MULTIPLE POINTS OF PATHS OF BROWNIAN MOTION IN THE PLANE *

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Let $(\Omega, \mathcal{E}, Pr)$ be a probability space, i.e., $\Omega = \{\omega\}$ is a set of elements ω, $\mathcal{E} = \{E\}$ is a Borel field of subsets of Ω called "events", and Pr is a countably additive measure defined on $\mathcal{E}$ and satisfying $Pr(\Omega)=1$. $Pr(E)$ is called the "probability" of the event E.

A (mathematical) *one-dimensional Brownian motion*[1,5] is a real-valued function $x(t, \omega)$ of the two variables t and ω, defined for all non-negative real numbers t, $0 \leqslant t < \infty$, and for all $\omega \in \Omega$, which has the following properties:

(B_1) $x(0; \omega) \equiv 0$;

(B_2) for any real numbers s, t with $0 \leqslant s < t < \infty$, the "increment" $x(t, \omega) - x(s, \omega)$ is $\mathcal{E}$-measurable in ω and has a normal distribution with mean 0 and variance $t-s$, i.e.,**

$$E_{x,s,t,\alpha} \equiv \{\omega \mid x(t,\omega) - x(s,\omega) < \alpha\} \in \mathcal{E} \tag{1}$$

and

$$Pr(E_{x,s,t,\alpha}) = \left(1/\sqrt{2\pi(t-s)}\right) \int_{-\infty}^{\alpha} e^{-u^2/2(t-s)} du \tag{2}$$

for every real number α.

(B_3) for any real numbers s_i, t_i $(i=1, \ldots, m)$ with $0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \ldots \leqslant s_m < t_m < \infty$, the increments $x(t_i, \omega) - x(s_i, \omega)$, $i = 1, \ldots, m$ are independent in the sense of probability theory, i. e.,

$$Pr\left(\bigcap_{i=1}^{m} E_{x,s_i,t_i,\alpha_i}\right) = \prod_{i=1}^{m} Pr\left(E_{x,s_i,t_i,\alpha_i}\right) \tag{3}$$

for any real α_i, $i = 1, \ldots, m$.

A 2-dimensional or *plane Brownian motion* is an ordered pair of two mutually independent one-dimensional Brownian motions, i.e., a pair of one-dimensional Brownian motions $x(t, \omega)$ and $y(t, \omega)$ with the property that

$$Pr(E_{x,s,t,\alpha} \cap E_{y,s',t',\alpha'}) = Pr(E_{x,s,t,\alpha}) \cdot Pr(E_{y,s',t',\alpha'}) \tag{4}$$

for any real numbers s, t, α, s', t', α', with $0 \leqslant s < t$, $0 \leqslant s' < t'$.

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** $\{\omega \mid \ldots\}$ denotes the set of ω having the properties following the vertical bar; and similarly $\{z \mid \ldots\}$ etc.

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If we consider $z(t, \omega) = [x(t, \omega), y(t, \omega)]$ as a point in a Euclidean plane then, for each fixed ω, $z(t, \omega)$ may be considered as a function of t, defined for $0 \leqslant t < \infty$, and assuming as values points (or vectors) in the plane.

It is easy to see that this definition of a plane Brownian motion is independent of the choice of the rectangular coordinate system; i.e., the motion is isotropic, it is invariant vis-à-vis rotations of the coordinate system.

It is further assumed[1] that the Borel field $\mathcal{E}$ is already extended in such a manner that the subset C of Ω consisting of all ω for which $x(t, \omega)$ is a continuous function of t for $0 \leqslant t < \infty$ is $\mathcal{E}$-measurable and satisfies $P(C) = 1$.

For any point z in the plane, for any $\omega \in \Omega$ and any real numbers a, b with $0 \leqslant a < b < \infty$, let us put

$$L_{a,b}(z'; \omega) = \{z' + z(t, \omega) \mid a \leqslant t \leqslant b\}, \tag{5}$$

$$L_a(z'; \omega) = \{z' + z(t, \omega) \mid a \leqslant t < \infty\}, \tag{6}$$

$$L(z'; \omega) = L_0(z'; \omega) = \{z' + z(t, \omega) \mid 0 \leqslant t < \infty\} \tag{7}$$

where the $+$ sign in the above formula (as well as $+$ and $-$ in similar context in the sequel) refers to vector addition in the plane. Furthermore, when $z' = 0$, i.e., coincides with the origin, we use the abbreviations

$$L_{a,b}(\omega) = L_{a,b}(0; \omega), \quad L_a(\omega) = L_a(0; \omega), \quad L(\omega) = L(0; \omega). \tag{8}$$

$L_{a,b}(z'; \omega)$ is called the (a, b) *path* of the plane Brownian motion starting from z', and $L(z'; \omega)$ is called the *path* of the plane Brownian motion starting from z'.

