ON POWER SERIES DIVERGING EVERYWHERE ON THE CIRCLE OF CONVERGENCE

A. Dvoretzky and P. Erdös

1. Lusin [4] (see also Dienes [1, pp. 463, 464] or Landau [3, §15]) constructed a power series

(1) $$\sum_{n=0}^{\infty} a_n z^n$$

which satisfies the condition

(2) $$\lim_{n\to\infty} a_n = 0$$

and diverges at every point of the unit circle C. Recently, Herzog [2] gave an example of such a series whose coefficients are real, nonnegative, and satisfy not only (2), but even the stronger condition $a_n = O(n^{-1/3})$. The theorem which we are about to state and prove implies the existence of a series (1) which diverges everywhere on C and satisfies, e.g., the condition $0 < a_n < (n \log n)^{-1/2}$ $(n = 3, 4, \cdots)$.

THEOREM 1. *Let* $\{b_n\}$ *be a sequence of complex numbers satisfying the conditions*

(3) $$|b_n| \geq |b_{n+1}| \quad (n = 0, 1, \cdots)$$

and

(4) $$\sum_{n=0}^{\infty} |b_n|^2 = \infty.$$

Then there exists a power series (1), *with*

(5) $$a_n \text{ equal to either } b_n \text{ or } 0 \quad (n = 0, 1, \cdots),$$

which diverges everywhere on C.

The monotonicity condition (3) cannot be entirely dispensed with, since every power series $\sum_1^\infty c_n z^{t_n}$ with $c_n \to 0$ and $\sum_1^\infty t_n/t_{n+1} < \infty$ converges on a set which is everywhere dense on C. Condition (4) probably cannot be relaxed at all; indeed, it has been conjectured that every power series $\sum b_n z^n$ satisfying (4) converges almost everywhere on C.

Received February 23, 1955.

2. We proceed to the simple constructive proof of Theorem 1. Obviously we may assume that

(6)
$$\lim_{n\to\infty} b_n = 0.$$

LEMMA. *Let the sequence* $\{b_n\}$ $(n = 0, 1, \cdots)$ *satisfy conditions* (3), (4) *and* (6). *Then there exists an increasing sequence of integers* k_i $(i = 1, 2, \cdots)$ *which satisfies the condition*

(7)
$$\sum_{i=1}^{\infty} \frac{1}{k_{i+1} - k_i} = \infty,$$

and for which

(8)
$$1 < \sum_{n=k_i}^{k_{i+1}-1} |b_n| < 2 \quad (i = 1, 2, \cdots).$$

Indeed, let k_1 be the smallest positive integer ν for which $|b_\nu| < 1$ and, having determined $k_1, \cdots, k_i$, let $k_{i+1} > k_i$ be determined by the inequalities

$$\sum_{n=k_i}^{k_{i+1}-2} |b_n| \le 1, \qquad \sum_{n=k_i}^{k_{i+1}-1} |b_n| > 1$$

(such an integer k_{i+1} exists, since (4) implies that $\sum_0^\infty |b_n| = \infty$. Then (8) clearly holds, and we have, for $i > 1$,

$$\sum_{n=k_i}^{k_{i+1}-1} |b_n|^2 \le |b_{k_i}| \sum_{n=k_i}^{k_{i+1}-1} |b_n| < 2|b_{k_i}| < 2\frac{2}{k_i - k_{i-1}};$$

therefore (7) follows from (4).

Remark. Under the hypothesis (3), condition (4) is not only sufficient but also necessary for the existence of a sequence $\{k_i\}$ satisfying the conclusion of the lemma. Indeed, if $\{k_i\}$ is any such sequence, then

$$\sum_{n=k_i}^{k_{i+1}-1} |b_n|^2 \ge |b_{k_{i+1}}| \sum_{n=k_i}^{k_{i+1}-1} |b_n| > |b_{k_{i+1}}| > \frac{1}{k_{i+2} - k_{i+1}},$$

and therefore (7) implies (4). (Only the first of the inequalities (8) was used, here.)

We are now ready to construct the series (1) of the theorem. For $i = 1, 2, \cdots$, let z_i be the point on C whose argument is

$$2\pi i \sum_{j=1}^{i} \frac{1}{k_{j+1} - k_j}$$

Because of (6) and (7), the sequence $\{z_i\}$ $(i = 1, 2, \ldots)$ is everywhere dense on C, and therefore each point on C belongs to infinitely many of the arcs

$$A_i: \ |z| = 1, \ \arg z_i \leq \arg z \leq \arg z_{i+1}.$$

Consider now the numbers $b_n z_i^n$ $(k_i \leq n < k_{i+1})$, and let $Q_i^{(\nu)}$ $(\nu = 0, 1, 2)$ denote the sum $\sum' |b_n|$, extended over those indices n in the range $k_i \leq n < k_{i+1}$ for which $|\arg b_n z_i^n - 2\nu\pi/3| \leq \pi/3$. Clearly

$$Q_i^{(0)} + Q_i^{(1)} + Q_i^{(2)} \geq \sum_{n=k_i}^{k_{i+1}-1} |b_n|;$$

hence, for at least one $\nu = \nu_i$, we obtain from (8) the inequality $1/3 < Q_i^{(\nu_i)} < 2$. We now define the coefficients a_n as follows: for $n < k_1$, $a_n = 0$; for $k_i \leq n < k_{i+1}$, we choose

$$\begin{aligned} a_n &= b_n \quad \text{if } |\arg b_n z_i^n - 2\nu_i\pi/3| \leq \pi/3, \\ a_n &= 0 \quad \text{otherwise.} \end{aligned}$$

