Intersections of prescribed power, type, or measure

by

F. Bagemihl (Princeton, N. J.) and **P. Erdös** (Notre Dame, Ind.)

In 1914, Mazurkiewicz [5] showed that there exists a set of points in the plane, which intersects every straight line in the plane in precisely two points. Recently, Bagemihl [1] proved a general intersection theorem in the theory of sets, which, when applied to the plane, yields the following generalization of Mazurkiewicz's result: With every straight line s, associate a cardinal number $\mathfrak{q}_s \geqslant 2$ so that the sum of fewer than $2^{\aleph_0}$ of the numbers $\mathfrak{q}_s$ is always less than $2^{\aleph_0}$. Then there exists a set of points which intersects every straight line s in exactly $\mathfrak{q}_s$ points.

In the present paper, after extending the general intersection theorem alluded to above, we obtain several theorems dealing with plane point sets which intersect every straight line in a set of prescribed power, order type, or measure. In particular, we show that the aforementioned $\mathfrak{q}_s$ may be chosen arbitrarily in the range $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Free use is made of the well-ordering theorem.

THEOREM 1. *Let α be an arbitrary, fixed ordinal number, and S be a set with*

$$(1) \qquad \overline{\overline{S}} \leqslant \aleph_\alpha .$$

To every $s \in S$ let there correspond a set L_s such that, for every $S' \subseteq S - \{s\}$ with $\overline{\overline{S'}} < \aleph_\alpha$,

$$(2) \qquad \overline{\overline{L_s - \sum_{s' \in S'} L_{s'}}} \geqslant \aleph_\alpha ,$$

and put $P = \sum_{s \in S} L_s$.

Suppose that for every $s \in S$ there exists a cardinal number $\mathfrak{l}_s$ with $1 \leqslant \mathfrak{l}_s \leqslant \aleph_\alpha$, such that the following holds: If $D \subset P$, $\overline{\overline{D}} < \aleph_\alpha$, and S_D is the set of elements $s' \in S$ for which $\mathfrak{l}_{s'} < \aleph_\alpha$ and $\overline{\overline{L_{s'} D}} \geqslant \mathfrak{l}_{s'}$, then

$$(3) \qquad \overline{\overline{S_D}} < \aleph_\alpha .$$

With every $s \in S$ let there be associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ satisfying

$$(4) \qquad \mathfrak{l}_s \leqslant \mathfrak{q}_s \leqslant \aleph_\alpha .$$

Then there exists a set $P^* \subseteqq P$ (*with* $\overline{\overline{P^*}} \leqslant \aleph_\alpha$) *such that* $\overline{\overline{L_s P^*}} = \mathfrak{q}_s$ *for every* $s \in S$.

We first prove

LEMMA 1. *Let* $D \subset P$, $\overline{\overline{D}} < \aleph_\alpha$, *and*

$$\overline{\overline{L_s D}} \leqslant \mathfrak{q}_s \quad \text{for every} \quad s \in S. \tag{5}$$

Suppose that for some element $e \in S$,

$$\overline{\overline{L_e D}} < \mathfrak{q}_e. \tag{6}$$

Denote by S_D^* *the set of elements* $s' \in S$ *for which* $\overline{\overline{L_{s'} D}} = \mathfrak{q}_{s'} < \aleph_\alpha$. *Then there exists an* $a \in L_e - (D + \sum_{s' \in S_D^*} L_{s'})$ *such that*

$$\overline{\overline{L_s(D + \{a\})}} \leqslant \mathfrak{q}_s \tag{7}$$

for every $s \in S$. (Such an a will be called an *admissible element* of L_e relative to D).

Proof: It is easy to see from (4) and (3) that $\overline{\overline{S_D^*}} \leqslant \overline{\overline{S_D}} < \aleph_\alpha$. Therefore, by (2), $\overline{\overline{L_e - \sum_{s' \in S_D^*} L_{s'}}} \geqslant \aleph_\alpha$, so that $L_e - (D + \sum_{s' \in S_D^*} L_{s'})$ contains at least one element — call it a.

Now let $s \in S$. Then s satisfies at least one of the following conditions: 1. $s \in S_D^*$, 2. $\overline{\overline{L_s D}} < \mathfrak{q}_s$, 3. $\mathfrak{q}_s = \aleph_\alpha$. If 1, then (7) follows from the fact that a non-$\in \sum_{s' \in S_D^*} L_{s'}$; if 2, then (7) is obvious; if 3, then (7) follows from (5). This completes the proof of the lemma.