For almost all ω, $L_{a,b}(z'; \omega)$ is a continuous image of the finite closed interval $\{t \mid a \leqslant t \leqslant b\}$ and is hence a compact subset of the plane.

A point z_0 in the plane is called a *k-multiple point* or a *multiple point of multiplicity* k, $(k = 2, 3, \ldots)$ of $L_{a,b}(z'; \omega)$ [resp. of $L_a(z'; \omega)$] if there exist k real numbers $t_1, \ldots, t_k$ with $a \leqslant t_1 < \ldots < t_k \leqslant b$ [resp. $a \leqslant t_1 < \ldots < t_k < \infty$] for which $z_0 = z' + z(t_i; \omega)$, $i = 1, \ldots, k$. It is clear that z_0 is a k-multiple point of $L_{a,b}(z'; \omega)$ [resp. $L_a(z'; \omega)$] if and only if $z_0 - z'$ is a k-tuple point of $L_{a,b}(\omega)$ [resp. $L_a(\omega)$].

P. Lévy[4] proved that almost all paths $L(\omega)$ have double (= 2-multiple) points. In a previous paper[2], we proved that if one considers Brownian motion in higher dimensional space then Lévy's result remains valid in 3-space, but that in a space of higher dimension than 3 almost all paths are free from double points. The main purpose of the present note is to prove that almost all paths of Brownian motion in the plane have points of arbitrary high (finite) multiplicity. Our methods are, in part, similar to those previously used[2], but we have to rely much more heavily on considerations of a combinatorial nature. We state explicitly our main result:

THEOREM 1. *Let* M *be the set of all* ω *for which* L(ω) *contains* k-*multiple points for every* k = 2, 3, . . ., *then* Pr(M) = 1.

Since the proof is rather involved, we shall lead to the theorem through a sequence of Lemmas.

For any point z in the plane we denote by $|z|$ the distance of z from the origin. We denote by $p(\rho, \beta)$ the probability that $L_{0,1}(z'; \omega)$ have a point in common with the circle $|z| \leqslant \rho$ when z' is a point at distance β from the origin (because of the isotropy of the Brownian motion this probability is the same for all z' with $|z'| = \beta$). In other words: we put for every $\beta \geqq 0$ and $\rho > 0$

$$p(\rho,\beta) = Pr\left(\inf_{0\leq t\leq 1} |(\beta,0)+z(t;\omega)| \leqslant \rho\right). \tag{9}$$

When $\beta = 1$ we abbreviate the notation and put

$$p(\rho) = p(\rho, 1). \tag{10}$$

LEMMA 1. p (ρ, β) *is a monotone non-decreasing function of* ρ *and a monotone non-increasing function of* β.

Proof. The assertion about ρ is obvious. To prove rigorously the other assertion, we use the homogeneity property of the Brownian motion (cf. e.g.[5]). According to this property, a space-scale change of $1 : \lambda$ is exactly compensated by a time-scale change of $1 : \lambda^2$; thus for every $\lambda > 0$ we have

$$p(\rho,\beta) = Pr\left(\inf_{0\leq t\leq \lambda^2} |(\lambda\beta, 0) + z(t,\omega)| \leqslant \lambda\rho\right),$$

whence it follows for $\lambda \geqq 1$ that

$$p(\rho, \beta) \geqq p(\lambda\rho, \lambda\beta) \geqq p(\rho, \lambda\beta).$$

LEMMA 2. *Let* $0 < \rho < \beta < R$, *then*

$$Pr(|(\beta, 0) + z(t, \omega)| \leqslant \rho \text{ before } |(\beta, 0) + z(t, \omega)| \geqq R) = \log (R/\beta)/\log (R/\rho). \tag{11}$$

Thus (11) gives an explicit expression for the probability that $L((\beta,0),\omega)$, which[2] passes with probability 1 through both circles

$$|(\beta,0) + z(t,\omega)| \leqslant \rho \text{ and } |(\beta,0) + z(t,\omega)| \leqslant R,$$

encounters the first circle before it does the second.

