SOME REMARKS ON SET THEORY III

by
Paul Erdős

This paper continues the treatment of some problems which were discussed in two unnumbered earlier communications [1], [2] of the same title.

1. ON A PROBLEM OF TURÁN. Suppose that with every real number x there is associated a set $S(x)$ of real numbers, called the picture of x, and subject to the restriction that x is not contained in $S(x)$. A pair of points x and y is then called independent provided neither point is contained in the picture of the other; a set E is called independent if each pair of points in E is independent.

In an oral communication, P. Turán asked whether the hypothesis that each of the pictures $S(x)$ is finite implies the existence of an infinite independent set. Grünwald [5] showed that the answer is in the affirmative. Lázár [7] proved that there exists an independent set of power c. Fodor [3], [4] pointed out that Lázár's proof gives a stronger result: *if no point x is a limit point of its picture $S(x)$, there exists an independent set E with $\overline{\overline{E}} = c$* (throughout this paper, the symbol $\overline{\overline{E}}$ denotes the cardinal number of E).

The remainder of this section deals with other hypotheses on the pictures $S(x)$, and with the question whether or not these hypotheses imply the existence of independent sets of certain cardinalities.

THEOREM 1. *The hypothesis $\overline{\overline{S(x)}} < c$ does not imply the existence of an independent pair.*

Let Ω_c be the least ordinal of power c. We arrange the real numbers in a transfinite sequence

$$\{x_0, x_1, \ldots, x_\xi, \ldots\} (\xi < \Omega_c),$$

and for $x = x_\alpha$ we define

$$S(x) = \{x_\beta \mid \beta < \alpha\}.$$

The example proves the theorem.

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THEOREM 2. The hypothesis that each picture $S(x)$ is not everywhere dense does not imply the existence of an independent pair.

To prove this proposition, it is sufficient to consider the case where $S(0)$ is empty, while for $x \neq 0$ the picture $S(x)$ consists of the open interval $(-|x|, |x|)$ together with the point $-x$. The same example proves the following result:

THEOREM 3. The hypothesis that each picture $S(x)$ is not everywhere dense and has finite measure does not imply the existence of an independent pair.

I conjecture that if $S(x)$ is not everywhere dense and the measure of $S(x)$ is bounded, there exists an independent pair, probably even an infinite independent set. Also, that if $S(x)$ is not everywhere dense and has measure less than $1/2$, there exists an independent pair of points in the interval $[0, 1]$.

THEOREM 4. If, for each x, $\overline{\overline{S(x)}} < c$ and $S(x)$ is not everywhere dense, there exists an independent pair, but not necessarily an independent triplet.

To prove the first part of the theorem, let M be any countable, dense set, and let M' be the union of the pictures of the points in M. By the theorem of J. König (see, for example, p. 6 of [9]), $\overline{\overline{M'}} < c$, since M' is the union of countably many sets of power less than c. Let z be any point in the complement of M'. Because the picture $S(z)$ is not everywhere dense, its complement contains a point x in M, and the points x and z constitute an independent pair.

The following example shows that the hypothesis of the theorem does not imply the existence of an independent triplet. Let the set of real numbers be well-ordered according to type Ω_c, and for each x let $S(x)$ be the set of numbers y which precede x in the well-ordering and which satisfy the following additional condition:

$$\text{if } x \le 1, \text{ then } y \le 1,$$
$$\text{if } -1 < x < 1, \text{ then } y \ge -1,$$
$$\text{if } 1 \le x, \text{ then } |y| \ge 1$$

Clearly, no independent triplet exists.

We turn next to the hypothesis that no point is a two-sided limit point of its picture. That this hypothesis does not imply the existence of an independent pair is seen from the example given in connection with Theorems 2 and 3.

