. Then

$$\overline{\{b_1, b_2, \ldots\}} = \alpha,$$

and therefore there is a one-to-one mapping of the set of all ν in the range $0 \leqslant \nu < \alpha$ onto the set of intervals $I_1, I_2, \ldots$ in such a way that, if the interval associated with ν is $J_\nu = (b'_\nu, c'_\nu)$, then

$$b'_\mu < c'_\mu < b'_\nu < c'_\nu$$

whenever $0 \leqslant \mu < \nu < \alpha$. Now, using (14) and the definition of J_ν, we can find, by transfinite construction, numbers $x_\nu \in J_\nu B$ such that

$$\{x_\mu, x_\nu\} \in K_2 \quad (0 \leqslant \mu < \nu < \alpha).$$

Then, putting $X = \sum \{x_\nu\}$, we have $\bar{X} = \alpha$; $\Omega_2(X) \subset K_2$, and therefore $\alpha \in F_2(S)$. This proves Theorem 5.

Proof of Theorem 6. There is $A_1 \subset S$ such that $f'_1(S) = f_1(A_1)$, and $A \subset A_1$ such that $f'_2(A_1) = f_2(A)$. Then

$$f_1(A) \leqslant f_1(A_1) = f'_1(S) \leqslant f'_1(A) \leqslant f_1(A),$$
$$f_2(A) = f'_2(A_1) \leqslant f'_2(A) \leqslant f_2(A),$$

and therefore $f_1(A) = f'_1(A) = f'_1(S)$; $f_2(A) = f'_2(A)$. Now, by hypothesis, $f'_1(S) = \nu_0 + 1$. Then

$$|AL_1(a)| \leqslant \aleph_0 \quad (a \in A). \tag{16}$$

For otherwise we could find $a_0 \in A$ such that $|AL_1(a_0)| > \aleph_0$. Then, since $\nu_0 < f'_1(S) = f'_1(A)$, there is $X_0 \subset AL_1(a_0)$ such that $\bar{X}_0 = \nu_0$; $\Omega_2(X_0) \subset K_1$. Then, putting $X_0 + \{a_0\} = X_1$,

$$\bar{X}_1 = \nu_0 + 1 = f_1(A); \qquad X_1 \subset A; \qquad \Omega_2(X_1) \subset K_1,$$

which is a contradiction against the definition of $f_1(A)$.

Let α be arbitrary, and suppose that $\phi_\nu \in F'_2(A)$ $(\nu < \alpha)$. In order to prove that $(\sum \phi^*_\nu)^* \in F'_2(A)$, let us consider any set $A_2 \subset A$. By Lemma 2 there is a full set $B \subset A_2$. Then we can choose T, t_m, I_m, b_m, c_m, J_ν, b'_ν, c'_ν exactly as in the proof of Theorem 5, so that (15) holds, but with α^* in place of α. Then, using (16) and the definition of J_ν, we can find, by transfinite construction, sets $X_\nu \subset J_\nu B$ such that $\bar{X}_\nu = \phi_\nu$; $\Omega_2(X) \subset K_2$, where $X = \sum \nu < \alpha \square X_\nu$; $\bar{X} = (\sum \phi^*_\nu)^*$. Hence

$$(\sum \phi^*_\nu)^* \in F_2(B) \subset F_2(A_2).$$

Since A_2 is arbitrary, this implies that $(\sum \phi^*_\nu)^* \in F'_2(A)$. If, in particular, $\phi_\nu = \phi$ for all $\nu < \alpha$, then

$$\phi\alpha^* = (\phi^*\alpha)^* = (\sum \phi^*_\nu)^* \in F'_2(A). \tag{17}$$

Finally, since $1 \in F'_2(A)$, we may put $\phi = 1$ and find $\alpha^* \in F'_2(A)$.

There only remains to prove the statement about μ. It is sufficient to show that $f'_2(A)$ is of the form ω^μ, since then, in view of $f'_2(A) \geqslant 2$, we have $\mu > 0$.

