ON THE LAW OF THE ITERATED LOGARITHM. II

BY

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3. *The lower estimate in the law of the iterated logarithm*

From now on let $n_1 < n_2 < \ldots < n_\nu < \ldots$ be a fixed lacunary sequence satisfying $n_{\nu+1}/n_\nu \geqslant q > 1$; $\nu = 1, 2, 3, \ldots$. For the sake of simplicity let for $N \geqslant N_0$

$$\psi(N) = \sqrt{N \log \log N}$$

and for $M \geqslant 0$, $N \geqslant 1$

$$F(M, N; x) = \left| \sum_{\nu=M+1}^{M+N} \exp 2\pi i\, n_\nu x \right|.$$

For convenience's sake we introduce also $F(M, 0; x) = 0$ for $M \geqslant 0$.

We want to prove that

$$\limsup_{N\to\infty} \frac{F(0, N; x)}{\psi(N)} \geqslant 1 \tag{25}$$

almost everywhere. Obviously it will be sufficient to prove the following: given arbitrarily small numbers $\varepsilon > 0$ and $\eta > 0$

$$\limsup_{N\to\infty} \frac{F(0, N; x)}{\psi(N)} \geqslant 1 - \varepsilon$$

for every x; $0 \leqslant x \leqslant 1$ except possibly a set of measure at most η. Finally it is also clear that this second statement is a consequence of the following third one:

Lemma 9. *Let $\varepsilon > 0$, $\eta > 0$ be arbitrarily small and let the positive integer N be arbitrarily large. Then there exists a finite sequence of integers $N < N_1 < N_2 < \ldots < N_k$ such that*

$$\underset{1 \leqslant \nu \leqslant k}{\text{maximum}} \frac{F(0, N_\nu; x)}{\psi(N_\nu)} \geqslant 1 - \varepsilon$$

for every x; $0 \leqslant x \leqslant 1$ except possibly a set of measure at most η.

In the proof of this lemma we shall use the following trivial result:

Lemma 10. *Let $I_1, I_2, \ldots, I_m$ and $J_1, J_2, \ldots, J_n$ be arbitrary intervals on the real line. Then the intersection $(I_1 + I_2 + \ldots + I_m) \cap (J_1 + J_2 + \ldots + J_n)$ consists of intervals the number of which is less than $m + n$.*

Now let $\varepsilon > 0$ be given and let $a \geqslant a_0(\varepsilon)$, $u \geqslant 1$ be arbitrary integers the exact value of which will be determined at the end of the following proof. At present the condition $a \geqslant a_0(\varepsilon)$ assures only that Lemma 8 can be applied to any of the sums

$$F_k(x) = F(a^u + a^{u+1} + \dots + a^{u+k-1}, a^{u+k}; x) \tag{27}$$

where $k = 1, 2, 3, \dots$.

For the sake of simplicity let $\psi_k = \psi(a^{u+k})$, and let I denote the interval $0 \leqslant x \leqslant 1$. We define the set

$$E_1 = \left\{ x \mid x \varepsilon I;\ F_1(x) \geqslant \left(1 - \frac{\varepsilon}{2}\right) \psi_1 \right\}$$

and in general

$$E_k = \left\{ x \mid x \varepsilon I - (E_1 + E_2 + \dots + E_{k-1});\ F_k(x) \geqslant \left(1 - \frac{\varepsilon}{2}\right) \phi_k \right\} \tag{28}$$

for $k = 1, 2, 3, \dots$ (E_0 denotes the empty set). Our object is to obtain an upper estimate for the measure

$$\mu(I - (E_1 + E_2 + \dots + E_k)).$$

To this end we consider $E_k (k = 1, 2, 3, \dots)$ and estimate $\mu(E_k)$ from below.

Let us introduce the notation

$$m_k = n(a^u + a^{u+1} + \dots + a^{u+k})$$

where $n_1 = n(1)$, $n_2 = n(2)$, ... denotes the given lacunary sequence. Since $F_k(x)^2$ is a trigonometric polynomial of degree $2m_k$ the set

$$\left\{ x \mid x \varepsilon I;\ F_k(x) < \left(1 - \frac{\varepsilon}{2}\right) \psi_k \right\}$$

consists of at most $4m_k$ intervals. In particular $I - E_1$ consists of ϱ_1 intervals where $\varrho_1 < 4m_1$. In general it is true that the set $I - (E_1 + E_2 + \dots + E_k)$ consists of ϱ_k intervals where

$$\varrho_k < 4(m_1 + m_2 + \dots + m_k) < 4\, k m_k.$$

For, according to the definition of $E_1, E_2, \dots, E_k$ in (28) we have

$$I - (E_1 + E_2 + \dots + E_k) =$$
$$= [I - (E_1 + E_2 + \dots + E_{k-1})] \cap \left\{ x \mid x \varepsilon I;\ F_k(x) < \left(1 - \frac{\varepsilon}{2}\right) \phi_k \right\}.$$