THEOREM 5. *If no point is a two-sided limit point of its picture, and the power of $S(x)$ is less than c for each x, then there exists an infinite independent set, but not necessarily a non-denumerable independent set.*

In the proof, we can assume without loss of generality that there exist c points x such that x is not limit point of $S(x)$ from the right. Moreover, by König's theorem we can assume that there exists a positive constant p and a set I of c points x with the property that $S(x)$ has no points in the interval $(x, x + p)$; furthermore, the set I can be assumed to have diameter less than p. Let x_1 in I be a condensation point of I from the left. Since the power of $S(x_1)$ is less than c, there exists a point x_2 in I which does not lie in $S(x_1)$ and which is a condensation point of I from the left. By mathematical induction, there exists a decreasing sequence $\{x_i\}$ in I such that each point x_i lies in the complement of $S(x_j)$ for $j = 1, 2, \ldots, i - 1$. Because of the properties of the set I, each point x_i also lies in the complement of $S(x_j)$ for $j > 1$. In other words, the set $\{x_i\}$ is independent.

It remains to show that the hypothesis of the theorem does not imply the existence of an uncountable independent set. We well-order the real numbers and define $S(x)$ to consist of all numbers less than x which precede x in the well-ordering. If the set $\{z_{\alpha_\gamma}\}$ $(\alpha_1 < \alpha_2 < \ldots)$ is independent, then $z_{\alpha_1} > z_{\alpha_2} > \ldots$. Since every monotonic sequence of real numbers is at most denumerable, every independent set is at most denumerable, under our definition of $S(x)$.

THEOREM 6. *If the picture of every x is nowhere dense, there exists a denumerable independent set.*

On the one hand, I am unable to establish a stronger conclusion, even under the additional assumption that $S(x)$ is denumerable and that

$$2^{\aleph_0} = \aleph_1 ;$$

on the other hand, I do not know of an example which shows that the theorem cannot be improved.

Let $S^{-1}(x)$ denote the set of points in whose picture x occurs. Let M be any set of the second category. We will first prove the preliminary assertion that there exists a point x such that the set $M - S^{-1}(z)$ is also of the second category (here the symbol A-B denotes the set of points which belong to A but not to B). If the assertion is false, let $\{z_i\}$ be any set which is dense in M. Since each set $M - S^{-1}(z_i)$ is of the first category and

$$\bigcap_{i=1}^{\infty} S^{-1}(z_i) \supset M - \bigcup_{i=1}^{\infty} [M - S^{-1}(z_i)] ,$$

the set $\bigcap S^{-1}(z_i)$ is not empty, that is, it contains at least one point y. By construction, the picture $S(y)$ contains each of the points z_i. It follows that the picture $S(y)$ cannot be nowhere dense, contrary to the hypothesis of the theorem. This proves our preliminary assertion.

We now prove our theorem. Let M_1 be any set of the second category, and x_1 a point in M_1 such that $M_1 - S^{-1}(x_1)$ is also of the second category. Since $S(x_1)$ is nowhere dense, the set

$$M_2 = M_1 - S(x_1) - S^{-1}(x_1)$$

is also of the second category. Moreover, each pair composed of x_1 and a point in M_2 is independent. Let x_2 be a point in M_2 such that $M_2 - S^{-1}(x_2)$ is of the second category, and let

$$M_3 = M_2 - S(x_2) - S^{-1}(x_2) .$$

Clearly, the construction can be continued indefinitely, and the set $\{x_i\}$ thus obtained is independent.

2. ON **DISSECTIONS OF THE REAL-NUMBER CONTINUUM.** Sierpiński [8] constructed two disjoint sets A and B whose union is the set of all real numbers, and which have the additional property that every translation of A intersects B in a set of power less than c. The latter property can be described thus: for every real number z, the equation $x - y = z$ has fewer than c solutions with x in A and y in B.

THEOREM 7. <u>There exist two sets $A_i (i = 1, 2)$ whose union is the set of all real numbers, and such that each of the sets A_i has the property that for every real z the equation</u>

$$x + y = z \ (x,\ y \text{ in } A_i)$$

<u>has fewer than c solutions</u>.