Proof of Theorem 1. The case $\overline{\overline{S}} = 0$ is trivial. We may therefore assume that $\overline{\overline{S}} > 0$. Note (1) and (4), consider $\mathfrak{q}_s$ replicas of every $L_s (s \in S)$, well-order the resulting complex of $\sum_{s \in S} \mathfrak{q}_s \leqslant \aleph_\alpha^2 = \aleph_\alpha$ sets to form a sequence

$$L_0, L_1, \ldots, L_\xi, \ldots \qquad (\xi < \varrho), \tag{8}$$

where $1 \leqslant \varrho \leqslant \omega_\alpha$, and denote by $\mathfrak{q}_\xi$ $(\xi < \varrho)$ the respective cardinal numbers associated with the sets (8) according to (4).

We define sets $A_\xi \subset P$ $(\xi < \varrho)$, by induction on ξ, as follows: Let A_0 consist of a single element of L_0. Suppose that $0 < \xi < \varrho$, and that for every $\mu < \xi$ a set $A_\mu \subset P$, with $\overline{\overline{A_\mu}} \leqslant 1$, has been defined so that $\overline{\overline{L_s \sum_{\mu<\xi} A_\mu}} \leqslant \mathfrak{q}_s$ for every $s \in S$. Evidently $\overline{\overline{\sum_{\mu<\xi} A_\mu}} \leqslant \overline{\overline{\xi}} < \aleph_\alpha$. If $\overline{\overline{L_\xi \sum_{\mu<\xi} A_\mu}} = \mathfrak{q}_\xi$, let $A_\xi = 0$. If, however, $\overline{\overline{L_\xi \sum_{\mu<\xi} A_\mu}} < \mathfrak{q}_\xi$, let A_ξ consist of a single admissible element of L_ξ

relative to $\sum_{\mu<\xi} A_\mu$; the existence of such an element is guaranteed by Lemma 1. The sets A_ξ $(\xi<\varrho)$ thus defined are obviously mutually exclusive.

Put $P^*=\sum_{\xi<\varrho} A_\xi$. Then $P^*\subseteq P$ and $\overline{\overline{P^*}}\leqslant\bar{\varrho}\leqslant\aleph_a$.

Now let $s\in S$. It is clear from the definition of the sets A_ξ $(\xi<\varrho)$ that $\overline{\overline{L_s\sum_{\mu\leqslant\xi} A_\mu}}\leqslant \mathfrak{q}_s$ for every $\xi<\varrho$. If $\overline{\overline{L_s\sum_{\mu\leqslant\xi} A_\mu}}<\mathfrak{q}_s$ for every $\xi<\varrho$, then $\overline{\overline{L_s\sum_{\mu<\varrho} A_\mu}}\leqslant \mathfrak{q}_s$. There are $\mathfrak{q}_s$ values of $\xi<\varrho$ for which $L_s=L_\xi$, and for every such ξ, $\overline{\overline{L_sA_\xi}}=1$, so that $\overline{\overline{L_s\sum_{\mu<\varrho} A_\mu}}\geqslant\mathfrak{q}_s$. Hence $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$. If, however, for some $\xi'<\varrho$, $\overline{\overline{L_s\sum_{\mu\leqslant\xi'} A_\mu}}=\mathfrak{q}_s$, then $\mathfrak{q}_s\leqslant\overline{\overline{\xi'+1}}<\aleph_a$, and, from the definition of an admissible element, it follows that $\overline{\overline{L_s\sum_{\xi'<\xi<\varrho} A_\xi}}=0$, so that in this case too $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$. This completes the proof of Theorem 1.

The following theorem was presented by the authors (see [2]) to the American Mathematical Society, May 28, 1952:

A complex $\mathfrak{C}$ of cardinal numbers is said to be *strongly less than* $\aleph_\alpha$, if every sum of fewer than $\aleph_\alpha$ terms belonging to $\mathfrak{C}$ is less than $\aleph_\alpha$. *Let a be an arbitrary, fixed ordinal number, S and P be sets, and to every $s\in S$ let there correspond a subset, L_s, of P. Suppose that the following conditions are satisfied:*

(I) $\overline{\overline{S}}\leqslant\aleph_\alpha$.

(II) $\overline{\overline{L_s}}\geqslant\aleph_\alpha$ *for every* $s\in S$.

