THE COVERING OF n-DIMENSIONAL SPACE BY SPHERES

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1. Suppose that an n-dimensional cube of volume V is covered by a system of m equal spheres each of volume J, so that every point of the cube is in or on the boundary of one at least of the spheres. The density of the covering is defined to be mJ/V, that is, the ratio of the sum of the volumes of the spheres to that of the cube. The main object of this note is to show that ϑ^*, *the lower limit as V tends to infinity of the density of such a covering, satisfies*

$$\vartheta^* > \tfrac{16}{15} - \epsilon_n, \tag{1}$$

where $\epsilon_n \to 0$ as $n \to \infty$. Bambah and Davenport‡ have recently considered the similar lower limit ϑ, defined as above but with the additional restriction that the centres of the spheres should form an n-dimensional lattice. They proved that

$$\vartheta > \tfrac{4}{3} - \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Our method is very similar to that of Bambah and Davenport; both methods are based on the following construction. Suppose that the whole of n-dimensional space is covered by equal spheres S. Corresponding to

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‡ R. P. Bambah and H. Davenport, *Journal London Math. Soc.*, 27 (1952), 224–229.

each sphere S construct the convex polyhedron Π consisting of all points which are as near or nearer to the centre of S as they are to any other centre. These polyhedra obviously fit together and cover the whole of n-dimensional space simply; each point of space either lying in the interior of just one of the polyhedra or being common to the boundaries of two or more of the polyhedra. Further it is clear that each polyhedron is contained in the corresponding sphere. Both proofs split into two stages. First it is shown that no polyhedron has too many faces. Then it is shown that if a polyhedron with not too many faces is inscribed in a sphere, the volume of the polyhedron is less than that of the sphere by a certain factor. It then follows immediately that the density of the covering is not less than the reciprocal of this factor.

We remark that Bambah and Davenport make use of their assumption that the centres of the spheres form a lattice, in both parts of their proof. We do not make this assumption, and in the first part of our proof we can only obtain a relatively poor upper bound for the number of faces of the polyhedra. However for the second part of the proof we use a recent result of Rogers† showing that their inequality‡ holds without making use of the condition (derived from the assumption that the centres of the spheres form a lattice) that the foot of the perpendicular from the centre of the sphere to each face of the polyhedron falls inside the face. We shall need the result in the following form.

LEMMA 2. *Let $V(\Pi)$ be the volume of a convex polyhedron with N faces inscribed in an n-dimensional sphere of unit radius and volume*

$$J_n = \frac{\pi^{n/2}}{\Gamma(1+\frac{1}{2}n)}.$$

Then
$$\frac{J_n}{V(\Pi)} \geqslant \int_0^1 \frac{dt}{\{1-\bar{C}^\delta(1-t^\delta)\}^{n/2}},$$

where
$$\delta = \frac{2}{n-1}, \quad \bar{C} = \frac{nV(\Pi)}{NJ_{n-1}}.$$

If n and N tend to infinity in such a way that $N^{1/n}$ tends to a limit λ, then the upper limit of $V(\Pi)/J_n$ is less than or equal to $1-\lambda^{-2}$.

Since Rogers' proof is rather complicated (probably unnecessarily so) for sake of completeness we give in § 3 a simple proof of a slightly weaker result and show that it leads to the result that

$$\vartheta^* > \sqrt{(\tfrac{16}{15})} - \epsilon_n, \tag{2}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

† C. A. Rogers, *Journal London Math. Soc.*, in course of publication.

‡ *Loc. cit.*, inequality (14).

The problem of finding more precise estimates for the upper bound of the ratio $V(\Pi)/J_n$ when n and N are large seems to be difficult. It is probable that the estimate given by the above lemma is far from best possible. It would also be of interest to show that given n and N one can construct polyhedra for which the ratio $V(\Pi)/J_n$ is reasonably large. We have not been able to obtain a result significantly better than the following result, which may be obtained by considering coverings of the surface of a sphere by spherical caps. If

$$N > \frac{nJ_n}{J_{n-1}} 2^{(n-1)/2},$$

then there is a n-dimensional polyhedron Π with N faces inscribed in the unit sphere and with volume $V(\Pi)$ satisfying

$$\frac{V(\Pi)}{J_n} > \left[1-2\left(\frac{nJ_n}{NJ_{n-1}}\right)^{2/(n-1)}\right]^n.$$