$f'_2(A)$, being a non-zero ordinal number, can be written in the form

$$f'_2(A) = \omega^\mu(\mu_1 + 1).$$

Then $\omega^\mu \mu_1 < f'_2(A)$, and therefore, by applying (17) to $\phi = \omega^\mu \mu_1$; $\alpha = 2$, we find

$$\omega^\mu(\mu_1 + \mu_1) < f'_2(A) = \omega^\mu(\mu_1 + 1),$$

i.e. $\mu_1 = 0$. This completes the proof of Theorem 6.

Proof of Theorem 7. We assume that

$$\omega + m_1 \,\bar{\in}\, F_1(S); \qquad \omega . m_2 \,\bar{\in}\, F_2(S),$$

where m_1 and m_2 are given numbers, and we shall deduce a contradiction.

We have, for any A_1, $\omega m_2 \bar{\in} F_2(A_1)$, and therefore, by Theorem 5, $\omega \in F_1(A_1)$. As A_1 is arbitrary, $\omega \in F_1'(S)$, and so

$$\omega < f_1'(S) \leqslant f_1(S) \leqslant \omega + m_1.$$

Hence it follows that $f_1'(S)$ is not a limit number, and by Theorem 6 there is A and $\mu > 0$ such that

$$\omega^\mu = f_2(A) \leqslant f_2(S) \leqslant \omega m_2.$$

Then $\mu = 1$, $\omega \bar{\in} F_2(A)$, and a second application of Theorem 5 shows that $\omega + m_1 \in F_1(A)$ which is the desired contradiction. This proves Theorem 7.

Proof of Theorem 8. We require the following lemma.

LEMMA 3. *If the hypothesis of Theorem* 8 *holds, and*

$$|R_1(a)| \leqslant \aleph_0 \quad (a \in S), \tag{18}$$

then every $\alpha \in F_2'(S)$.

Proof of the lemma. Let $\beta > 1$, and assume that $\alpha \in F_2'(S)$ for every $\alpha < \beta$. We have $\sum 0 < \alpha < \beta \,\square\, \{\alpha\} = \{\alpha_1, \alpha_2, \ldots\}$. Let A be arbitrary. By Lemma 2 there is a full set $B \subset A$. Then there are numbers $b_m, c_m \in B$ such that $b_1 < c_1 < b_2 < c_2 < \ldots$. Put $I_m = (b_m, c_m)$. Since $\alpha_1 \in F_2'(S)$, there is $B_1 \subset BI_1$ such that $\bar{B}_1 = \alpha_1$; $\Omega_2(B_1) \subset K_2$. Generally, if $B_1, \ldots, B_{m-1}$ have already been found for some m, then we can find B_m such that

$$B_m \subset BI_m - BI_m \sum_{a \in B_r}^{1 \leqslant r < m} \square\, R_1(a),$$

$$\bar{B}_m = \alpha_m; \qquad \Omega_2(B_m) \subset K_2 \quad (m = 1, 2, \ldots).$$

Here we make use of (18). If $B' = \sum B_m$, then

$$B' \subset A; \qquad \bar{B}' = \sum \alpha_m \geqslant \beta; \qquad \Omega_2(B') \subset K_2.$$

Hence $\beta \in F_2(A)$. Since A is arbitrary, this implies $\beta \in F_2'(S)$, and the lemma follows by transfinite induction.

COROLLARY. By applying Lemma 3 to the set $S_1 = \sum x \in S \,\square\, \{-x\}$ we find that *if* $|L_1(b)| \leqslant \aleph_0$ $(b \in S)$, *then every* $\alpha^* \in F_2'(S)$.