To prove this theorem, let $\{a_\alpha\}$ $(\alpha < \Omega_c)$ be a Hamel basis [6]. Let the number

$$z = \sum_{k=1}^{n} \gamma_k \, a_{\alpha_k} \quad (\alpha_1 < \alpha_2 < \ldots \alpha_n ;\ \gamma_k \text{ rational};\ n = n_z)$$

belong to A_1 if $\gamma_n > 0$, to A_2 if $\gamma_n < 0$. Then, if $z = x + y$ and x and y belong to the same set A_i, no a_α with $\alpha > \alpha_n$ can occur in the representation of either x or y, and therefore there exist fewer than c choices for x and y. This proves the theorem. It is easily seen that our sets A_1 and A_2 also have the Sierpiński property.

In an oral communication, P. Lax proved that if

i) $A_1 \cup A_2$ is the set of all real numbers,

ii) $\overline{\overline{A_1}} = \overline{\overline{A_2}} = c$,

iii) m is a cardinal less than c,

then there exists a translation of A_1 which intersects A_2 in a set whose power is at least m (see [1], p. 646). Analogously we can prove that under Lax's conditions there exists a number z such that, for a suitable choice of i, the equation $z = x + y$ has at least m solutions with x and y both in A_i. In fact, we shall prove a slightly stronger result:

THEOREM 8. If

i) $A_1 \cup A_2$ is the set of all real numbers,

ii) $\overline{\overline{A_1}} = \overline{\overline{A_2}} = c$,

iii) m is a cardinal less than c ,

iv) for every z the equation $z = x + y$ has fewer than m solutions with x and y both in A_1 ,

then for some z the equation $z = x + y$ has c solutions with x and y both in A_2.

LEMMA. Let $m < n$ be two infinite cardinal numbers, let $\overline{\overline{S}} = n$, and let $\{S_\alpha\}$ $(1 \leq \alpha < \Omega_n)$ be a collection of n subsets of S, each of cardinality less than M. Then there exists a subcollection

$$\{S_{\alpha_k}\} \ (1 \leq k < \Omega_n)$$

of n subsets whose union is a proper subset of S.

To prove the lemma, we need to show that there exists a point x in S and a collection of n sets S_{α_k} $(1 \leq k < \Omega_n)$ none of which contains x. Suppose that such an x does not exist. Then to every x in S there corresponds a set $P(x)$ of ordinals $\alpha < \Omega_n$, with $\overline{\overline{P(x)}} < n$, and such that x belongs to S_α whenever α does not belong to $P(x)$.

Two cases arise. If n is not the union of m or fewer smaller cardinals, let $\{x_r\}$ $(1 \leq r < \Omega_m)$ be an arbitrary collection of elements of S. Since the union of the sets $P(x_r)$ has cardinality less than n, there exists an ordinal β $(1 \leq \beta < \Omega_n)$ which is not contained in this union. But this implies that x_r is in S_β for $1 \leq r < \Omega_m$, contrary to the hypothesis that $\overline{\overline{S_\beta}} < m$.

We proceed to the second case. If n is the union of m or fewer cardinals, let $\{n_i\}$ $(1 \leq i < \Omega_m)$ be an increasing sequence of cardinals whose limit is n. For each index i, let $S^{(i)}$ be composed of the points x for which the cardinality of $P(x)$ does not exceed n_i. Clearly, $\cup S^{(i)} = S$; also, since $m^2 = m$, there exists an i for which $S^{(i)}$ has power greater than m. Let S^* be a subset of $S^{(i)}$, with cardinality m. If x is in S^*, then the power of $P(x)$ is at most n_i; because $m \cdot n_i < n$, there again exists an ordinal β ($1 \leq \beta < \Omega_n$) such that β is not contained in $P(x)$ for any x in S^*. But this implies that $S_\beta \supset S^*$, contrary to the hypothesis that S_β has power less than m. The proof of the lemma is complete.

Before we go on with the proof of the theorem, some digressions on the lemma are appropriate. The proof of the lemma really depended on the following fact:

Let $m < n$, $\overline{\overline{S}} = n$, $S_\alpha \subset S$ for $1 \leq \alpha < \Omega_n$, and $\overline{\overline{S}}_\alpha < n$ for each α. Then there exists a cardinal $p < n$ and a subcollection of m sets S_{α_k} ($1 \leq k < \Omega_m$), each of power at most p.