2. Let r and ϵ be given positive numbers. Then, by the definition of ϑ^*, if s is a suitable large number, there will be a cube C of side $(s+2r+2)$ and volume $(s+2r+2)^n$ which is covered by a system of less than

$$(1+\epsilon)\vartheta^* s^n/J_n$$

spheres of radius 1. Now consider a movable sphere $S(X)$ of radius $r+1$ and volume $(r+1)^n J_n$ whose centre X lies in a cube C' of side s and volume s^n concentric with C. Let $m(X)$ be the number of those spheres of radius 1 covering C whose centres lie in $S(X)$. Then the integral of $m(X)$ over C' is clearly equal to the sum of the volumes of the intersection with C' of spheres of radius $(r+1)$ concentric with the spheres covering C. So this integral is less than

$$(1+\epsilon)\vartheta^* s^n (r+1)^n,$$

and consequently we can choose a sphere $S(X_1)$ of radius $r+1$ contained in C which contains less than

$$(1+\epsilon)\vartheta^*(r+1)^n$$

of the centres. Let S_0 be the concentric sphere of radius r. Now every point of S_0 belongs to at least one of the spheres of radius 1, and can only belong to those of the spheres whose centres belong to $S(X_1)$. Hence the sphere S_0 of radius r is covered by less than $(1+\epsilon)\vartheta^*(r+1)^n$ spheres of radius 1. Now it follows by continuity considerations that any sphere of radius r can be covered by some system of not more than $\vartheta^*(r+1)^n$ spheres of radius 1.

Let R be any positive number and consider a cube C of side $3R+4$. Let N be the positive integer such that C cannot be covered by $N-1$ spheres of unit radius but is covered by a certain system of N spheres S of unit radius. We take r to be any number with $0 < r < R$ and consider any sphere S' of radius r whose centre lies in the cube with side R concentric with C. We prove that the number m of those of the spheres S of unit radius covering C, which have a point in common with S', is at most

$$\vartheta^* (r+3)^n.$$

It is clear that all these spheres S which have a point in common with S' lie in the sphere S'' of radius $r+2$ concentric with S'. Now the result of the last paragraph shows that S'' can be covered by not more than $\vartheta^*(r+3)^n$ spheres of unit radius. So the cube C can be covered by not more than

$$\vartheta^*(r+3)^n+(N-m)$$

spheres of unit radius, namely the spheres covering S'' and the spheres S having no point in common with S'. Hence by our choice of N we have

$$m \leqslant \vartheta^*(r+3)^n$$

and our assertion is proved.

Now it is clear that if we let R tend to infinity and use an appropriate diagonal process we can construct a covering of the whole of n-dimensional space by spheres of unit radius having density ϑ^* and having the property that for each $r > 0$ the number of spheres of the system having a point in common with a sphere of radius r is not more than $\vartheta^*(r+3)^n$.

[We remark that so far we have not made much use of the fact that we are concerned with coverings of space by *spheres*; the whole of the above argument applies, with only trivial modifications, to coverings of space by congruent and similarly situated symmetrical convex bodies.]

We consider this system of spheres S of unit radius covering n-dimensional space. For each sphere S we take Π to be the convex polyhedron consisting of all points which are as near or nearer to the centre of S as they are to any other centre of one of the spheres of the covering. Then each point of space lies either in the interior of just one of the polyhedra or on the common boundary of two or more of the polyhedra. Now since each point of space is within unit distance of the nearest centre, it is clear that each polyhedron Π lies within the corresponding sphere S. Thus each face of one of the polyhedra Π arises as the perpendicular bisector of the segment joining the centre of the corresponding sphere S to the centre of some other sphere of the system. To different faces of Π correspond in this way different spheres of the system having the different faces in common with S. Hence by the result of the last paragraph, with $r = 1$, we see that Π has at most $\vartheta^* 4^n-1$ faces. Thus, if N is the integral part of $\vartheta^* 4^n-1$, and V_N is the volume of the largest polyhedron having N faces which can be inscribed in an n-dimensional sphere of unit radius, we have $V(\Pi) \leqslant V_N$. Since the density of the covering by the spheres of volume J_n is ϑ^* while that of the covering by the polyhedra of volume not more than V_N is 1, it is easy to see that

$$\vartheta^* \geqslant J_n/V_N. \tag{3}$$

In order to prove the result (1) of §1 it clearly suffices to suppose that for some $\epsilon > 0$ we have

$$\vartheta^* < \tfrac{16}{15}(1-\epsilon) \tag{4}$$

for infinitely many n and to obtain a contradiction. By virtue of (4) it is clear that $N^{1/n} \to 4$ as n tends to infinity through this sequence of values of n. So using Lemma 2 we have

$$\frac{V_N}{J_n} < \frac{1-(\frac{1}{4})^2}{1-\epsilon}$$

for all sufficiently large n of the sequence, contrary to (3) and (4). This completes the proof.

3. In this section we prove a weak form of Lemma 2 and indicate how it leads to the inequality (2). We first prove an elementary result.