We now turn to the proof of Theorem 8. Let us assume that both (ii) and (iii) of Theorem 8 are false. Then, by Lemma 3 and its corollary, given any $T \subset S$, $|T| > \aleph_0$, there are numbers $a, b \in T$ such that $|TR_1(a)| > \aleph_0$; $|TL_1(b)| > \aleph_0$. By means of repeated applications of this result we find numbers a_m, b_m and sets A_0, A_m, B_m such that $A_0 = S$,

$$a_1 \in A_0; \qquad B_1 = A_0 R_1(a_1),$$

$$b_1 \in B_1; \qquad A_1 = B_1 L_1(b_1),$$

and, generally,

$$a_m \in A_{m-1}; \qquad B_m = A_{m-1} R_1(a_m),$$

$$b_m \in B_m; \qquad A_m = B_m L_1(b_m).$$

Then $A_m \subset B_m \subset A_{m-1}$. If $0 < r < s$, then

$$a_s \in A_{s-1} \subset A_r \subset B_r \subset R_1(a_r); \qquad \{a_r, a_s\}_< \in K_1,$$

$$b_s \in B_s \subset B_{r+1} \subset A_r \subset L_1(b_r); \qquad \{b_s, b_r\}_< \in K_1,$$

$$a_s \in A_{s-1} \subset A_r \subset L_1(b_r); \qquad \{a_s, b_r\}_< \in K_1,$$

$$b_{s-1} \in B_{s-1} \subset B_r \subset R_1(a_r); \qquad \{a_r, b_{s-1}\}_< \in K_1.$$

Hence, if $X = \sum \{a_m, b_m\}$, then $\bar{X} = \omega + \omega^*$; $\Omega_2(X) \subset K_1$, and therefore $\omega + \omega^* \in F_1(S)$. This proves Theorem 8.

Proof of Theorem 9. Let $S(m)$ be the set of all $a \in S$ such that

$$|\sum \{a, x_2, ..., x_n\}_< \in K_\lambda \square \{\lambda\}| \leqslant m,$$

$$|\sum \{x_1, ..., x_{n-1}, a\}_< \in K_\lambda \square \{\lambda\}| \leqslant m.$$

Then $|\sum S(m)| = |S| > \aleph_0$, and therefore $|S(m_0)| > \aleph_0$ for some suitable m_0. An application of Lemma 2 shows that $S(m_0)$ contains an increasing sequence, and by applying to this sequence the generalized Ramsey theorem of (4) we find a set $S' = \{a_1, a_2, ...\}_< \subset S(m_0)$ such that in $\Omega_n(S')$ the distribution Δ is canonical, say $\Delta = \Delta^{(k)}_{\nu_1 \ldots \nu_k}$, where $0 \leqslant k \leqslant n$; $1 \leqslant \nu_1 < \nu_2 < ... < \nu_k \leqslant n$. If $k > 0$ and $\nu_k > 1$, then

$$\{a_1, a_{r+2}, ..., a_{r+n}\} \not\equiv \{a_1, a_{s+2}, ..., a_{s+n}\} \quad (.\Delta) \quad (0 \leqslant r < s \leqslant m_0),$$

which is a contradiction of the definition of $S(m_0)$. If $k > 0$ and $\nu_k = 1$, then $k = 1$,

$$\{a_r, a_{m_0+2}, ..., a_{m_0+n}\} \not\equiv \{a_s, a_{m_0+2}, ..., a_{m_0+n}\} \quad (.\Delta) \quad (1 \leqslant r < s \leqslant m_0+1),$$

which is again a contradiction of the definition of $S(m_0)$. Hence $k = 0$, and Theorem 9 follows.