(III) *If $s\in S$, there is a cardinal number $\mathfrak{n}_s\geqslant 1$ such that, if $s'\in S$ and $s'\neq s$, then $\overline{\overline{L_sL_{s'}}}\leqslant\mathfrak{n}_s$, and the complex of cardinal numbers $\mathfrak{n}_s$ $(s\in S)$ is strongly less than $\aleph_\alpha$.*

(IV) *There is a cardinal number $\mathfrak{m}\geqslant 1$ with the following properties:* (a) $\mathfrak{d}^{\mathfrak{m}}<\aleph_\alpha$ *for every $\mathfrak{d}<\aleph_\alpha$;* (b) *if $P'\subseteq P$, $\overline{\overline{P'}}=\mathfrak{m}$, and $\mathfrak{m}_{P'}$ is the number of elements $s\in S$ for which P' is a subset of L_s, then the complex of cardinal numbers $\mathfrak{m}_{P'}$ obtained by letting P' run through all the subsets of P having $\mathfrak{m}$ elements, is strongly less than $\aleph_\alpha$.*

Suppose that with every $s\in S$ there is associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ such that

(V) $\mathfrak{m}+\mathfrak{n}_s-1\leqslant\mathfrak{q}_s\leqslant\aleph_\alpha$.

Then there exists a subset P^* of P (with $\overline{\overline{P^*}}\leqslant\aleph_\alpha$) such that $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$ for every $s\in S$.

We shall now show that this theorem, which is a generalization of one due to Bagemihl [1], is contained in Theorem 1, by proving that (I)-(V) imply (1)-(4) (that the converse is not true will be evident from the proof).

Condition (1) follows trivially from (I).

To prove (2), suppose that $S'\subseteq S-\{s\}$ and $\overline{\overline{S'}}<\aleph_\alpha$. If $s'\in S'$, then, according to (III), $\overline{\overline{L_{s'}L_s}}\leqslant\mathfrak{n}_{s'}$, and (III) also implies that

$$\overline{\overline{L_s\sum_{s'\in S'} L_{s'}}}\leqslant\sum_{s'\in S'}\overline{\overline{L_{s'}L_s}}\leqslant\sum_{s'\in S'}\mathfrak{n}_{s'}<\aleph_\alpha.$$

According to (II), then,

$$\overline{\overline{L_s - L_s \sum_{s' \in S'} L_{s'}}} = \overline{\overline{L_s - \sum_{s' \in S'} L_{s'}}} \geqslant \aleph_\alpha,$$

which is (2).

Now to prove (3), take every $\mathfrak{l}_s$ in Theorem 1 to be the $\mathfrak{m}$ in (IV). Let $D \subset P$, $\overline{\overline{D}} < \aleph_\alpha$, and S_D be the set of elements $s' \in S$ for which $\overline{\overline{L_{s'} D}} \geqslant \mathfrak{l}_{s'}$ $(=\mathfrak{m})$. There are not more than $\overline{\overline{D}}^{\mathfrak{m}}$ subsets of D of $\mathfrak{m}$ elements. By (a) of (IV), $\overline{\overline{D}}^{\mathfrak{m}} < \aleph_\alpha$, and by (b) of (IV), $\overline{\overline{S_D}} < \aleph_\alpha$, which proves (3).

Finally, according to (V), (4) is certainly satisfied if $\mathfrak{l}_s = \mathfrak{m}$.

The following examples show that if one of the hypotheses (1)-(4) of Theorem 1 is not satisfied, then the conclusion of this theorem may no longer be true:

$\overline{(1)}$, (2), (3), (4): Let $S = \{\mu\}_{\mu < \omega_{\alpha+1} + \omega_\alpha}$, $P = \{(\xi, \eta)\}_{\xi < \omega_{\alpha+1}, \eta < \omega_\alpha}$, $L_\xi = \{(\xi, \eta)\}_{\eta < \omega_\alpha}$ for every $\xi < \omega_{\alpha+1}$, $L_{\omega_{\alpha+1}+\eta} = \{(\xi, \eta)\}_{\xi < \omega_{\alpha+1}}$ for every $\eta < \omega_\alpha$. Then $\mathfrak{l}_\mu = 1$, and we take $\mathfrak{q}_\mu = 1$, for every $\mu \in S$.

If $\overline{\overline{L_\xi P^*}} = 1$ for every $\xi < \omega_{\alpha+1}$, then $\overline{\overline{L_{\omega_{\alpha+1}+\eta} P^*}} = \aleph_{\alpha+1}$ for some $\eta < \omega_\alpha$, contradicting $\mathfrak{q}_{\omega_{\alpha+1}+\eta} = 1$.