Hence using Lemma 10 we obtain $\varrho_k < \varrho_{k-1} + 4m_k$, which proves the above estimate.

Now let $e_{k+1} (k \geqslant 1)$ be the union of those intervals of

$$I - (E_1 + E_2 + \dots + E_k)$$

the length of which is less than $\delta_k = m_k^{-1} a^{-(u+k+1)/2}$. Then we have

$$\mu(e_{k+1}) \leqslant \varrho_k\, \delta_k < 4\, k m_k\, \delta_k = 4\, k a^{-(u+k+1)/2}. \tag{29}$$

The set $I - (E_1 + E_2 + \dots + E_k) - e_{k+1}$ consists of intervals the length of which is at least

$$\delta_k > 1/n(1 + a^u + a^{u+1} + \dots + a^{u+k}) \sqrt{a^{u+k+1}}.$$

Hence the condition $\beta - \alpha \geqslant 1/n_1 \sqrt{N}$ of Lemma 8 is satisfied for every interval of the set $I-(E_1+E_2+\ldots+E_k)-e_{k+1}$. Since $a \geqslant a_0(\varepsilon)$ we may use Lemma 8 in order to estimate $\mu(E_{k+1})$:

$$\mu(E_{k+1}) \geqslant \frac{\mu[I-(E_1+E_2+\ldots+E_k)-e_{k+1}]}{(u+k+1)\log a}$$
$$\geqslant \frac{\mu[I-(E_1+E_2+\ldots+E_k)]}{(u+k+1)\log a} - \mu(e_{k+1}).$$

According to Lemma 8 we also have

$$\mu(E_1) \geqslant \frac{1}{(u+1)\log a} = \frac{\mu(I)}{(u+1)\log a}.$$

From these last inequalities we obtain by induction on k;

$$\mu[I-(E_1+E_2+\ldots+E_k)] \leqslant \prod_{\nu=1}^{k}\left(1-\frac{1}{(u+\nu)\log a}\right)+\sum_{\nu=2}^{k}\mu(e_\nu).$$

By (29),

$$\sum_{\nu=2}^{k}\mu(e_\nu) \leqslant 4\,a^{-u/2}\sum_{k=1}^{\infty} k a^{-(k+1)/2}$$
$$= O\,a^{-u/2}.$$

Finally it follows that

$$\mu[I-(E_1+E_2+\ldots+E_k)] \leqslant$$
$$\prod_{\nu=1}^{k}\left(1-\frac{1}{(u+\nu)\log a}\right)+O(a^{-u/2}).$$

Let us introduce the notation

$$N_k = a^u + a^{u+1} + \ldots + a^{u+k}\ (k \geqslant 0)$$

and let us define the sets $E_k'\ (k \geqslant 1)$ as

$$E_k' = \{x \mid x \varepsilon I;\ F(0, N_{k-1}; x) \geqslant \sqrt{2}\phi(N_{k-1})\}. \tag{30}$$

The measure $\mu(E_k')$ can be estimated by Lemma 7, and it follows that

$$\mu(E_k') < \frac{18\log\log N_{k-1}}{(\log N_{k-1})^2} < \frac{1}{2(u+k-1)^{3/2}}$$

(provided $a \geqslant a_0$). Hence

$$\sum_{\nu=1}^{k}\mu(E_\nu') < \frac{1}{2}\left(\frac{1}{u^{3/2}}+\frac{1}{(u+1)^{3/2}}+\ldots\right) < \frac{1}{\sqrt{u-1}}.$$

Using our previous estimates we see that

$$\left\{\begin{aligned} &\mu[I-(E_1+E_2+\ldots+E_k)]+\mu(E_1'\cup E_2'\cup\ldots\cup E_k') \\ &\qquad < \prod_{\nu=1}^{k}\left(1-\frac{1}{(u+\nu)\log a}\right)+\frac{1}{\sqrt{u-1}}+O(a^{-u/2}).\end{aligned}\right. \tag{31}$$

Having this inequality Lemma 9 can be proved easily as follows.