LEMMA 1. *Let F be a closed convex $(n-1)$-dimensional set lying in the unit sphere S. Let P be the pyramid with vertex O and base F, and let P^* be the set of all points of S which lie on the same half ray through O as some point of F. Then the volumes V and V^* of P and P^* are connected by the inequality*

$$V^* \geqslant V[1-C^\delta]^{-1/2}, \tag{5}$$

where
$$\delta = \frac{2}{n-1}, \quad C = \frac{nV}{J_{n-1}}.$$

Proof. Let r be the distance from O of the point O' of F nearest to O. Then since F is convex it is easy to see that F is contained in the sphere with centre O' and radius $\sqrt{(1-r^2)}$. So, if A is the $(n-1)$-dimensional area of F and h is the perpendicular distance from O to the space in which F lies, we have

$$J_{n-1}(1-r^2)^{(n-1)/2} \geqslant A = \frac{nV}{h} > nV.$$

Hence
$$r \leqslant \left[1-\left(\frac{nV}{J_{n-1}}\right)^{2/(n-1)}\right]^{1/2} = [1-C^\delta]^{1/2}.$$

Now let O'' be the point on the boundary of S on the line OO' produced. The spherical pyramid P^* clearly contains both the pyramid P with vertex O and base F and the pyramid with vertex O'' and base F. Thus

$$V^* \geqslant V + \frac{1-r}{r}V = \frac{1}{r}V \geqslant V[1-C^\delta]^{-1/2},$$

as required.

We use this result in conjunction with a method of Bambah and Davenport† to prove the following weaker form of Lemma 2.

LEMMA 2A. *Under the conditions of Lemma 2 we have*

$$\frac{J_n}{V(\Pi)} \geqslant \frac{1}{\sqrt{(1-\bar{C}^\delta)}}, \tag{6}$$

and the upper limit of $V(\Pi)/J_n$ *is less than or equal to* $\sqrt{(1-\lambda^{-2})}$.

Proof. Suppose that Π is a convex polyhedron with N faces and volume $V(\Pi)$ inscribed in the unit sphere S. We may clearly suppose that Π is placed in S so that it contains O. Let $V_1, \ldots, V_N$ be the volumes of the pyramids $P_1, \ldots, P_N$ with O as vertex having the faces $F_1, \ldots, F_N$ of Π as their bases. Then clearly

$$V(\Pi) = \sum_{\nu=1}^{N} V_\nu.$$

Let $V_1^*, \ldots, V_N^*$ be the volumes of the spherical pyramids $P_1^*, \ldots, P_N^*$ consisting of those points of S which lie on half rays from O which meet the faces $F_1, \ldots, F_N$. Then we have

$$J_n = V(S) = \sum_{\nu=1}^{N} V_\nu^*.$$

By Lemma 1 we have

$$V_\nu^* \geqslant \frac{J_{n-1} C_\nu}{n\sqrt{(1-C_\nu^\delta)}}, \tag{7}$$

where $$\delta = \frac{2}{n-1}, \quad C_\nu = \frac{nV_\nu}{J_{n-1}}.$$

The mean value $\bar{C}$ of $C_1, \ldots, C_N$ is given by

$$\bar{C} = \frac{1}{N}\sum_{\nu=1}^{N} C_\nu = \frac{1}{N}\sum_{\nu=1}^{N}\frac{nV_\nu}{J_{n-1}} = \frac{nV(\Pi)}{NJ_{n-1}}.$$

The right-hand side of (7) is clearly a convex increasing function of C_ν. Hence by Jensen's inequality

$$J_n = \sum_{\nu=1}^{N} V_\nu^* \geqslant \frac{NJ_{n-1}\bar{C}}{n\sqrt{(1-\bar{C}^\delta)}}.$$

Thus $$\frac{J_n}{V(\Pi)} \geqslant \frac{1}{\sqrt{(1-\bar{C}^\delta)}}, \tag{8}$$

and the first part of the lemma is proved.

Now suppose that n and N tend to infinity so that $N^{1/n}$ tends to a limit λ. Suppose, as we may, that

$$J_n/V(\Pi) < \kappa.$$

† *Loc. cit.*, §3.

where κ is independent of n. Then

$$\frac{nJ_n}{\kappa NJ_{n-1}} < \frac{nV(\Pi)}{NJ_{n-1}} = \bar{C} < \frac{nJ_n}{NJ_{n-1}}.$$

So $\bar{C}^{\delta}$ converges to λ^{-2} as n tends to infinity and the right-hand side of (8) converges to $1/\sqrt{(1-\lambda^{-2})}$. This proves the second part of the lemma.

It is clear that the result (2) may be deduced from Lemma 2A in just the same way that the result (1) is deduced in §2 from Lemma 2.

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