(1), $\overline{(2)}$, (3), (4): Let $S = \{\mu\}_{\mu < \omega_\alpha + 1}$, $P = \{\xi\}_{\xi < \omega_\alpha}$, $L_\xi = \{\xi\}$ for every $\xi < \omega_\alpha$, $L_{\omega_\alpha + 1} = \{\xi\}_{\xi < \omega_\alpha}$. Then $\mathfrak{l}_\mu = 1$, and we take $\mathfrak{q}_\mu = 1$, for every $\mu \in S$.

We have $\overline{\overline{L_{\omega_\alpha+1} P^*}} = \aleph_\alpha \neq \mathfrak{q}_{\omega_\alpha+1}$.

(1), (2), $\overline{(3)}$, (4): Let P be the set of points of a projective plane in which every line contains $\aleph_\alpha$ points, and let the sets L_s be the lines in this plane. Take every $\mathfrak{l}_s = 1$; then (3) is not satisfied. Take also every $\mathfrak{q}_s = 1$.

If we consider any two points of P^*, these two points determine a line L_s, and thus $\overline{\overline{L_s P^*}} \geqslant 2 > \mathfrak{q}_s$.

(1), (2), (3), $\overline{(4)}$: In the preceding example, take every $\mathfrak{l}_s = 2$.

If we take every $\mathfrak{q}_s = 1$ or $> \aleph_\alpha$, then it is obvious that P^* does not exist.

Theorem 1 can be applied, *e. g.*, to the points and straight lines of a Euclidean plane. In this case we interpret S as a set of indices for the set of straight lines in the plane and L_s as the set of points which constitute the straight line $s \in S$, P as the set of points in the plane, and $\aleph_\alpha = 2^{\aleph_0}$. Conditions (1) and (2) are evidently satisfied, and (3) clearly holds if we take every $\mathfrak{l}_s = 2$. Thus we obtain

COROLLARY 1. *With every straight line s in a Euclidean plane associate a cardinal number $\mathfrak{q}_s$ such that $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Then there exists a set of points which intersects every straight line s in precisely $\mathfrak{q}_s$ points.*

This result was obtained by Mazurkiewicz [5] for the case $\mathfrak{q}_s=2$ for every s; by Bagemihl [1] for the case $\mathfrak{q}_s \geqslant 2$ for every s, the complex of cardinal numbers $\mathfrak{q}_s$ being strongly less than $2^{\aleph_0}$; and in the above form, independently and practically simultaneously (early in 1952), by Sierpiński [8] and by the present authors.

As is easily seen, condition (3) is also satisfied under the weaker assumptions of

COROLLARY 2. *With every point p of the set, P, of points of a Euclidean plane associate a set, S_p, of $\mathfrak{k}_p$ straight lines in this plane, each of which contains the point p, and let the complex of cardinal numbers $\mathfrak{k}_p$ $(p \in P)$ be strongly less than $2^{\aleph_0}$. Let $S' = \sum_{p \in P} S_p$. With every $s \in S'$ associate a cardinal number $\mathfrak{q}_s$ such that $1 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, and with every s non-$\in S'$ associate a cardinal number $\mathfrak{q}_s$ such that $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Then there exists a set of points which intersects every straight line s in the plane in precisely $\mathfrak{q}_s$ points.*

Let the word *curve* mean any set of points $(x;y)$ satisfying an equation of the form $y=f(x)$ where f is a single-valued function of a real variable (*cf.* p. 11 of [9]). Take S' in Corollary 2 to be the set of all straight lines parallel to the y-axis, and let $\mathfrak{q}_s=1$ for every $s \in S'$, $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$ for every s non-$\in S'$. Then $\mathfrak{k}_p=1$ for every $p \in P$, and Corollary 2 yields

COROLLARY 3. *With every straight line s (in a Euclidean plane) which is not parallel to the y-axis associate a cardinal number $\mathfrak{q}_s$ such that $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Then there exists a curve which intersects every straight line s which is not parallel to the y-axis in precisely $\mathfrak{q}_s$ points.*

Corollary 2 suggests the following problems dealing with the Euclidean plane:

What is a necessary and sufficient condition on a set, S^*, of straight lines so that, if with every $s \in S^*$ there is associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ in the range $1 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, and if with every s non-$\in S^*$ there is associated in an arbitrary manner a cardinal nurm ber $\mathfrak{q}_s$ in the range $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, there exists a set of points which intersects every straight line s in precisely $\mathfrak{q}_s$ points?