According to the definition of $F(M, N; x) \geqslant 0$ we have

$$\frac{F(0, N_\nu; x)}{\psi(N_\nu)} \geqslant \frac{F_\nu(x)}{\psi_\nu} \cdot \frac{\psi_\nu}{\psi(N_\nu)} - \frac{F(0, N_{\nu-1}; x)}{\sqrt{2}\,\psi(N_{\nu-1})} \cdot \frac{\sqrt{2}\,\psi(N_{\nu-1})}{\psi(N_\nu)}.$$

An elementary computation shows that

$$\frac{\psi_\nu}{\psi(N_\nu)} \geqslant \frac{\log u}{(1+(2/a))\log(u+1)} > \frac{1}{(1+(2/a))\,(1+(1/u))}$$

for any $\nu = 1, 2, 3, \ldots; a \geqslant 2$ and $u \geqslant 3$. Similarly one shows that $\psi(N_{\nu-1})/\psi(N_\nu) < \sqrt{2/(a-1)}$ for any $\nu = 1, 2, 3, \ldots; a \geqslant 2$ and $u \geqslant 0$. Hence we have

$$\frac{F(0, N_\nu; x)}{\psi(N_\nu)} \geqslant \frac{F_\nu(x)}{\psi_\nu} \cdot \frac{1}{(1+(2/a))\,(1+(1/u))} - \frac{F(0, N_{\nu-1}; x)}{\sqrt{2}\,\psi(N_{\nu-1})} \cdot \sqrt{\frac{4}{(a-1)}}$$

for any $\nu = 1, 2, 3, \ldots; a \geqslant 2$ and $u \geqslant 3$.

If we restrict ourselves to those x's which belong to the set

$$E = (E_1 + E_2 + \ldots + E_k) \cap (I - E_1' \cup E_2' \cup \ldots \cup E_k')$$

then by (30) $F(0, N_{\nu-1}; x) < \sqrt{2}\,\psi(N_{\nu-1})$ for every $\nu = 1, 2, \ldots, k$ and by (28) $F_\nu(x)/\psi_\nu \geqslant 1 - \varepsilon/2$ for a suitable $\nu = \nu(x) \leqslant k$. Hence on the set E we have

$$\underset{1 \leqslant \nu \leqslant k}{\text{maximum}} \frac{F(0, N_\nu; x)}{\psi(N_\nu)} \geqslant \left(1 - \frac{\varepsilon}{2}\right) \frac{1}{(1+(2/a))\,(1+(1/u))} - \sqrt{\frac{4}{a-1}}.$$

Moreover according to (31) we have

$$(^{**}) \qquad \mu(E) \geqslant 1 - \prod_{\nu=1}^{k} \left(1 - \frac{1}{(u+\nu)\log a}\right) - \frac{1}{\sqrt{u-1}} - O(a^{-u/2}).$$

Now the truth of Lemma 9 is clear: Given $\varepsilon > 0$, $\eta > 0$ and N, first we choose $a = a(\varepsilon)$ such that $a \geqslant a_0(\varepsilon)$, $a > N$,

$$\sqrt{\frac{4}{a-1}} \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{1+(2/a)} \geqslant 1 - \frac{\varepsilon}{4}.$$

Then $N_\nu > N$ $(\nu \geqslant 1)$ is satisfied. Next we choose $u = u(q, \varepsilon, \eta) \geqslant 3$ such that $\frac{1}{1+(1/u)} \geqslant 1 - \frac{\varepsilon}{4}$, and the sum of the last two terms in (**) is numerically les than $\eta/2$.

Finally we choose $k = k(q, \varepsilon, \eta)$ such that

$$\prod_{\nu=1}^{k} \left(1 - \frac{1}{(u+\nu)\log a}\right) \leqslant \frac{\eta}{2}.$$

Then (26) holds on the set E and $\mu(E) \geqslant 1 - \eta$. This proves the lower estimate in the law of the iterated logarithm.

4. *The upper estimate in the law of the iterated logarithm*

It will be sufficient to prove the following statement:

Lemma 11. *Given arbitrarily small numbers $\varepsilon > 0$, $\eta > 0$ there exists* *an* $N_0 = N_0(q, \varepsilon, \eta)$ *such that*

$$F(0, N; x) \leqslant (1+\varepsilon)\phi(N) \tag{32}$$

for every $N \geqslant N_0$ *and every* $x \varepsilon I$ *except possibly on a set of measure at most* η.