Remark. The hypothesis $|S| > \aleph_0$ cannot be replaced by $|S| = \aleph_0$, as is shown by the distribution $\Delta_1^{(1)}$ in $\Omega_n(S)$, where $S = \{1, 2, ...\}$. In this case, for $a \in S$,

$$|\sum \{a, x_2, ..., x_n\}_< \in K_\lambda \square \{\lambda\}| = 1,$$

$$|\sum \{x_1, ..., x_{n-1}, a\}_< \in K_\lambda \square \{\lambda\}| = \max(0, a-n+1).$$

Proof of Theorem 10. We choose a full set $A \subset S$, and we put

$$A_1 = A_2 = ... = A_{n-1} = A.$$

Choose $\{a_1, ..., a_{n-1}\}_< \subset A$. Then there is a full set $A_n \subset A_{n-1}$ such that

$$\{a_1, ..., a_{n-1}, x\}_< \in K(1, 2, ..., n-1) \quad (x \in A_n),$$

where, generally, $K(\rho_1, ..., \rho_{n-1})$ denotes one of the K_λ. We choose $a_n \in A_n$. Generally, if $m \geqslant n$, and if the a_r and A_r have been chosen for

$$1 \leqslant r \leqslant m-1,$$

we can find a full set $A_m \subset A_{m-1}$ such that

$$\{a_{\rho_1}, \ldots, a_{\rho_{n-1}}, x\}_< \in K(\rho_1, \ldots, \rho_{n-1})$$

for $1 \leqslant \rho_1 < \ldots < \rho_{n-1} < m$; $x \in A_m$.

We now choose $a_m \in A_m$. This defines a sequence a_m such that

$$\{a_{\rho_1}, a_{\rho_2}, \ldots, a_{\rho_n}\}_< \in K(\rho_1, \ldots, \rho_{n-1}) \quad (1 \leqslant \rho_1 < \rho_2 < \ldots < \rho_n). \tag{19}$$

By the generalized Ramsey theorem there is a set

$$S' = \{x_1, x_2, \ldots\}_< \subset \{a_1, a_2, \ldots\}$$

such that in $\Omega_n(S')$ the distribution Δ is canonical, say $\Delta = \Delta^{(k)}_{\nu_1 \ldots \nu_k}$. Then, by (19), $k < n$; $1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k < n$, and Theorem 10 is proved.

Some counter-examples for extensions of Ramsey's theorem

1. Ramsey's theorem makes an assertion about distributions of $\Omega_n(S)$ when $n < \aleph_0$, $|\Delta| < \aleph_0$. The question arises how much of the theorem remains true when $n = \aleph_0$. The following example shows that there is very little scope for a reasonable extension in such a direction, even for $|\Delta| = 2$ and arbitrarily large cardinals $|S|$. In this final section the conventions about the use of the letters S, A, B are no longer valid.

EXAMPLE 1. *Let $|S| \geqslant \aleph_0$. Then there exists a distribution of the set T of all infinite subsets of S into two classes such that, given any $S' \in T$, there are infinite subsets A, B of S' satisfying $A \not\equiv B$ $(.\Delta)$.*

Proof. Let $X < Y$ be a well-ordering relation of the set T. Let K_1 be the set of all $X \in T$ such that $X' < X$ for at least one $X' \subset X$, and put $K_2 = T - K_1$. Then the distribution Δ whose classes are K_1 and K_2 has the required property. For let $S' \in T$, and let A be the first infinite subset of S'. Then, whenever $A' \subset A$ and $A' \in T$, we have $A' \subset A \subset S'$ and hence, by definition of A, $A \leqslant A'$. Therefore $A \in K_2$. We can write

$$A = \{a_1, a_2, a_3, \ldots\}_{\neq} + C,$$

where, for all m, $a_m \,\bar{\in}\, C$. Put $A_m = \{a_2, a_4, \ldots, a_{2m}, a_1, a_3, a_5, \ldots\} + C$. Then there exists $\min A_m = A_{m_0}$, and we have

$$A_{m_0} < A_{m_0+1}; \qquad A_{m_0} \subset A_{m_0+1}.$$

Then $B = A_{m_0+1} \in K_1$, and we have $A \not\equiv B(.\Delta)$. This proves the assertion.

Clearly, in Example 1 we may replace the set T of all infinite subsets of S by the set of all subsets of S having a given fixed infinite cardinal not exceeding $|S|$.