What is the answer if $\mathfrak{q}_s=1$ $(s \in S^*)$ instead of $\mathfrak{q}_s$ being chosen arbitrarily in the range $1 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$?

We have been able to solve the following problem:

What is a necessary and sufficient condition on a set, S^*, of straight lines so that, if with every $s \in S^*$ there is associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ in the range $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, there exists a set of points, P^*, which intersects every straight line $s \in S^*$ in precisely $\mathfrak{q}_s$ points and every straight line s non-$\in S^*$ in less than 2 points?

The answer is: *S^* is the set of straight lines joining all pairs of points of a point set M having the property that if a straight line contains at least 2 points of M, it contains $2^{\aleph_0}$ points of M.*

To see that the condition is necessary, take $\mathfrak{q}_s=2^{\aleph_0}$ $(s\in S^*)$; then P^* is such a set M. To show that the condition is sufficient, in Theorem 1 take $\aleph_\alpha=2^{\aleph_0}$, $S=S^*$, $P=M$, $L_s=$ the subset of M which is contained in the straight line $s\in S^*$, $\mathfrak{l}_s=2$ $(s\in S^*)$; the set P^* of Theorem 1 is then obviously one that has the properties required of the set P^* in the problem.

In what follows we still deal with the Euclidean plane, but when we speak of a straight line we shall tacitly assume that one of the two possible orientations has been assigned to it, and when we speak of a subset of a straight line we shall regard the subset as ordered by the orientation of the line, so that the (order) type of such a subset is well-defined.

THEOREM 2. *With every straight line s associate in an arbitrary manner a finite or an enumerable order type $\tau_s\neq 0, 1$. Then there exists a set of points whose intersection with every straight line s is a set of type τ_s.*

Proof: Well-order the set of straight lines to form a sequence

$$s_0, s_1, \dots, s_\xi, \dots \qquad (\xi<\omega_\gamma),$$

where ω_γ is the initial number of $Z(2^{\aleph_0})$, and let the sequence of associated types be

$$\tau_0, \tau_1, \dots, \tau_\xi, \dots \qquad (\xi<\omega_\gamma).$$

Let T_0 be a set of points on s_0 such that $\overline{T}_0=\tau_0$. Let $0<\xi<\omega_\gamma$, and suppose that, for every $\mu<\xi$, a set, T_μ, of points on s_μ has been defined, such that $\overline{T}_\mu=\tau_\mu$, and so that at most 2 points of s_ξ belong to $\sum\limits_{\mu<\xi} T_\mu=T'_\xi$. We have

$$\overline{\overline{T'_\xi}}=\overline{\overline{\sum_{\mu<\xi} T_\mu}}\leqslant\sum_{\mu<\xi}\overline{\overline{T}}_\mu\leqslant\overline{\xi}\,\aleph_0<2^{\aleph_0}.$$

Consequently, there are fewer than $2^{\aleph_0}$ straight lines such that each contains at least 2 points of the set T'_ξ. Therefore the set, V_ξ, of points on s_ξ which are not on any of these lines different from s_ξ is everywhere dense on s_ξ, so that every interval of s_ξ contains a subset of V_ξ of any given finite or enumerable type.

If s_ξ contains no point of T'_ξ, let T_ξ be any subset of V_ξ such that $\overline{T}_\xi=\tau_\xi$. If s_ξ has precisely one point, p, in common with T'_ξ, write $\tau_\xi=\sigma_\xi+1+\varrho_\xi$, and let T_ξ consist of the points of a subset, of type σ_ξ, of V_ξ preceding p, p itself, and the points of a subset, of type ϱ_ξ, of V_ξ succeeding p. Finally, if s_ξ has two points, p, p', in common with T'_ξ, write $\tau_\xi=\sigma_\xi+1+\varrho_\xi+1+\zeta_\xi$, and define T_ξ in the obvious manner. Denote by T the union of the sets T_ξ ($\xi < \omega_\gamma$) thus defined. Evidently T intersects every s in a set of type τ_s, and the proof of the theorem is complete.

Call an order type τ a *subtype of the continuum*, if the linear continuum contains an ordered subset of type τ. We shall prove

THEOREM 3. *With every straight line s associate in an arbitrary manner a subtype, $\tau_s \neq 0, 1$, of the continuum. Then, if $2^{\aleph_0} = \aleph_1$, there exists a set of points whose intersection with every straight line s is a set of type τ_s and measure* 0.