Proof. For a point ζ in the plane let $u(\zeta) = Pr(|\zeta+z(t,\omega)| \leqslant \rho$ before $|\zeta+z(t,\omega)| \geqq R)$, then[3] $u(\zeta)$ is a harmonic function of ζ in the ring $\rho \leqslant |\zeta| \leqslant R$ and its boundary values are $u(\zeta) = 0$ for $|\zeta| = R$ and $u(\zeta) = 1$ for $|\zeta| = \rho$. Consequently

$$u(\zeta) = \frac{\log(R/|\zeta|)}{\log(R/\rho)}, \quad \rho \leqslant |\zeta| \leqslant R.$$

LEMMA 3. *For every* $0 < \rho < 1$ *we have**

$$p(\rho) < c_1/\log(1/\rho). \tag{12}$$

Proof. By Lemmas 1 and 2 we have

$$\begin{aligned} p(\rho) &\leqslant Pr(|(1,0) + z(t,\omega)| \leqslant \rho \text{ before } |(1,0)+z(t,\omega)| \geqq 2) + \\ &+ Pr\left(\sup_{0\leq t\leq 1} |(1,0)+z(t,\omega)| \geqq 2\right) \cdot Pr\left(\inf_{0\leq t\leq 1} |(2,0) + z(t,\omega)| \leqslant \rho\right) = \\ &= \log 2/\log(2/\rho) + \delta p(\rho, 2) \leqslant \log 2/[\log 2 + \log(1/\rho)] + \delta p(\rho) \end{aligned}$$

with

$$\delta = Pr\left(\sup_{0\leq t\leq 1} |(1,0) + z(t,\omega)| \geqq 2\right) < 1.$$

Hence

$$p(\rho) \leqslant \frac{1}{1-\delta} \cdot \frac{\log 2}{\log 2+\log(1/\rho)}, < \frac{1}{1-\delta} \cdot \frac{\log 2}{\log(1/\rho)}$$

i.e., (12).

* $c_1, c_2, \ldots, c_{17}$ are finite positive constants.

LEMMA 4. *For every* $0 < \rho < \beta < 1$ *we have*

$$p(\rho, \beta) < [c_1 + \log (1/\beta)]/\log (1/\rho). \tag{13}$$

Proof. As in the proof of Lemma 3 we have

$$p(\rho, \beta) \leqslant Pr(|(\beta, 0) + z(t, \omega)| \leqslant \rho \text{ before } |(\beta, 0) + z(t, \omega)| \geqq 1) + Pr(\sup_{0 \leqq t \leqq 1} |(\beta, 0) + z(t, \omega)| \geqq 1) \cdot Pr(\inf_{0 \leqq t \leqq 1} |(1, 0) + z(t, \omega)| \leqslant \rho) < \log (1/\beta)/\log (1/\rho) + p(\rho),$$

and (13) follows from (12).

LEMMA 5. *For* $0 < \rho < 1/2$ *we have*

$$p(\rho) > c_2/\log (1/\rho). \tag{14}$$

Proof. Let $N > 1$ be an integer and put

$$t_i = 1/2 + i/2N \quad (i = 1, \ldots, N). \tag{15}$$

Let $E_i = \{\omega \mid |(1, 0) + z(t_i, \omega)| \leqslant \rho\}$ be the event that the Brownian motion starting at distance 1 from the origin be within distance ρ from the origin at the time t_i. Let p_i denote the probability that E_i occur and p_{ij} denote the probability that both E_i and E_j occur $(i, j = 1, \ldots, N)$. From (2), (4) and (15) we have

$$p_i = (1/2\pi t_i) \int\int_{u^2+v^2 \leqq \rho^2} e^{-((u-1)^2+v)^2/2t_i} \, du \, dv$$

$$> (1/2\pi) \cdot \pi \rho^2 \cdot e^{-(\rho+1)^2 - \rho^2}$$

or

$$p_i > c_3 \rho^2. \tag{16}$$

Similarly, we have for $1 \leqslant i < j \leqslant N$

$$p_{ij} \leqslant p_i \cdot Pr(|z(t_j, \omega) - z(t_i, \omega)| \leqslant 2\rho) = p_i \cdot \left(1/2\pi(t_j - t_i)\right) \int\int_{u^2+v^2 \leqq 4\rho^2} e^{-(u^2+v^2)/2(t_j - t_i)} \, du \, dv$$