2. Let $|S| = \aleph_0$, and let Δ be a distribution of the set of all finite subsets of S. Suppose that $|\Delta| < \aleph_0$. Then, by means of N successive applications of Ramsey's theorem it is easy to find an infinite set $S'_N \subset S$ such that, for every $n \leqslant N$, we have $|\Delta| = 1$ in $\Omega_n(S'_N)$. The question arises whether the

set S'_N can be chosen so that it is independent of N, say $S'_N = S'$ for all N, so that now simultaneously for all n we have $|\Delta| = 1$ in $\Omega_n(S')$. The following example shows that such a set S' need not exist.

EXAMPLE 2. *If* $|S| = \aleph_0$ *then there exists a distribution* Δ *of the set* $T = \sum_n \Omega_n(S)$, *where* $|\Delta| = 2$, *such that the following condition holds. If* $S' \in \Omega_{\aleph_0}(S)$ *then, for every sufficiently large* n, *there are sets* A, $B \in \Omega_n(S')$ *such that* $A \not\equiv B$ $(.\Delta)$.

Proof. Let $S = \{1, 2, 3,...\}$. Denote by K_1 the set of all $A \in T$ such that $|A| \geqslant m \in A$ for at least one m, and put $K_2 = T - K_1$. Let Δ be the distribution whose classes are K_1 and K_2. Now suppose that $S' = \{a_1, a_2,...\}_< \subset S$. Then, for $n \geqslant a_1$, we have

$$\{a_1, a_2,..., a_n\} \not\equiv \{a_{n+1}, a_{n+2},..., a_{2n}\} \quad (.\Delta).$$

3. The last example quite naturally leads to the question whether a distribution exists which has a property similar to that described in Example 2, but with respect to a set S which is not denumerable. This question has only been decided for $|S| \leqslant 2^{\aleph_0}$, and the general case seems to be well worth studying. In the case $|S| \leqslant 2^{\aleph_0}$ several examples of distributions are known which have the required property. In order to throw more light on the problem we give three such examples, in the hope that perhaps one of the methods used might turn out to be capable of an extension to cardinals exceeding that of the continuum.

EXAMPLE 3A. *If* $|S| = 2^{\aleph_0}$, *then there exists a distribution* Δ *of the set* $T = \sum_n \Omega_n(S)$, *where* $|\Delta| = 2$, *such that the following condition holds. If* $S' \in \Omega_{\aleph_0}(S)$, *then, for infinitely many* n, *there are sets* A, $B \in \Omega_n(S')$ *such that*

$$A \not\equiv B \ (.\Delta).$$

Proof. Let S be the set of all sequences $a = (a^{(1)}, a^{(2)},...)$, where $a^{(r)} \in \{0, 1\}$. Let K_1 be the set of all sets $\{a_1, a_2,..., a_m\}_{\neq} \subset S$ such that

$$a_l = (a_l^{(1)}, a_l^{(2)},...); \qquad a_1^{(m)} = a_2^{(m)} = ... = a_m^{(m)},$$

and put $K_2 = T - K_1$. Now consider any set $S' = \{b_1, b_2,...\}_{\neq} \subset S$, where $b_l = (b_l^{(1)}, b_l^{(2)},...)$ $(l = 1, 2,...)$. Then there are infinitely many $r > 1$ such that integers s, t can be found satisfying $1 \leqslant s < t$; $b_s^{(r)} \neq b_t^{(r)}$. For every such choice of r, s, t we have

$$A = \{b_s, b_t, b_{t+1},..., b_{t+r-2}\} \in K_2.$$

Furthermore, among the $2r-1$ numbers $b_1^{(r)}$, $b_2^{(r)}$,..., $b_{2r-1}^{(r)}$ there are r which are equal to each other. Hence there is a set $B = \{b_{s_1}, b_{s_2},..., b_{s_r}\} \in K_1$ for suitable s_ρ such that $1 \leqslant s_1 < s_2 < ... < s_r \leqslant 2r-1$.