The proof of this lemma consists of three steps. Let $a > 1$ be a parameter which will tend to 1 at the end of our proof. We start from the following:

Lemma 12. *Let* $N \geqslant N_0(a)$ *be an arbitrary integer. Let the integers* $n = n(N) \geqslant 0$, $\Lambda = \Lambda(n) \geqslant 1$ *and* $1 \leqslant \lambda = \lambda(n) \leqslant \Lambda(n)$ *be defined by the inequalities* $[a^n] \leqslant N < [a^{n+1}]$, $2^\Lambda \leqslant [a^{n+1}] - [a^n] < 2^{\Lambda+1}$ *and* $2^{\lambda-1} \leqslant [a^n]^{1/3} < 2^\lambda$. *Then there exist integers* $N^* < 2[a^n]^{1/3} < 2N^{1/3}$ *and* m_l *satisfying*

$$0 \leqslant m_l < 2^{\Lambda-l+1} \ (l = \lambda, \lambda+1, \ldots, \Lambda+1)$$

such that

$$\left\{ \begin{aligned} F(0, N; x) \leqslant F(0, [a^n]; x) &+ \sum_{l=\lambda}^{\Lambda} F([a^n] + m_{l+1} 2^{l+1}, 2^l; x) \\ &+ F([a^n] + m_\lambda 2^\lambda, N^*; x). \end{aligned} \right. \tag{33}$$

Proof of Lemma 12. According to the definition of $F(M, N; x) \geqslant 0$ we have

$$F(M, N; x) \leqslant F(M, N'; x) + F(M + N', N - N'; x) \tag{34}$$

for any $M \geqslant 0$ and $0 \leqslant N' \leqslant N$. Since $[a^n] \leqslant N$ we obtain immediately

$$F(0, N; x) \leqslant F(0, [a^n]; x) + F([a^n], N - [a^n]; x). \tag{35}$$

Using the definition of n, Λ and $\lambda \leqslant \Lambda$ we see that

$$0 \leqslant N - [a^n] < [a^{n+1}] - [a^n] < 2^{\Lambda+1}$$

and so

$$N - [a^n] = \varepsilon_\Lambda 2^\Lambda + \varepsilon_{\Lambda-1} 2^{\Lambda-1} + \ldots + \varepsilon_\lambda 2^\lambda + N^*,$$

where $\varepsilon_l = 0, 1$ $(l = \lambda, \lambda+1, \ldots, \Lambda)$ and $N^* < 2^\lambda \leqslant 2[a^n]^{1/3}$.

Now we return to (35) and apply (34) repeatedly to obtain the inequality

$$\begin{aligned} F(0, N; x) &\leqslant F(0, [a^n]; x) + F([a^n], \varepsilon_\Lambda 2^\Lambda + \ldots + \varepsilon_\lambda 2^\lambda + N^*; x) \\ &\leqslant F(0, [a^n]; x) + F([a^n], \varepsilon_\Lambda 2^\Lambda; x) \\ &+ \sum_{l=\lambda}^{\Lambda-1} F([a^n] + \varepsilon_\Lambda 2^\Lambda + \ldots + \varepsilon_{l+1} 2^{l+1}, \varepsilon_l 2^l; x) \\ &+ F([a^n] + \varepsilon_\Lambda 2^\Lambda + \ldots + \varepsilon_\lambda 2^\lambda, N^*; x). \end{aligned}$$

Next we introduce the notations $m_{\Lambda+1} = 0$ and

$$m_l = \varepsilon_\Lambda 2^{\Lambda-l} + \varepsilon_{\Lambda-1} 2^{\Lambda-l-1} + \ldots + \varepsilon_l$$

for $l = \lambda, \lambda+1, \ldots, \Lambda$. Then the inequalities $0 \leqslant m_l < 2^{\Lambda-l+1}$ $(l = \lambda, \lambda+1, \ldots$ $\ldots, \Lambda+1)$ are satisfied.

The relations $F(M, 0; x) = 0$ and $F(M, N; x) \geqslant 0$ imply that

$$\begin{aligned} F([a^n] + \varepsilon_\Lambda 2^\Lambda + \ldots + \varepsilon_{l+1} 2^{l+1}, \varepsilon_l 2^l; x) \\ = F([a^n] + m_{l+1} 2^{l+1}, \varepsilon_l 2^l; x) \\ \leqslant F([a^n] + m_{l+1} 2^{l+1}, 2^l; x). \end{aligned}$$

Hence indeed (33) holds.