and

$$p_{ij} < c_4[N/(j-i)] \rho^2 p_i. \tag{17}$$

Now, the events E_i all imply that $L_{0,1}((1, 0); \omega)$ has points within distance ρ from the origin, an event whose probability was defined by (9) and (10) as $p(\rho)$. Hence, by (17),

$$p(\rho) \geqq \sum_{i=1}^{N} p_i - \sum_{1 \leqq i < j \leqq N} p_{ij}$$

$$= \sum_{i=1}^{N} p_i (1 - \sum_{j=i+1}^{N} p_{ij}) > \sum_{i=1}^{N} p_i (1 - c_4 \rho^2 N \sum_{i=i+1}^{N} \frac{1}{j-i}) \geqq (1 - c_4 \rho^2 N \sum_{j=1}^{N-1} \frac{1}{j}) \sum_{i=1}^{N} p_i$$

whence

$$p(\rho) > (1 - c_5 \rho^2 N \log N) \sum_{i=1}^{N} p_i. \tag{18}$$

Let $c' = 1/(1+8c_5)$ and put

$$N = \left[c'/\rho^2 \log(1/\rho)\right] \tag{19}$$

(the square brackets denoting the integral part).

Then we have, for $0<\rho<1/e$,

$$\rho^2 N \log N \leqslant [c'/\log(1/\rho)] \cdot 2\log(1/\rho) < 1/2c_5. \tag{20}$$

Let now $\rho_0>0$ be such that $\rho_0<1/e$ and satisfy furthermore the condition $2\rho_0^2<c'$. Then for every $0<\rho\leqslant\rho_0$ the integer N calculated by (19) is greater than 2, we may thus apply (18) and (20) to obtain

$$p(\rho) > {}^1/_2 \sum_{i=1}^{N} p_i \qquad (0<\rho<\rho_0).$$

Thus, by (16) and (19),

$$p(\rho) > (c_3/2)\,\rho^2\left[c'/\rho^2\log(1/\rho)\right] > c_6/\log(1/\rho)$$

for $0<\rho\leqslant\rho_0$. Since $p(\rho)>p(\rho_0)>0$ for $\rho>\rho_0$, (14) is valid for $0<\rho<1/2$ and the Lemma is established.

LEMMA 6. *For every* $0<\rho<1$ *we have*

$$Pr\left(\inf_{1/2\leq t\leq 1} |z(t,\omega)| \leqslant \rho\right) < c_7/\log(1/\rho). \tag{21}$$

Proof. From (2), (4), (13) and the isotropy of the Brownian movement we have

$$Pr\left(\inf_{1/2\leq t\leq 1} |z(t,\omega)| \leqslant \rho\right) \leqslant Pr(|z({}^1/_2,\omega)| \leqslant \rho) + \int_\rho^\infty p(\rho,\beta)\, d_\beta\, Pr(|z({}^1/_2,\omega)| \leqslant \beta)$$

$$< (1/\pi)\int\int_{u^2+v^2\leq 1/4} e^{-u^2-v^2} du\, dv + \int_\rho^\infty \frac{c_1+\log(1/\beta)}{\log(1/\rho)} \cdot 2\beta e^{-\beta^2} d\beta$$

$$< \rho^2 + [2/\log(1/\rho)] \int_0^\infty [c_1+\log(1/\beta)]\beta e^{-\beta^2} d\beta$$

$$= \rho^2 + c_8/\log(1/\rho)$$

since the last integral is convergent. This proves (21) for $0<\rho<1$.

LEMMA 7. *For every* $0<\rho<1/2$ *we have*

$$Pr\left(\inf_{1/2\leq t\leq 1} |z(t,\omega)| \leqslant \rho\right) > c_9/\log(1/\rho). \tag{22}$$

Proof. By Lemma 1 and the homogeneity property we have

$$Pr\left(\inf_{1/2\leq t\leq 1} |z(t,\omega)| \leqslant \rho\right) \geqq Pr(|z(1/2,\omega)| \leqslant 1/\sqrt{2}) \cdot Pr\left(\inf_{0\leq t\leq 1/2} |(1/\sqrt{2},0) + \right.$$

$$\left. + z(t,\omega)| \leqslant \rho\right) = c_{10}\, p(\sqrt{2}\,\rho).$$

In view of (14), (22) follows for $0<\rho<1/(2\sqrt{2})$ and hence also for $0<\rho<1/2$.