Let us term a linear perfect set *sparse*, if it is a dyadic discontinuum (cf. [3], p. 134), D, whose dyadic schema at the n-th stage consists of 2^n mutually exclusive closed intervals, the length of the largest of which is $o(4^{-n})$.

Suppose that s_1, s_2, s_3 are three straight lines and E_1, E_2 are sets of points on s_1, s_2, respectively. Consider the set of all straight lines each of which is different from s_3 and is determined by a point of E_1 and a point of E_2. This set of lines intersects s_3 in a set of points which we shall call the *mutual projection of E_1 and E_2 on s_3* and denote by $(E_1, E_2; s_3)$.

LEMMA 2. *Let s_1, s_2, s_3 be three straight lines and D_1, D_2 be sparse perfect sets on s_1, s_2, respectively. Then*

$$\operatorname{meas}(D_1, D_2; s_3) = 0.$$

Proof: Denote by p_{13}, p_{23} the points of intersection (if nonexistent, to be disregarded in what follows) of s_1 and s_3, s_2 and s_3, respectively. Let $\varepsilon > 0$, p be an arbitrary, but fixed, point, C be a circle with radius ε^{-1} and center p, and C_{13} and C_{23} be circles of radius ε and with the respective centers p_{13}, p_{23}. Choose ε so small that C_{13} and C_{23} lie in the interior of C, and hold ε fixed for the time being. Denote by R_ε the region inside C and outside C_{13} and C_{23}, and let $D_1^\varepsilon = D_1 R_\varepsilon$, $D_2^\varepsilon = D_2 R_\varepsilon$. Suppose that I_1^n, I_2^n signify the parts in R_ε of any two intervals of the n-th stage of the dyadic schemata representing D_1, D_2, respectively. Then it is not difficult to see that

$$\operatorname{meas}\big(R_\varepsilon \cdot (I_1^n, I_2^n; s_3)\big) < c_\varepsilon \cdot (\operatorname{meas} I_1^n + \operatorname{meas} I_2^n) = c_\varepsilon \cdot o(4^{-n}),$$

where c_ε is a constant depending only on ε. Since at most 4^n pairs of intervals of the n-th stage come into question,

$$\operatorname{meas}\big(R_\varepsilon \cdot (D_1^\varepsilon, D_2^\varepsilon; s_3)\big) < 4^n c_\varepsilon \cdot o(4^{-n}),$$

and letting $n \to \infty$ we see that

$$\operatorname{meas}\big(R_\varepsilon \cdot (D_1^\varepsilon, D_2^\varepsilon; s_3)\big) = 0.$$

Now let $\{\varepsilon_k\}$ be a sequence of sufficiently small positive numbers tending monotonically to 0. Then

$$0=\sum_{k=1}^{\infty}\operatorname{meas}\left(R_{\varepsilon_k}\cdot(D_1^{\varepsilon_k},D_2^{\varepsilon_k};s_3)\right)\geqslant\operatorname{meas}\sum_{k=1}^{\infty}\left(R_{\varepsilon_k}\cdot(D_1^{\varepsilon_k},D_2^{\varepsilon_k};s_3)\right)$$
$$=\operatorname{meas}(D_1,D_2;s_3), \qquad \text{q. e. d.}$$

Proof of Theorem 3. Well-order the set of straight lines to form a sequence

$$s_0,s_1,\dots,s_\xi,\dots \qquad (\xi<\omega_1),$$

and let the sequence of associated types be

$$\tau_0,\tau_1,\dots,\tau_\xi,\dots \qquad (\xi<\omega_1).$$

Let D_0 be a sparse perfect set on s_0. Since D_0 is perfect, it contains a subset T_0 of type τ_0, and since D_0 is of measure 0, so is T_0. Let $0<\xi<\omega_1$, and suppose that, for every $\mu<\xi$, a sparse perfect set D_μ, and a set $T_\mu\subset D_\mu$, with $\overline{T}_\mu=\tau_\mu$, have been defined on s_μ in such a manner that at most 2 points of s_ξ belong to $\sum_{\mu<\xi}T_\mu=T'_\xi$. According to Lemma 2,

$$0=\operatorname{meas}\sum_{\mu<\nu<\xi}(D_\mu,D_\nu;s_\xi)=\operatorname{meas}\sum_{\mu<\nu<\xi}(T_\mu,T_\nu;s_\xi),$$