Our next object is to prove (32) for the subsequence $N=[a^n]$ $(n=1,2,\dots)$. For this purpose we need the following:

Lemma 13. *Let $1<a\leqslant 2$ and let E_n denote the set*

$$E_n=\{x|x\varepsilon I;\ F(0,[a^n];x)\geqslant a^{1/4}\phi([a^n])\}. \tag{36}$$

Then given $\eta>0$, arbitrarily small, there exists a $n_0=n_0(q,a,\eta)$ such that $\sum_{n\geqslant n}\mu(E_n)\leqslant\eta/2$.

Proof of Lemma 13. If $n\geqslant n_0(q,a)$ then we can apply Lemma 7 with $(\alpha,\beta)=I$, $N=[a^n]$ and $t=a$. Hence for $n\geqslant n_0(q,a)$ we have

$$\mu(E_n)<\frac{18\log\log[a^n]}{(\log[a^n])^a}<c(a)\frac{\log n}{n^a}$$

where $c(a)>0$ depends only on a. Since $a>1$ the series $\sum n^{-a}\log n$ is convergent and so $\sum_{n\geqslant n_0}\mu(E_n)\leqslant\eta/2$ if $n_0\geqslant n_0(q,a,\eta)$ is large enough.

Finally we have to fill up the gaps in the lacunary sequence

$$[a^n],\qquad (n=1,2,3,\dots).$$

Having Lemma 12 this will be accomplished by proving the following:

Lemma 14. *Let $1<a\leqslant 2$, $n\geqslant n_0(a)$ and let $\Lambda(n)\geqslant 1$, $\lambda(n)\geqslant 1$ be the integers defined previously. Define the set $E_{lm}=E_{lm}^n$*

$$E_{lm}=\{x|x\varepsilon I;\ F([a^n]+m2^{l+1},2^l;x)\geqslant 2^{2+\frac{l-\Lambda}{4}}(a-1)^{1/4}\phi([a^n])\}. \tag{37}$$

Then given $\eta>0$, arbitrarily small, there exists a $n_0=n_0(q,a,\eta)$ such that

$$\sum_{n\geqslant n_0}\sum_{\lambda\leqslant l\leqslant\Lambda}\sum_{0\leqslant m<2^{\Lambda-l}}\mu(E_{lm})\leqslant\eta/2.$$

Proof of Lemma 14. Let $n\geqslant n_0(a)$ so large that $\Lambda(n)\geqslant\lambda(n)\geqslant 2$ and $\log\log[a^n]\geqslant 4$. According to the definition of $\Lambda(n)$ we have

$$2^\Lambda\leqslant[a^{n+1}]-[a^n]<a^{n+1}-[a^n]<(a-1)[a^n]+a,$$

and so

$$2^{\Lambda-1}\leqslant 2^{\Lambda-1}+(2^{\Lambda-1}-a)<(a-1)[a^n].$$

Similarly, according to the definition of $\lambda(n)$ we have for $l\geqslant\lambda$

$$3\log\log 2^l\geqslant 3\log\log[a^n]^{1/3}=\log\log[a^n]+(2\log\log[a^n]-\log 27)>\log\log[a^n]$$

Hence for $n\geqslant n_0(a)$ we have the inequalities

$$3\log\log 2^l\geqslant\log\log[a^n], \tag{38}$$

$$(a-1)[a^n]\geqslant 2^{\Lambda-1}, \tag{39}$$

$$\log\log[a^n]\geqslant 4. \tag{40}$$

From (18) it follows that

$$\int_0^1 F(M,N;x)^{2p}dx<c(q)p(pN)^p$$

for any $M \geqslant 0$, $N = 1, 2, 3, \ldots$ and $p \leqslant 3 \log\log N$. Therefore if $l \geqslant \lambda$ we obtain

$$\mu(E_{lm}) \leqslant \int\limits_{E_{lm}} \left(\frac{F([a^n]+m\,2^{l+1}, 2^l; x)}{2^{2+\frac{l-\Lambda}{4}} (a-1)^{1/2} \phi([a^n])} \right)^{2p} dx \leqslant \int\limits_0^1 (\ldots)^{2p}\, dx$$