We claim that the series (1) thus constructed diverges everywhere on C. Indeed:

$$\left| \sum_{n=k_i}^{k_{i+1}-1} a_n z_i^n \right| \geq \tfrac{1}{2} \sum_{n=k_i}^{k_{i+1}-1} |a_n z_i^n| = \tfrac{1}{2} Q_i^{(\nu_i)} > 1/6.$$

Moreover, writing

$$P_i(z) = z^{-k_i} \sum_{n=k_i}^{k_{i+1}-1} a_n z^n$$

we have, for the derivative of this polynomial in $|z| \leq 1$,

$$\begin{aligned} |P_i'(z)| &\leq \sum_{m=0}^{k_{i+1}-k_i-1} m |a_{k_i+m}| \, |z^{m-1}| \\ &\leq (k_{i+1} - k_i - 1) \sum_{m=k_i}^{k_{i+1}-1} |a_m| \end{aligned}$$

$$< (k_{i+1} - k_i) \sum_{n=k_i}^{k_{i+1}-1} |b_n| < 2(k_{i+1} - k_i).$$

Hence the variation of $P_i(z)$ on the arc A_i is smaller than

$$2(k_{i+1} - k_i)|z_{i+1} - z_i| < 2(k_{i+1} - k_i)\frac{1}{24(k_{i+1} - k_i)} = 1/12.$$

Since $|P_i(z_i)| > 1/6$ we have therefore, for every z on A_i,

$$\left|\sum_{n=k_i}^{k_{i+1}-1} a_n z^n\right| = |P_i(z)| > 1/6 - 1/12 = 1/12.$$

Since every point on C belongs to infinitely many arcs A_i, it is clear that the series (1) cannot converge anywhere on C. This completes the proof.

3. Our theorem admits some easy extensions. We mention the following.

Under the assumptions of the theorem there exists a series (1) *which satisfies* (5) *and for which*

$$\limsup_{N \to \infty} \sum_{n=0}^{N} a_n z^n = \infty$$

everywhere on C.

Indeed, it is clear from the lemma that there also exists an increasing sequence of integers k_i which satisfies (7) and for which

$$B_i = \sum_{n=k_i}^{k_{i+1}-1} |b_n| \to \infty$$

as $i \to \infty$. Operating with such a sequence $\{k_i\}$, we get $|P_i(z)| > B_i/12$ for all z on A_i, and the result follows.

It should also be remarked that though we can not dispense altogether with the condition of monotonicity, we can easily relax it in various ways. For instance, it is sufficient to assume that there exist disjoint blocks (N_i, N_{i+1}) such that $|b_n|$ is monotone within each block $N_i < n < N_{i+1}$, the sum $\sum |b_n|$ over each block is bounded away from zero, and the series $\sum |b_n|^2$, extended over all blocks, is divergent. The monotonicity within the block can also be replaced by certain weaker requirements.

The following is another obvious consequence of Theorem 1: *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ $(n = 0, 1, \cdots)$ *be two sequences of complex numbers such that* $\sum |\alpha_n - \beta_n|^2$

is a monotone divergent series. Then there exists a power series (1) *with* a_n *equal either to* α_n *or to* β_n ($n = 0, 1, \cdots$), *which diverges everywhere on* C.

All the results above extend also to Dirichlet series $\sum a_n e^{-\lambda_n s}$ satisfying the restriction $\lambda_{n+1} - \lambda_n = O(1)$, as well as to Laplace integrals $\int_0^\infty a(s) e^{st} \phi(t) dt$ satisfying the condition that, for some H and all T,

$$\int_T^{T+H} \phi(t) dt > 1.$$

4. The following result is somewhat connected with the main problem of this paper.

THEOREM 2. *If* $\sum |b_n| = \infty$, *then there exists a power series* (1) *which satisfies* (5) *and which diverges everywhere on a residual set on* C.

The condition $\sum' |b_n| = \infty$ is clearly necessary in order that (1) fail to converge uniformly and absolutely. The proof of the present result is even simpler than that of Theorem 1; in particular, the lemma is not needed. We choose a sequence $\{z_i\}$ which is dense on C, and we write

$$\{y_i\} = \{z_1, z_2, z_1, z_2, z_3, z_1, \cdots\}.$$

Then, having determined the sequence $\{k_i\}$ according to (8), we choose the coefficients a_n so that $|P_i(y_i)| > 1/6$. Since then $|P_i(z)| \geq 1/6$ on an arc of C through y_i, the result follows. A slight modification yields a series whose partial sums are unbounded on a residual set on C.

REFERENCES

1. P. Dienes, *The Taylor series*, Oxford, 1931.

2. F. Herzog, *A note on power series which diverge everywhere on the unit circle*, Michigan Math. J. 2 (1953-1954), 175-177.

3. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 2nd edition, Berlin, 1929.

4. N. Lusin, *Über eine Potenzreihe*, Rend. Circ. Mat. Palermo 32 (1911), 386-390.

The Hebrew University
Jerusalem