The following example is due to N. G. de Bruijn.

EXAMPLE 3B. *Let* $|S| = 2^{\aleph_0}$. *Then there exists a distribution* Δ *of the set* $T = \sum_n \Omega_n(S)$, *where* $|\Delta| = 2$, *such that the following condition holds. If* $\{a, b\}_{\neq} \subset S$, *then there is a positive integer* $n = n(a, b)$ *such that*

$$\{a\}+C \not\equiv \{b\}+C \ (.\Delta) \quad \textit{for all } C \in \Omega_{n-1}(S-\{a,b\}).$$

If $\{a_1, a_2, ...\}_{\neq} \subset S$, *then the numbers* $n(a_s, a_t)$ *are unbounded for* $1 \leqslant s < t$.

Proof. Define S as in Example 3A, but modify the definition of K_1 as follows. Let K_1 be the set of all $\{a_1, ..., a_m\}_{\neq} \subset S$ such that $a_1^{(m)}+a_2^{(m)}+...+a_m^{(m)}$ is even, and put $K_2 = T-K_1$. If we now define $n(a, b)$ to be the least number m such that $a^{(m)} \neq b^{(m)}$, then the desired conditions hold.

EXAMPLE 3C. *Let* $|S| = 2^{\aleph_0}$. *Then there exists a distribution* Δ *of the set* $T = \sum \Omega_n(S)$, *where* $|\Delta| = 2$, *such that the following condition holds. If* $S' \in \Omega_{\aleph_0}(S)$, *then, given any sufficiently large integer* m, *there are sets* $A, B \in \Omega_m(S')$ *such that* $A \not\equiv B \ (.\Delta)$.

Proof. Let S be the set of all real numbers x in the range $0 < x < 1$. Denote by K_1 the set of all sets $\{x_1, x_2, ..., x_m\}_< \subset S$ such that $m(x_m - x_1) \leqslant 1$, and put $K_2 = T-K_1$. Now let $S' = \{a_1, a_2, ...\}_{\neq} \subset S$. As is well known, there is an increasing sequence of positive integers $r_1, r_2, ...$ such that the sequence $b_m = a_{r_m}$ is monotonic. Then, for $m > |b_2 - b_1|^{-1}$, we have

$$A = \{b_1, b_2, ..., b_m\} \in K_2.$$

On the other hand, since, for fixed m, $b_{r+m} - b_{r+1} \to 0$ as $r \to \infty$, we have $B = \{b_{r+1}, b_{r+2}, ..., b_{r+m}\} \in K_1$ if r is sufficiently large.

4. Our next example, due to Sierpiński, shows that Theorems 4–8 cannot be strengthened very much.

EXAMPLE 4A. *Let* S *be a set of real numbers. Then there is a distribution of* $\Omega_2(S)$ *into two classes* K_1, K_2 *such that, in the notation defined on p. 427,*

$$F_1(S) \subset \sum_\alpha \{\alpha\}; \qquad F_2(S) \subset \sum_\alpha \{\alpha^*\}.$$

Proof. Let $a \preccurlyeq b$ be a well-ordering relation in S. Define K_1 to be the set of all $\{a, b\}_< \subset S$ such that $a \preccurlyeq b$, and put $K_2 = \Omega_2(S) - K_1$. If, now, $\Omega_2(A) \subset K_1$ then, in A, the relations $x < y$ and $x \preccurlyeq y$ are equivalent, and therefore $\bar{A}$ is an ordinal number. Similarly, if $\Omega_2(B) \subset K_2$, then $(\bar{B})^*$ is an ordinal number. In either case, the ordinal number is at most denumerable.

If, in particular, $|S| = \aleph_0$, then the same construction, which now no longer requires Zermelo's axiom, leads to a distribution of an even more special character.