because the sum contains at most $\aleph_0$ terms (this is where we make use of the assumption $2^{\aleph_0}=\aleph_1$). Hence, the point set

$$V_\xi=s_\xi-\sum_{\mu<\nu<\xi}(T_\mu,T_\nu;s_\xi)$$

is of positive measure in every interval of s_ξ. Consequently, every interval of s_ξ contains a perfect subset of V_ξ, and therefore also contains a sparse perfect subset of V_ξ, which in turn contains a subset having as its type any given subtype of the continuum. The rest of the proof is verbally identical with the last paragraph of the proof of Theorem 2. It is merely necessary to note that, if D,D',D'' are sparse perfect sets on some straight line, and every point of D precedes every point of D', and every point of D' precedes every point of D'', then the union $D+D'+D''$ is also a sparse perfect set.

It would be interesting to know whether or not the assumption $2^{\aleph_0}=\aleph_1$ is necessary in Theorem 3.

THEOREM 4. *With every straight line s associate in an arbitrary manner a number m_s such that $0\leqslant m_s\leqslant\infty$. Then, if $2^{\aleph_0}=\aleph_1$, there exists a set of points whose intersection with every straight line s is a set of measure m_s.*

Proof: Well-order the set of straight lines to form a sequence

$$s_0,s_1,\dots,s_\xi,\dots \qquad (\xi<\omega_1),$$

and let the sequence of associated measures be

$$m_0,m_1,\dots,m_\xi,\dots \qquad (\xi<\omega_1).$$

Let M_0 be a point set, of measure m_0, on s_0. Let $0<\xi<\omega_1$, and suppose that, for every $\mu<\xi$, M_μ is a point set, of measure m_μ, on s_μ. Since $\overline{\overline{\xi}}\leqslant\aleph_0$, the intersection of $\sum_{\mu<\xi} M_\mu$ with s_ξ is an at most enumerable set of points, and is therefore of measure 0, so that it is possible to define M_ξ as a subset, of measure m_ξ, of $s_\xi-\sum_{\mu<\xi} M_\mu$. Evidently the set $M=\sum_{\xi<\omega_1} M_\xi$ intersects each s in a set of measure m_s, q. e. d.

Instead of assuming that $2^{\aleph_0}=\aleph_1$, it is sufficient to assume that every linear set of power less than $2^{\aleph_0}$ has measure 0.

Added during printing. Following a kind suggestion of K. Gödel's, we are able to show that the assumption $2^{\aleph_0}=\aleph_1$ is unnecessary in Theorem 3. Specifically, we shall prove

THEOREM 5. *With every straight line s in the plane, associate in an arbitrary manner a subtype, $\tau_s\neq 0, 1$, of the continuum. Then there exists a set of points whose intersection with every straight line s (assumed, for simplicity, to be oriented in the positive sense relative to a set of Cartesian coordinate axes) is a set of type τ_s.*

Proof. Well-order the set of straight lines to form a sequence

$$s_0, s_1, \ldots, s_\xi, \ldots \qquad (\xi<\omega_\gamma),$$

where ω_γ is the initial number of $Z(2^{\aleph_0})$, and let the sequence of associated types be

$$\tau_0, \tau_1, \ldots, \tau_\xi, \ldots \qquad (\xi<\omega_\gamma).$$

Every s_ξ has an equation either of the form

$$\text{(i)} \quad y=a_\xi x+b_\xi \quad \text{or} \quad \text{(ii)} \quad x=c_\xi,$$

where a_ξ, b_ξ, c_ξ are real numbers, the "constants belonging to s_ξ".

Let M be the system of algebraically independent real numbers constructed by J. von Neumann [6]. The set M contains a perfect subset (cf. S. Ruziewicz and W. Sierpiński [7], p. 18). Every perfect set contains $2^{\aleph_0}$ mutually exclusive perfect subsets (see, *e. g.*, C. Kuratowski and W. Sierpiński [4], p. 195).

Let $I_0, I_1, \ldots, I_n, \ldots$ $(n<\omega)$ be the set of all nonempty open intervals with rational endpoints, and let $B_0, B_1, \ldots, B_n, \ldots$ $(n<\omega)$ be $\aleph_0$ mutually exclusive, bounded, perfect subsets of M. For every $n<\omega$, there exists a one-to-one transformation $t'=r_n t+r'_n$, where r_n, r'_n are rational numbers and $r_n\neq 0$, under which the image of B_n is a perfect subset, C_n, of I_n. The sets C_n $(n<\omega)$ are mutually exclusive, and $\sum_{n<\omega} C_n$ is an algebraically independent system of real numbers (cf. J. von Neumann [6], p. 140). Each C_n contains $2^{\aleph_0}$ mutually exclusive perfect subsets P_n^ξ $(\xi<\omega_\gamma)$. For every $\xi<\omega_\gamma$, we define $P_\xi=\sum_{n<\omega} P_n^\xi$; then every nonempty open interval of real numbers contains a perfect subset of P_ξ.