$$< c(q)\, p \left(\frac{p 2^l}{2^{4+\frac{l-\Lambda}{2}} (a-1)\,[a^n] \log\log [a^n]} \right)^p$$

for any $p \leqslant 3 \log\log 2^l$. Hence by (38) $p = [\log\log [a^n]]$ is an admitted value of p. Using the inequalities (39) and (40) we see that

$$\mu(E_{lm}) < c(q)\, p \left(\frac{p}{2^{3+\frac{\Lambda-\lambda}{2}} \log\log [a^n]} \right)^p < c(q)\, p e^{-2p}\, 2^{2(l-\Lambda)}$$

$$< c(q) (\log n)\, 2^{2(l-\Lambda)}\, e^2 (\log [a^n])^{-2}.$$

Consequently

$$\mu(E_{lm}) < c(q, a)\, 2^{2(l-\Lambda)} \frac{\log n}{n^2}$$

for any $m \geqslant 0$ and $l \geqslant \lambda$.

Summing over m, $0 \leqslant m < 2^{\Lambda - l}$ we get

$$\sum_{0 \leqslant m < 2^{\Lambda-l}} \mu(E_{lm}) < c(q, a)\, 2^{l-\Lambda} \frac{\log n}{n^2},$$

and

$$\sum_{\lambda \leqslant l \leqslant \Lambda} \sum_{0 \leqslant m < 2^{\Lambda-l}} \mu(E_{lm}) < 2c(q, a) \frac{\log n}{n^2}.$$

The series $\sum n^{-2} \log n$ being convergent the statement of Lemma 14 follows immediately.

Now given $\varepsilon > 0$ and $\eta > 0$, arbitrarily small, first we choose $a = a(\varepsilon) > 1$ such that

$$1 < a^{1/2} \leqslant 1 + \frac{\varepsilon}{3} \text{ and } 4(a-1) \sum_{k=0}^{\infty} 2^{-k/4} \leqslant \frac{\varepsilon}{3}.$$

Next we choose $N_0 = N_0(q, \varepsilon, \eta)$ so large that $N_0^{-2/5} \leqslant \varepsilon/3$, and $n(N_0) = n_0(q, \varepsilon, \eta)$ satisfies the requirements of Lemma 13 and 14. Then the set

$$E = I - \bigcup_{n \geqslant n_0} [E_n \cup (\bigcup_l \bigcup_m E_{lm})]$$

has measure $\mu(E) \geqslant 1 - \eta$.

Let $N \geqslant N_0$ be arbitrary. Then we have by Lemma 12, inequality (33)

$$\frac{F(0, N; x)}{\phi(N)} \leqslant \frac{F(0, [a^n]; x)}{\phi([a^n])} + \sum_{l=\lambda}^{\Lambda} \frac{F([a^n]+m_{l+1}\, 2^{l+1}, 2^l; x)}{\phi([a^n])}$$

$$+ \frac{F([a^n]+m_\lambda 2^\lambda, N^*; x)}{\sqrt{N}}.$$

If $x \varepsilon E$ then we have by (36) and (37)

$$\frac{F(0, N; x)}{\phi(N)} \leqslant a^{1/2} + 4(a-1)^{1/2} \sum_{l=\lambda}^{\Lambda} 2^{\frac{l-\lambda}{4}} + \frac{N^*}{\sqrt{N}}.$$

Using the conditions on $a=a(\varepsilon)$ and the inequality $N^*<2N^{1/s}$ we obtain $F(0, N; x)/\phi(N)<1+\varepsilon$ for every $N \geqslant N_0(q, \varepsilon, \eta)$ and every $x \varepsilon E$, where $\mu(E) \geqslant 1-\eta$. This proves Lemma 11. Hence our theorem has been proved.

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BIBLIOGRAPHY

1. Kac, M., Probability methods in some problems of analysis and number theory. Bull. Amer. Math. Soc., **55**, 641–665 (1949).
2. Salem, R. and A. Zygmund, La loi du logarithm itéré pour les séries trigonométriques lacunaires. Bull. des Sciences Mathématiques, **74**, 209–224 (1950).
3. Gál, I. S. and J. F. Koksma, Sur l'ordre de grandeur des fonctions sommables. Indagationes Mathematicae, **12**, 192–207 (1950).
4. Gál, I. S., Sur la majoration des suites de fonctions. Proc. Kon. Ned. Akad. van Wet., **54**, 243–251 (1951).