EXAMPLE 4B. *Let* S *be a denumerable set of real numbers. Then there is a distribution of* $\Omega_2(S)$ *into two classes* K_1, K_2 *such that*

$$F_1(S) \subset \{\omega, 1, 2, ...\}, \qquad F_2(S) \subset \{\omega^*, 1, 2, ...\}.$$

The classes K_λ can be defined without the use of Zermelo's axiom. If, in addition, S is well ordered according to magnitude, then the stronger result $F_2(S) \subset \{1, 2, ...\}$ holds.

Proof. Let $S = \{x_1, x_2, ...\}_{\neq}$, and take as K_1 the set of all $\{x_r, x_s\}_<$ such that $r < s$, while $K_2 = \Omega_2(S) - K_1$. If we now assume that $\omega+1 \in F_1(S)$, then there is $A \subset S$ such that $\bar{A} = \omega+1$; $\Omega_2(A) \subset K_1$. Then there is $x_{m_0} \in A$ such that $x_m < x_{m_0}$; $x_m \in A$ for infinitely many m. Then, by definition of K_1, $m < m_0$ for infinitely many m, which is impossible. Similarly, if $1+\omega^* \in F_2(S)$, then $\bar{B} = 1+\omega^*$; $\Omega_2(B) \subset K_2$ for some $B \subset S$, and there is $x_{m_1} \in B$ such that $x_m > x_{m_1}$; $x_m \in B$ for infinitely many m. Then, by definition of K_2, $m < m_1$ for infinitely many m, which is impossible. This proves our assertions.

5. The construction in Example 4 A required the axiom of choice. The following example shows that a weaker result can be obtained without the use of that axiom.

EXAMPLE 5. *Let S be the set of all real numbers. Then, without the use of Zermelo's axiom, a distribution of $\Omega_2(S)$ into two classes K_1, K_2 can be defined which has the property that neither $F_1(S)$ nor $f_2(S)$ contains any order type which is dense in an interval.*

Proof. Let K_λ be the set of all sets $\{a, b\}_< \subset S$ such that

$$2^{2r+\lambda} \leqslant b-a < 2^{2r+\lambda+1}$$

for some suitable integer r, positive, negative, or zero, where $\lambda = 1$ or $\lambda = 2$. Now suppose that $\Omega_2(A) \subset K_\lambda$, and that A is dense in the open interval (a_0, b_0). Then we can find real numbers a, b and an integer r such that

$$a_0 < a < b < b_0, \qquad b-a = 2^{2r+\lambda},$$

and hence numbers a', $b' \in A$ such that

$$a < a' < b' < b; \qquad 2^{2r+\lambda-1} < b'-a' < 2^{2r+\lambda}.$$

Then $\{a', b'\} \,\bar{\in}\, K_\lambda$, and we have obtained a contradiction.

6. The next example shows that Ramsey's theorem becomes false in the case of denumerably many classes, even if we replace the hypothesis $|S| = \aleph_0$ by $|S| = 2^{\aleph_0}$.

EXAMPLE 6. *If $|S| = 2^{\aleph_0}$, then there exists a distribution Δ of $\Omega_2(S)$ into $\aleph_0$ classes such that, given any set $S' \in \Omega_{\aleph_0}(S)$, there is a sequence of sets $A_m \in \Omega_2(S')$ satisfying $A_r \not\equiv A_s \,(.\Delta)$ for $1 \leqslant r < s$.*

Proof. Let S be the set of all real numbers x in $0 < x < 1$, and denote, for $\lambda = 0, 1, 2, ...$, by K_λ the set of all sets $\{a, b\}_< \subset S$ such that

$$\lambda \leqslant (b-a)^{-1} < \lambda+1.$$

Then the distribution whose classes are the K_λ has the desired property. For if $S' \in \Omega_{\aleph_0}(S)$ then there are sets $A_m = \{a_m, b_m\}_< \subset S'$ such that $b_m - a_m \to 0$ and $A_m \in K_{\lambda_m}$; $\lambda_m \to \infty$ as $m \to \infty$.