Denote by R_0 a subset, of type τ_0, of P_0, and let T_0 be the set of points on s_0 whose abscissas or ordinates form the set R_0 according as s_0 is of the form (i) or (ii). Now suppose that $0<\xi<\omega_\gamma$, and that we have defined, for every $\mu<\xi$, the set $R_\mu\subseteq P_\mu$ and the set T_μ on s_μ in such a manner that the orthogonal projection of T_μ onto the x- or the y-axis is R_μ according as s_μ is of the form (i) or (ii), that $\overline{T}_\mu=\tau_\mu$, and that at most 2 points of s_ξ belong to $\sum_{\mu<\xi} T_\mu=T'_\xi$. Denote by K_ξ the set of all constants belonging to at least one s_μ $(\mu\leqslant\xi)$; obviously $\overline{\overline{K_\xi}}<2^{\aleph_0}$. Consequently, the cardinal number of the set, D_ξ, of elements of P_ξ which are algebraically dependent on the system of real numbers $K_\xi+\sum_{\mu<\xi} R_\mu$ is less than $2^{\aleph_0}$. Hence, every nonempty open interval of real numbers contains a perfect subset of $Q_\xi=P_\xi-D_\xi$. Let V_ξ be the set of points on s_ξ whose abscissas or ordinates form the set Q_ξ according as s_ξ is of the form (i) or (ii). No straight line different from s_ξ and determined by two points of T'_ξ can intersect s_ξ in a point of V_ξ, for otherwise the system of real numbers $K_\xi+\sum_{\mu<\xi} R_\mu+Q_\xi$ would not be algebraically independent, contradicting the definition of Q_ξ. The proof of our theorem can now be completed in accordance with the last paragraph of the proof of Theorem 2 above (if we define R_ξ as the orthogonal projection of T_ξ onto the appropriate coordinate axis).

Added in proof. We are indebted to A. Rosenthal for calling our attention to his papers *Über Gebilde mit einzigem Ordnungsindex*, Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München, 1922, p. 221-240, and *Über die Nichtexistenz von Kontinuen in gewissen Mengen mit einziger Ordnungszahl*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, mathematisch-natur-wissenschaftliche Klasse, 1934, p. 49-56, the first of which contains, in addition to other results, a special case of our Theorem 1. A paper by J. Moneta, *Application du théorème du continu*, Cahiers Rhodaniens 4 (1952), p. 29-42 in which an unnecessary appeal is made to the (unproved) relation $2^{\aleph_0}=\aleph_1$, contains a particular case of our Corollary 2.

INSTITUTE FOR ADVANCED STUDY
UNIVERSITY OF NOTRE DAME

References

[1] F. Bagemihl, *A theorem on intersections of prescribed cardinality*, Annals of Mathematics 55 (1952), p. 34-37.

[2] F. Bagemihl and P. Erdös, *Intersections of prescribed cardinality*, Bulletin of the American Mathematical Society 58 (1952), p. 619-620.

[3] F. Hausdorff, *Mengenlehre*, 3-d ed. (reprint), New York 1944.

[4] C. Kuratowski et W. Sierpiński, *Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaires*, Fund. Math. 8 (1926), p. 193-200.

[5] S. Mazurkiewicz, *Sur un ensemble plan*, Comptes rendus des séances de la Société des Sciences de Varsovie, Classe III, 4 (1914), p. 382-384.

[6] J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann. 99 (1928), p. 134-141.

[7] S. Ruziewicz et W. Sierpiński, *Sur un ensemble parfait qui a avec toute sa translation au plus un point commun*, Fund. Math. 19 (1932), p. 17-21.

[8] W. Sierpiński, *Une généralisation des théorèmes de S. Mazurkiewicz et F. Bagemihl*, Fundamenta Mathematicae 40, p. 1-2.

[9] — *Hypothèse du continu*, Monografie Matematyczne 4, Warszawa-Lwów 1934.

Reçu par la Rédaction le 17. 3. 1953