If we assume the generalized continuum hypothesis $2^{\aleph_k} = \aleph_{k+1}$, we can strengthen the lemma as follows:

Let $m < n$ be two infinite cardinal numbers; let $\overline{\overline{S}} = n$, and let $\{S_\alpha\}$ ($1 \leq \alpha < \Omega_n$) be a collection of n subsets of S, each of power less than m. Then there exists a subcollection $\{S_{\alpha_k}\}$ ($1 \leq k < \Omega_n$) of n of these subsets such that the complement relative to S of their union has power n.

We well-order the elements a_α ($1 \leq \alpha < \Omega_n$) of S. With each S_α we associate the element of at least index which is not contained in it and which does not correspond to any S_β with $\beta < \alpha$. There may be some points of S which are not associated with any set S_α; but the set of these points has power at most m; we now suppose that these points are deleted from S. The remaining set S^* clearly has the power n, since with each set S_α there is associated a point of S^*; that is, with each x in S^* there is associated an S_{α_x}, x not in S_{α_x}, $\overline{\overline{S_{\alpha_x}}} < m$. A pair of points x and y of S^* is independent provided $x \notin S_{\alpha_y}$ and $y \notin S_{\alpha_x}$. By a previous result of mine (see Theorem V, pp. 133-137 of [2]), there exists an independent set S^{**} of power n. Let $\{S_{\alpha_k}\}$ ($1 \leq k < \Omega_n$) be the collection of sets S_α associated with points x in S^{**}. The union of the complements of the sets S_{α_k} contains S^{**}; therefore it has power n, and the modified version of our lemma is extablished.

We now return to the theorem. Suppose that for every real number t the equation $t = x + y$ has fewer than m solutions ($m < c$) with x and y in A_1. If the power of A_1 is less than c, then for every t the equation $t = x + y$ clearly has c solutions with x and y in A_2, and the theorem holds. We may therefore assume that the power of A_1 is c.

Let $A_1(t)$ denote the set of real numbers x in A_1 for which either $t - x$ or $1 - t - x$ is in A_1. Clearly, $A_1(t)$ has power less than m, since by assumption there exist fewer than m solutions of the equation $t = x + y$ with x and y in A_1. Since T can run through the set of all real numbers, we thus obtain c sets $A_1(t)$ in A_1. And since $A_1(t)$ has power less than m and A_1 has power c, our lemma implies the existence of a z in A_1 and a set $\{t_\beta\}$ $(1 \leq \beta < \Omega_c)$ of real numbers such that z is not in $A(t_\beta)$ for $1 \leq \beta < \Omega_c$. But this means that, for each β, $t_\beta - z$ and $1 - t_\beta - z$ are in A_2. We conclude that the equation

$$1 - 2z = u + v$$

has c solutions with

$$u = t_\beta - z \in A_2,$$

$$v = 1 - t_\beta - z \in A_2 .$$

This proves the theorem.

REFERENCES

1. P. Erdös, Some remarks on set theory, Ann. of Math. ser. 2 vol. 44 (1943) pp. 643-646.

2. ________, Some remarks on set theory, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 127-141.

3. G. Fodor, On two problems concerning the theory of binary relations, Publ. Math. Debrecen vol. 1 (1950) pp. 199-200.

4. ________, On a theorem in the theory of binary relations, Compositio Math. vol. 8 (1951) p. 250.

5. G. Grünwald, Über einen mengentheoretischen Satz, Mat. Fiz. Lapok vol. 44 (1937) pp. 51-53.

6. G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$, Math. Ann, vol. 60 (1905), pp. 459-462.

7. D. Lázár, On a problem in the theory of aggregates, Compositio Math. vol. 3 (1946) p. 304.

8. W. Sierpiński, Sur les translations des ensembles linéaires, Fund. Math. vol. 19 (1932) pp. 22-28.

9. ____________, Hypothèse du Continu, Warsaw, 1934.

University of Notre Dame