The last example is a best-possible one, in view of the following theorem, which is a special case of (**8**), Theorem 1.

If $|S| > 2^{\aleph_0}$; $\Omega_2(S) = K_1+K_2+\ldots$, *then there is always a set* $S' \subset S$ *such that* $|S'| > \aleph_0$; $\Omega_2(S') \subset K_{\lambda'}$ *for some* λ'.

7. Our next example shows that in Theorem 10 we cannot replace the set S of real numbers by any arbitrary non-denumerable abstract set in which an order has been defined. Some properties of the order type of the real continuum are essential for the truth of Theorem 10.

EXAMPLE 7. *There exists a non-denumerable ordered set S and a distribution* Δ *of* $\Omega_2(S)$ *into classes* K_λ *such that, for every* $a \in S$,

$$|\sum \{a,x\}_< \in K_\lambda \square \{\lambda\}| \leqslant \aleph_0,$$

but, at the same time, $\{x,y\} \not\equiv \{x,z\}\ (.\Delta)$

whenever $\{x,y,z\}_< \subset S$.

Proof. Let ω_1 be the first non-denumerable ordinal number, and consider an abstract set S, ordered according to the order type ω_1^*. For $\lambda \in S$, let K_λ be the set of all sets $\{a,\lambda\}_< \subset S$, for varying a. Then the assertion holds.

8. In conclusion we prove a theorem which belongs to the present chapter in so far as it asserts the existence of a distribution having a special property, although the property in question is not concerned with Ramsey's theorem but with the following theorem of van der Waerden (**5**). *Given any positive integers k and l, there is a positive integer m which has the following property. If* Δ *is any distribution of the set* $\{1,2,\ldots,m\}$, *and* $|\Delta| \leqslant k$, *there are positive integers a and d satisfying*

$$a \equiv a+d \equiv a+2d \equiv \ldots \equiv a+ld\ (.\Delta). \tag{20}$$

We define *van der Waerden's function* $W(k,l)$ as being the least value of such a number m. The existing proofs of van der Waerden's theorem lead to upper estimates of W which are far beyond the range of explicit expressions in terms of common algebraic operations. The following example of a distribution gives what seems to be the first non-trivial lower estimate of W, namely $W(k,l) > (2lk^l)^{\frac{1}{2}}$.

EXAMPLE 8. *Let k and l be integers not less than* 2, *and let* m_0 *be the largest integer such that* $m_0^2 \leqslant 2lk^l$. *Then there exists a distribution* Δ *of*

$$S_0 = \{1,2,\ldots,m_0\},$$

where $|\Delta| \leqslant k$, *such that* (20) *does not hold for any positive integers a, d.*

Proof. We assume that, given any distribution Δ ($|\Delta| \leqslant k$) of the set $S = \{1,\ldots,m\}$, there are positive numbers a, d satisfying (20), and we shall deduce that $m^2 > 2lk^l$.

We have $m \geqslant l+1$. For any integer d in the range $1 \leqslant d \leqslant (m-1)/l$, the number of increasing arithmetic progressions of $l+1$ terms with common difference d and terms in S is $m-ld$. Hence the total number of such progressions, for varying d, is

$$M = \sum_{d=1}^{r} (m-ld),$$

where r is the integer satisfying $r \leqslant (m-1)/l < r+1$. The number of functions f on S into the set $\{1, 2, ..., k\}$ is k^m, and the number of those f which take a given value κ_0 at the $l+1$ places corresponding to the terms of a fixed progression is k^{m-l-1}. Hence, in view of our initial assumption,

$$kMk^{m-l-1} \geqslant k^m,$$

$$k^l \leqslant M = \tfrac{1}{2}r(2m-l-lr) < \frac{m-1}{2l}[2m-(m-1)] < \frac{m^2}{2l},$$

which is the desired conclusion.

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