The next Lemma is rather complicated, but it is quite close to the theorem we wish to prove.

LEMMA 8. *Let* k > 1 *be a fixed positive integer. Let* H *be a positive number and* n *a positive integer and let*

$$\rho = e^{-Hn^{2/k}} \tag{23}$$

For $\nu_1, \nu_2 = 1, 2, \ldots,$ n *put* $\nu = (\nu_1 - 1)n + \nu_2$, *and, for* $\nu = 1, \ldots, n^2$, *let*

$$z_\nu = \Big((1/5) + (\nu_1/2n), (1/5) + (\nu_2/2n)\Big) \tag{24}$$

and let $S_\nu(\rho)$ *denote the circle* $\{z \mid |z - z_\nu| < \rho\}$. *Let* $F_\nu = F_\nu(H)$ *be the event that there exist* k *numbers* t_i, $i = 1, \ldots, k$ *satisfying*

$$0 < t_1 \leqslant 1, \quad 1/2 \leqslant t_i - t_{i-1} \leqslant 1 \quad (i = 2, \ldots, k) \tag{25}$$

for which

$$x(t_i, \omega) \in S_\nu(\rho) \qquad (i = 1, \ldots, k).$$

Let $F^{(n)} = F^{(n)}(H)$ *be the union of the events* F_ν, $\nu = 1, \ldots, n^2$.

Then

$$\liminf_{n=\infty} Pr(F^{(n)}) > c_{11}/H^k \tag{26}$$

for all sufficiently large H.

Proof. Let q_ν be the probability that F_ν occur and $q_{\nu,\nu'}$ be the probability that both F_ν and $F_{\nu'}$ occur ($\nu, \nu' = 1, 2, \ldots, n^2$).

We may assume $H > 1$ so that $\rho < 1/e$ and the estimates of the previous Lemmas become applicable. Since $|z_\nu| < 1$ we have from Lemmas 1, 5, and 7*

$$q_\nu \geqq Pr\big(\inf_{0 < t \leqslant 1} |z(t, \omega) - z_\nu| < \rho\big) \cdot [1/3 \cdot Pr(\inf_{1/2 \leqq t \leqq 1} |z(t, \omega)| < \rho)]^{k-1}$$

$$> [c_6/\log(1/\rho)] \cdot [1/3 \cdot (c_9/\log(1/\rho)]^{k-1}.$$

Since, by (23), $[\log(1/\rho)]^k = H^k n^2$, this gives

$$q_\nu > c_{12}/H^k n^2 \quad (\nu = 1, 2, \ldots, n^2). \tag{27}$$

For $\nu \neq \nu'$ let P_1 be the probability that $L_{0,1}(\omega)$ pass through at least one of the circles $S_\nu(\rho)$ and $S_{\nu'}(\rho)$; let P_2 be an upper bound for the probability that $L_{0,1}(z; \omega)$ with $z \in S_\nu(\rho)$ encounter $S_{\nu'}(\rho)$; and let P_3 be an upper bound for the probability that a Brownian motion starting at $z' \in S_\nu(\rho)$ encounters $S_\nu(\rho)$ again for some $1/2 \leqslant t \leqslant 1$. By obvious symmetry considerations we have for $\nu \neq \nu'$ the inequality

$$q_{\nu,\nu'} < P_1(P_2 + P_3)^{2k-1}.$$

Now by Lemmas 1 and 4 we have

$$P_1 < p(\rho, |z_\nu|) + p(\rho, |z_{\nu'}|) \leqslant 2p(\rho, |z_1|) < 2\,[c_1 + \log(1/|z_1|)]\,/\,\log(1/\rho).$$

* The conditional probability of being in $S_\nu(\rho)$ at time t_i, given that the path is at $z' \in S_\nu(\rho)$ at time t_{i-1}, is $\geqq$ the probability that $z' + x(t_i - t_{i-1}, \omega)$ is in the intersection of $S_\nu(\rho)$ and a circle of radius ρ about z'; this intersection contains a sector of opening $2\pi/3$ and hence, because of the isotropy property, is greater than the second factor.

Also

$$P_2 < p(\rho, |z_\nu - z_{\nu'}| - 2\rho).$$

Now, by (24), $|z_\nu - z_{\nu'}| \geqq 1/2n$ while, by (23), we have $\rho < 1/4n$ for all $n > n_0 = n_0(k)$. Hence for all $n > n_0$ we have from (13)

$$P_2 < [c_1 + \log(2/|z_\nu - z_{\nu'}|)]/\log(1/\rho).$$

Finally, by (21),

$$P_3 < Pr\,(\inf_{1/2 \leqq t \leqq 1} |z(t,\omega)| \leqslant 2\rho) < c_7/\log(1/2\rho).$$

Combining these estimates, we have for $n > n_0$ and $\nu \neq \nu'$

$$q_{\nu,\nu'} < c_{13} \left\{ [1 + \log(1/|z_\nu - z_{\nu'}|)]/\log(1/\rho) \right\}^{2k},$$

or, by (23) and (24),

$$q_{\nu,\nu'} < [c_{14}/H^{2k} n^4] \log^{2k}(1/|z_\nu - z_{\nu'}|). \tag{28}$$

From (27) and (28) we have

$$Pr(F^{(n)}) \geqq \sum_{\nu=1}^{n^2} q_\nu - \sum_{1 \leqq \nu < \nu' \leqq n^2} q_{\nu,\nu'}$$

$$> c_{12}/H^k - c_{14}/H^{2k} \cdot 1/n^4 \cdot \sum_{1 \leqq \nu < \nu' \leqq n^2} \log^{2k}(1/|z_\nu - z_{\nu'}|).$$

Now this last sum is smaller than twice

$$n^2 \sum_{\nu=2}^{n^2} \log^{2k}(1/|z_\nu - z_1|) < n^2 \sum_{\nu=2}^{n} (2\nu+1) \log^{2k}(1/|z_\nu - z_1|)$$

$$= n^2 \sum_{\nu=2}^{n} (2\nu+1) \log^{2k}[2n/(\nu-1)]$$

$$< 5n^2 \sum_{j=1}^{n} j \log^{2k}(2n/j)$$

$$< 5n^3 \max_{1 \leqq u \leqq n} u \log^{2k}(2n/u)$$

$$< c_{15} n^4$$

(c_{15} depends on k, but k is fixed throughout).

Hence

$$Pr(F^{(n)}) > c_{12}/H^k - c_{16}/H^{2k} \tag{29}$$

for all $n > n_0$. Taking $H^k > 2c_{16}/c_{12}$ we obtain (26).

Proof of Theorem 1. Let $k > 1$ be fixed. For every $\omega \in \Omega$ let

$$g_k(\omega) = \inf \sum_{i=2}^{k} |z(t_i, \omega) - z(t_{i-1}, \omega)|$$

where the *inf* is taken over all sequences $t_1, \ldots, t_k$ satisfying (25). Since the path is continuous for almost all ω, $g_k(\omega)$ is easily seen to be a random variable. As the event $F^{(n)}$ of Lemma 8 implies $g_k(\omega) < 2k\rho$ and since $\rho \to 0$ as $n \to \infty$ we have

$$Pr(g_k(\omega) < \varepsilon) > c_{17} > 0$$

for every $\varepsilon > 0$ (where c_{17} is, of course, independent of ε). Hence, $Pr(g_k(\omega) = 0) \geqq c_{17}$ which implies, for all continuous paths, the existence of a k-multiple point of $L_{0,n}(\omega)$. For every integer $j = 1, 2, \ldots$ let G_j denote the event $L_{(j-1)k, jk}(\omega)$ has a k-multiple point. Then $Pr(G_j) = Pr(G_1) > 0$ and the events G_j are independent. Therefore, with probability 1 infinitely many of the events G_j occur, and hence there is probability 1 that $L(\omega)$ have k-multiple points. q. e. d.

Using the homogeneity property of the Brownian motion, we deduce immediately

THEOREM 2. *Let* a, b *be any positive numbers with* $0 \leqslant a < b < \infty$ *then, with probability* 1, *the* (a, b)*-path* $L_{a,b}(\omega)$ *has multiple points of arbitrarily high (finite) multiplicity.*

It also follows that there exist k-multiple points for which the intervals between returns to the point are arbitrarily large.

In view of the fact[4] that, for almost all ω, $L(\omega)$ is dense in the entire plane we have from Theorem 2 the following

THEOREM 3. *For almost all* ω *the set of k-multiple points of* $L(\omega)$ *is dense everywhere in the plane for all* $k = 2, 3, \ldots$.

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